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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 19 (2009), No. 1, 73 - 82

ABOUT SOME BIVARIATE OPERATORS OF SCHURER TYPE

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Abstract. In this paper, we will obtain a form of Bernstein-Schurer bivariate operators and finally we will give an approximation theorem for them.

1. Introduction

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\Delta_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x, y \geq 0, x + y \leq 1\}$. For $m \in \mathbb{N}$, the operator $B_m : C([0, 1] \times [0, 1]) \to C(\Delta_2)$ defined for any function $f \in C([0, 1] \times [0, 1])$ by

$$(B_m f)(x, y) = \sum_{\substack{k,j=0 \\ k+j \leq m}} p_{m,k,j}(x, y) f\left(\frac{k}{m}, \frac{j}{m}\right) \tag{1.1}$$

for any $(x, y) \in \Delta_2$, where

$$p_{m,k,j}(x, y) = \frac{m!}{k!j!(m - k - j)!} x^k y^j (1 - x - y)^{m-k-j}, \tag{1.2}$$

for any $k, j \in \mathbb{N}_0$, $k + j \leq m$ and any $(x, y) \in \Delta_2$ is named the Bernstein bivariate operator (see [10]).

Keywords and phrases: Bivariate operators of Schurer type, modulus of continuity of the bivariate functions, mixed modulus of smoothness, approximation theorem.
(2000)Mathematics Subject Classification: 41A10, 41A25, 41A36, 41A63

Let $e_{ij} : \Delta_2 \to \mathbb{R}$ be the test functions, defined by $e_{ij}(x,y) = x^i y^j$ for any $(x,y) \in \Delta_2$, where $i,j \in \mathbb{N}_0$. In the paper [9] the following representation for the polynomials $B_m e_{pq}$ is proved.

Lemma 1.1. *The operators $(B_m)_{m\geq 1}$ verify for any $(x,y) \in \Delta_2$ and any $m \in \mathbb{N}$, $p,q \in \mathbb{N}_0$ the following equality*

$$(B_m e_{pq})(x,y) = \frac{1}{m^{p+q}} \sum_{i=0}^{p} \sum_{j=0}^{q} m^{[i+j]} S(p,i) S(q,j) x^i y^j, \tag{1.3}$$

*where $S(p,i)$, $S(q,j)$ are the Stirling's numbers of second kind and $m^{[k]} =$
$= m(m-1)\dots(m-k+1)$, $k \in \mathbb{N}_0$, $m^{[0]} = 1$.*

Let $I_1, I_2 \subset \mathbb{R}$ be given intervals and $f : I_1 \times I_2 \to \mathbb{R}$ be a bounded function. The function $\omega_{total}(f; \cdot, *) : [0,\infty) \times [0,\infty) \to \mathbb{R}$, defined for any $(\delta_1,\delta_2) \in [0,\infty) \times [0,\infty)$ by

$$\omega_{total}(f;\delta_1,\delta_2) = \sup\left\{|f(x,y) - f(x',y')| : (x,y),(x',y') \in I_1 \times I_2,\ |x-x'| \leq \delta_1, |y-y'| \leq \delta_2\right\} \tag{1.4}$$

is called the first order modulus of smoothness of function f or total modulus of continuity of function f. For some further informations on this measure of smoothness see for example [5] or [14]. The following result is given in [13].

Theorem 1.1. *Let $L : C(I_1 \times I_2) \to B(I_1 \times I_2)$ be a constant reproducing linear positive operator. For any $f \in C(I_1 \times I_2)$, any $(x,y) \in I_1 \times I_2$ and any $\delta_1, \delta_2 > 0$, the following inequality*

$$|(Lf)(x,y) - f(x,y)| \leq \left(1 + \delta_1^{-1}\sqrt{(L(\cdot - x)^2)(x,y)}\right) \cdot \left(1 + \delta_2^{-1}\sqrt{(L(* - y)^2)(x,y)}\right) \omega_{total}(f;\delta_1,\delta_2) \tag{1.5}$$

holds, where " $\cdot$ " and " $*$ " stand for the first and the second variable.

