"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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CONSIDERATIONS ON TIME MINIMIZATION IN TRANSPORTATION PROBLEM WITH IMPURITIES

GHEORGHE DOGARU and CRISTINA NISTOR

Abstract. The transportation problems with impurities in goods are very important from practical point of view because of higher frequency. An extension of time transportation problem is considered when goods can have some impurities and the final mixture of goods arrived at destination have some specifications. This time transportation problem is in connection with linear lexicographical transportation problem with impurities. In this paper is also presented an algorithm to solve bottleneck transportation problem making a connection with linear lexicographical transportation problem with impurities. The algorithm optimality conditions are similar to those given by H. Issermann, but there are some modifications caused by impurities.

1. INTRODUCTION

The transportation problems (T.P.) with impurities in goods are of high importance from the practical point of view due to the higher frequency of this kind of problems.

The problem of transportation in minimum time has been studied by Hammer 1969, Garfinkel and Rao 1971. Also, a procedure for time minimization in a transportation problem was developed by J. K. Sharma and Kanti Swarup and is based on moving from a basic feasible solution to another till the last solution is arrived at.

In many situations from real world, impurities have some features depending on source and recipient requirements. Such transportation problems of cost minimization with linear objective functions can be solve efficiently by K.B.Haley and A.J.Smith algorithm.

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Recently, using duality results for fractional programming, S.Chandra and P.K. Saxena have modified the algorithm of K.B.Haley and A.I.Smith to solve fractionary transportation problem after cost minimization of problem with impurities.

The main objective of this paper is to elaborate a time minimization algorithm different than the bottleneck type algorithm used by Issermann.

The originality of the paper consists in the fact that we use the linear lexicographical method that generalizes the method of minimum path using graphs. We use the couples (i, j) instead of using separately the i and j indices. The advantage of the algorithm is that it has a high convergence speed (the run time is low). The high convergence speed is given by the use of (i, j) couples and the chosen form for θ.

In the following we study the bottleneck transportation problem with impurities. This problem appears in connection with perishable goods, provision in maximal emergency situations, or when military equipments or units battle are sent from their military base on battle-front.

The bottleneck transportation problems was studied by P.L.Hammer, W. Ssware, R. S. Garfinkel, H. Issermann etc, but these don't take in consideration impurities in goods.

Next we present an algorithm to solve bottleneck transportation problem doing a connection with linear lexicographical transportation problem (L.T.P.) and integrates the results of K. B. Haley, S. Chandra and P. K. Saxena, as well as H. Isermann. This algorithm takes in consideration the special structure of transportation problem and depends strongly of optimality considerations which are similar to those given by H. Isermann, but there are some modifications caused by impurities.

2. STATEMENT OF THE PROBLEM

The mathematical formulation of the problem is as follows:

(T.P.) $$Min\ t = \max_{i,j}\{t_{ij} \mid x_{ij} > \mathbf{0}\} \qquad (1)$$

with restrictions

$$\sum_j x_{ij} = a_i \qquad (2)$$

$$\sum_i x_{ij} = b_j \qquad (3)$$

$$\sum_i f_{ijk} x_{ij} \le q_{jk} \qquad (4)$$

$$x_{ij} \geq \mathbf{0} \tag{5}$$

$i=1,2,\ldots,M$, $j=1,2,\ldots,N$, $k=1,2,\ldots,P$,

where a_i, b_j, x_{ij}, t_{ij} are the classical well-known notations, f_{ijk} is the unit from impurity *k* in goods transported from source *i* to destination *j*, q_{ijk} are *k* type impurities quantity admitted at destination *j*, a_i and b_j are nonnegative, and

$$\sum_{i=1}^{M} a_i = \sum_{j=1}^{N} b_j \ .$$

Without restrictions (4) about impurities, the problem (T