The purpose of this paper is to give a representation for the bivariate operators and GBS operators of Schurer type, to establish a convergence theorem for these operators. We also give an approximation theorem for these operators in terms of the first modulus of smoothness and of the mixed modulus of smoothness.

2. The construct of the bivariate operators of Schurer type. Approximation and convergence theorems

Let $p \in \mathbb{N}_0$ be given and $m \in \mathbb{N}$. The operator $\tilde{B}_{m,p} : C([0,1+p] \times [0,1+p]) \to C(\Delta_2)$ defined for any function $f \in C([0,1+p]\times[0,1+p])$ and any $(x,y) \in \Delta_2$ by

$$(2.1) \qquad (\tilde{B}_{m,p}f)(x,y) = \sum_{\substack{k,j=0 \\ k+j \leq m+p}} p_{m+p,k,j}(x,y) f\left(\frac{k}{m}, \frac{j}{m}\right),$$

is a bivariate operator of Schurer type. Clearly, this operator is linear and positive. For $p = 0$, we obtain the Bernstein bivariate operator (1.1).

Lemma 2.1. *The operators $(\tilde{B}_{m,p})_{m\geq 1}$ verify for any $(x,y) \in \Delta_2$ the following equality:*
(2.2)

$$(\tilde{B}_{m,p}e_{ij})(x,y) = \frac{1}{(m+p)^{i+j}} \sum_{\nu_1=0}^{i} \sum_{\nu_2=0}^{j} (m+p)^{[i+j]} S(i,\nu_1) S(j,\nu_2) x^{\nu_1} y^{\nu_2},$$

for any $i, j \in \mathbb{N}_0$.

Proof. We use the equalities

$$m^{i+j}(\tilde{B}_{m,p}e_{ij})(x,y) = (m+p)^{i+j}(B_{m+p}e_{ij})(x,y),$$

for any $i, j \in \mathbb{N}_0$ and Lemma 1.1. □

Lemma 2.2. *The operators* $(\tilde{B}_{m,p})_{m\geq 1}$ *verify for any* $(x,y)\in\Delta_2$ *the following equalities:*

(2.3)
$$(\tilde{B}_{m,p}e_{00})(x,y)=1,$$

(2.4)
$$m^2(\tilde{B}_{m,p}(\cdot-x)^2)(x,y)=[-m+p(p-1)]x^2+(m+p)x,$$

(2.5)
$$m^2(\tilde{B}_{m,p}(*-y)^2)(x,y)=[-m+p(p-1)]y^2+(m+p)y,$$

(2.6)
$$m^4(\tilde{B}_{m,p}(\cdot-x)^2(*-y)^2)(x,y)=[2m^2-(5p^2-10p+4)m+ \\ +p(p-1)(p-2)^2]x^2y^2+[m^2-(p^2-4p+2)m-p(p-1)(p-2)]\cdot \\ \cdot xy(1-x)(1-y)+(p-1)^2(m+p)xy,$$

(2.7)
$$m^6(\tilde{B}_{m,p}(\cdot-x)^4(*-y)^2)(x,y)=xy(m+p)(m+p-1)+ \\ +xy^2(m+p)[-m+(p-1)(p-2)]+x^2y(m+p)(m+p-1)\cdot \\ \cdot[3m+7(p-2)]+x^2y^2(m+p)[-3m^2+(3p^2-30p+41)m+ \\ +7(p-1)(p-2)(p-3)]+6x^3y(m+p)(m+p-1)[-m+ \\ +(p-2)(p-3)]+6x^3y^2(m+p)[3m^2-(6p^2-26p+26)m+ \\ +(p-1)(p-2)(p-3)(p-4)]+x^4y(m+p)[3m^2-(6p^2-26p+ \\ +26)m+(p-1)(p-2)(p-3)(p-4)]+x^4y^2[-15m^3+(45p^2- \\ -165p+130)m^2-(15p^4-130p^3+375p^2-404p+120)m+ \\ +p(p-1)(p-2)(p-3)(p-4)(p-5)],$$

and a similar relation for $m^6(\tilde{B}_{m,p}(\cdot-x)^2(*-y)^2)(x,y)$.

Proof. We apply Lemma 2.1 and the equalities $(\tilde{B}_{m,p}(\cdot-x)^2)(x,y)= =(\tilde{B}_{m,p}e_{20})(x,y)-2x(\tilde{B}_{m,p}e_{10})(x,y)+(\tilde{B}_{m,p}e_{00})(x,y)$,
$(\tilde{B}_{m,p}(\cdot-x)^2(*-y)^2)(x,y)=(\tilde{B}_{m,p}e_{22})(x,y)-2y(\tilde{B}_{m,p}e_{21})(x,y)+ +y^2(\tilde{B}_{m,p}e_{20})(x,y)-2x(\tilde{B}_{m,p}e_{12})(x,y)+4xy(\tilde{B}_{m,p}e_{11})(x,y)- -2xy^2(\tilde{B}_{m,p}e_{10})(x,y)+x^2(\tilde{B}_{m,p}e_{02})(x,y)-2x^2y(\tilde{B}_{m,p}e_{01})(x,y)+ +x^2y^2(\tilde{B}_{m,p}e_{00})(x,y)$ and
$(\tilde{B}_{m,p}(\cdot-x)^4(*-y)^2)(x,y)=(\tilde{B}_{m,p}e_{42})(x,y)-2y(\tilde{B}_{m,p}e_{41})(x,y)+ +y^2(\tilde{B}_{m,p}e_{40})(x,y)-4x(\tilde{B}_{m,p}e_{32})(x,y)+8xy(\tilde{B}_{m,p}e_{31})(x,y)-$

$- 4xy^2(\tilde{B}_{m,p}e_{30})(x, y) + 6x^2(\tilde{B}_{m,p}e_{22})(x, y) - 12x^2y(\tilde{B}_{m,p}e_{21})(x, y) +$
$+ 6x^2y^2(\tilde{B}_{m,p}e_{20})(x, y) - 4x^3(\tilde{B}_{m,p}e_{12})(x, y) + 8x^3y(\tilde{B}_{m,p}e_{11})(x, y) -$
$- 4x^3y^2(\tilde{B}_{m,p}e_{10})(x, y) + x^4(\tilde{B}_{m,p}e_{02})(x, y) - 2x^4y(\tilde{B}_{m,p}e_{01})(x, y) +$
$+ x^4y^2(\tilde{B}_{m,p}e_{00})(x, y).$ □

Lemma 2.3. *The operators $(\tilde{B}_{m,p})_{m\geq 1}$ verify for any $(x, y) \in \Delta_2$ the following estimations*

$$4m^2(\tilde{B}_{m,p}(\cdot - x)^2)(x, y) \leq m + 4p^2 \tag{2.8}$$

and

$$4m^2(\tilde{B}_{m,p}(* - y)^2)(x, y) \leq m + 4p^2. \tag{2.9}$$

Proof. Using the relation (2.4), we can write $(\tilde{B}_{m,p}(\cdot - x)^2)(x, y) = \frac{x(1-x)}{m} + \frac{p(p-1)x^2+px}{m^2} \leq \frac{1}{4m} + \frac{p^2}{m^2} = \frac{m+4p^2}{4m^2}$, taking into account that $x(1 - x) \leq \frac{1}{4}$ and $p(p - 1)x^2 + px \leq p^2$, for any $x \in [0, 1]$. □

Theorem 2.1. *If $f \in C([0, 1+p]\times[0, 1+p])$, then for any $(x, y) \in \Delta_2$ and any $m \in \mathbb{N}$, we have*

$$|f(x, y) - (\tilde{B}_{m,p}f)(x, y)| \leq \left(1 + \delta_1^{-1}\sqrt{\frac{m + 4p^2}{4m^2}}\right) \cdot \tag{2.10}$$

$$\cdot \left(1 + \delta_2^{-1}\sqrt{\frac{m + 4p^2}{4m^2}}\right) \omega_{total}(f; \delta_1, \delta_2),$$

for any $\delta_1, \delta_2 > 0$ and

$$|f(x, y) - (\tilde{B}_{m,p}f)(x, y)| \leq 4\omega_{total}\left(f; \sqrt{\frac{m + 4p^2}{4m^2}}, \sqrt{\frac{m + 4p^2}{4m^2}}\right). \tag{2.11}$$

Proof. The relation (2.10) results from Theorem 1.1 and Lemma 2.2; choosing by $\delta_1 = \delta_2 = \sqrt{\frac{m+4p^2}{4m^2}}$, we obtain the relation (2.11). □

Corollary 2.1. *If $f \in C([0, 1 + p] \times [0, 1 + p])$, then*

$$\lim_{m\to\infty} \tilde{B}_{m,p}f = f \tag{2.12}$$

uniformly on Δ_2.

3. APPROXIMATION AND CONVERGENCE THEOREMS FOR GBS OPERATORS OF SCHURER TYPE

In the following, let X and Y be compact real intervals. A function $f : X \times Y \to \mathbb{R}$ is called B-continuous (Bögel-continuous) function at $(x_0, y_0) \in X \times Y$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f\left[(x,y),(x_0,y_0)\right] = 0.$$

Here $\Delta f\left[(x,y),(x_0,y_0)\right] = f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0)$ denotes a so-called mixed difference of f.

A function $f : X \times Y \to \mathbb{R}$ is called B-differentiable (Bögel-differentiable) function at $(x_0, y_0) \in X \times Y$ if it exists and if the limit is finite:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f\left[(x,y),(x_0,y_0)\right]}{(x-x_0)(y-y_0)}.$$

The limit is named the B-differential of f at the point (x_0, y_0) and is noted by $D_B f(x_0, y_0)$.

The definitions of B-continuity and B-differentiability were introduced by K. Bögel in the papers [6], [7] and [8].

The function $f : X \times Y \to \mathbb{R}$ is B-bounded on $X \times Y$ if there exists $K > 0$ such that

$$\left|\Delta f\left[(x,y),(s,t)\right]\right| \le K$$

for any $(x,y),(s,t) \in X \times Y$.

We shall use the functions sets $B(X \times Y) = \{f : X \times Y \to \mathbb{R} | f$ bounded on $X \times Y\}$ with the usual sup-norm $\|\cdot\|_\infty$, $B_b(X \times Y) = \{f : X \times Y \to \mathbb{R} | f$ B-bounded on $X \times Y\}$ and we set $\|f\|_B = \sup_{(x,y),(s,t)\in X\times Y} \left|\Delta f\left[(x,y),(s,t)\right]\right|$ where $f \in B_b(X \times Y)$, $C_b(X \times Y) = \{f : X \times Y \to \mathbb{R} | f$ B-continuous on $X \times Y\}$ and $D_b(X \times Y) = \{f : X \times Y \to \mathbb{R} | f$ B-differentiable on $X \times Y\}$.

Let $f \in B_b(X \times Y)$. The function $\omega_{\text{mixed}}(f; \cdot, \cdot) : [0,\infty) \times [0,\infty) \to \mathbb{R}$, defined by

(3.1)
$$\omega_{\text{mixed}}(f; \delta_1, \delta_2) = \sup\left\{\left|\Delta f[(x,y),(s,t)]\right| : |x-s| \le \delta_1, |y-t| \le \delta_2\right\}$$

for any $(\delta_1, \delta_2) \in [0,\infty)\times[0,\infty)$ is called the mixed modulus of smoothness.

For related topics, see [1], [2], [3] and [4].

Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator. The operator $UL : C_b(X \times Y) \to B(X \times Y)$ defined for any function $f \in C_b(X \times Y)$ and any $(x, y) \in X \times Y$ by

$$(ULf)(x,y) = \left(L(f(\cdot,y) + f(x,*) - f(\cdot,*))\right)(x,y) \tag{3.2}$$

is called GBS operator ("Generalized Boolean Sum" operator) associated to the operator L, where "$\cdot$" and "$*$" stand for the first and respectively the second variable. Let $e_{ij} : X \times Y \to \mathbb{R}$ be the test functions, defined by $e_{ij}(x, y) = x^i y^j$, for any $(x, y) \in X \times Y$, where $i, j \in \mathbb{N}_0$. The following theorem is proved in [3].

Theorem 3.1. *Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator and $UL : C_b(X \times Y) \to B(X \times Y)$ the associated GBS operator. Then for any $f \in C_b(X \times Y)$, any $(x, y) \in (X \times Y)$ and any $\delta_1, \delta_2 > 0$, we have*

(3.3)

$$\begin{aligned}
&|f(x,y) - (ULf)(x,y)| \le |f(x,y)|\,|1 - (Le_{00})(x,y)| + \\
&+ \left[(Le_{00})(x,y) + \delta_1^{-1}\sqrt{\left(L(\cdot - x)^2\right)(x,y)} + \delta_2^{-1}\sqrt{\left(L(* - y)^2\right)(x,y)} + \right.\\
&\left. + \delta_1^{-1}\delta_2^{-1}\sqrt{\left(L(\cdot - x)^2(* - y)^2\right)(x,y)}\right]\omega_{mixed}(f;\delta_1,\delta_2).
\end{aligned}$$

For B-differentiable functions, we have (see [11]):

Theorem 3.2. *Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator and $UL : C_b(X \times Y) \to B(X \times Y)$ the associated GBS operator. Then for any $f \in D_b(X \times Y)$ with $D_B f \in B(X \times Y)$, any $(x, y) \in X \times Y$ and any $\delta_1, \delta_2 > 0$, we have*

(3.4)

$$\begin{aligned}
&|f(x,y) - (ULf)(x,y)| \le \\
&\le |f(x,y)||1 - (Le_{00})(x,y)| + 3\|D_B f\|_\infty \sqrt{\left(L(\cdot - x)^2(* - y)^2\right)(x,y)} + \\
&+ \left[\sqrt{(L(\cdot - x)^2(* - y)^2)(x,y)} + \delta_1^{-1}\sqrt{(L(\cdot - x)^4(* - y)^2)(x,y)} + \right.\\
&+ \delta_2^{-1}\sqrt{(L(\cdot - x)^2(* - y)^4)(x,y)} + \\
&\left. + \delta_1^{-1}\delta_2^{-1}\left(L(\cdot - x)^2(* - y)^2\right)(x,y)\right]\omega_{mixed}(D_B f;\delta_1,\delta_2).
\end{aligned}$$

Lemma 3.1. *There exists a natural number $m_1 \in \mathbb{N}$ such that*

$$(\tilde{B}_{m,p}(\cdot - x)^2(* - y)^2)(x,y) \le \frac{1}{4m^2}, \tag{3.5}$$

$$(\tilde{B}_{m,p}(\cdot - x)^4(* - y)^2)(x,y) \le \frac{1}{4m^3} \tag{3.6}$$

and

$$(\tilde{B}_{m,p}(\cdot - x)^2(* - y)^4)(x,y) \le \frac{1}{4m^3}, \tag{3.7}$$

for any $m \in \mathbb{N}$, $m \ge m_1$ and any $(x,y) \in \Delta_2$.

Proof. Using the relation (2.6), we obtain that $16m^4(\tilde{B}_{m,p}(\cdot - x)^2(* - y)^2)(x,y) \le 3m^2 - 2(p^2 - 3p + 1)m + p(p-1)(p^2 - p + 2)$, from where the relation (3.5) results. The relation (2.7) can be write $m^6(\tilde{B}_{m,p}(\cdot - x)^4(* - y)^2)(x,y) = Am^3 + Bm^2 + Cm + D$, where A, B, C, D are real numbers depending on p, x, y; here, $A = 3x(1-x)[xy(1-x)(1-y) + 4x^2y^2] \le \frac{15}{64}$, from where the relation (3.6) results. The relation (3.7) follows analogously. We used the inequalities $x(1-x) \le 1/4$, for any $x \in [0,1]$, $xy(1-x)(1-y) \le 1/16$ and $xy \le 1/4$, for any $(x,y) \in \Delta_2$. □

Theorem 3.3. *If $f \in C_b([0,1+p] \times [0,1+p])$, then for any $(x,y) \in \Delta_2$ and any $m \in \mathbb{N}$, $m \ge m_1$, the following inequalities*

$$|(U\tilde{B}_{m,p}f)(x,y) - f(x,y)| \le \left(1 + \delta_1^{-1}\sqrt{\frac{m+4p^2}{4m^2}} + \delta_2^{-1}\sqrt{\frac{m+4p^2}{4m^2}} + \delta_1^{-1}\delta_2^{-1}\frac{1}{2m}\right)\omega_{mixed}(f;\delta_1,\delta_2) \tag{3.8}$$

for any $\delta_1, \delta_2 > 0$ and

$$|(U\tilde{B}_{m,p}f)(x,y) - f(x,y)| \le \frac{5}{2}\omega_{mixed}\left(f; \sqrt{\frac{m+4p^2}{m^2}}, \sqrt{\frac{m+4p^2}{m^2}}\right) \tag{3.9}$$

hold, where

$$(U\tilde{B}_{m,p}f)(x,y) = \sum_{\substack{k,j=0\\ k+j\le m+p}} p_{m+p,k,j}(x,y)\left(f\left(\frac{k}{m}, y\right) + f\left(x, \frac{j}{m}\right) - \right.$$

$$-f\left(\frac{k}{m},\frac{j}{m}\right)\biggr).$$

Proof. For the first inequality, we apply Theorem 3.1 and Lemma 3.1. The inequality (3.9) is obtained from (3.8) by choosing $\delta_1 = \delta_2 = \sqrt{\frac{m+4p^2}{m^2}}$. □

Corollary 3.1. *If $f \in C_b([0,1+p]\times[0,1+p])$, then*

$$\lim_{m\to\infty} U\tilde{B}_{m,p}f = f \tag{3.10}$$

uniformly on Δ_2.

Proof. It results from the relation (3.9). □

Theorem 3.4. *Let the function $f \in D_b([0,1+p]\times[0,1+p])$ with $D_Bf \in B([0,1+p]\times[0,1+p])$. Then, for any $(x,y)\in\Delta_2$ and for any $m\in\mathbb{N}$, $m\geq m_1$, we have*

(3.11)

$$|(U\tilde{B}_{m,p}f)(x,y) - f(x,y)| \leq \frac{3}{2m}\|D_Bf\|_\infty +$$
$$+\frac{1}{2m}\left(1+\delta_1^{-1}\frac{1}{\sqrt{m}}+\delta_2^{-1}\frac{1}{\sqrt{m}}+\delta_1^{-1}\delta_2^{-1}\frac{1}{2m}\right)\omega_{mixed}(D_Bf;\delta_1,\delta_2)$$

for any $\delta_1,\delta_2>0$ and

$$|(U\tilde{B}_{m,p}f)(x,y) - f(x,y)| \leq \frac{3}{2m}\|D_Bf\|_\infty + \tag{3.12}$$
$$+\frac{7}{4m}\omega_{mixed}\left(D_Bf;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{m}}\right).$$

Proof. It results from Theorem 3.2 and Lemma 3.1. □

Remark 3.1. *Other construction for bivariate operators of Schurer type can be found in* [5].

Remark 3.2. *For $p=0$, we find some results obtained in the paper* [12].

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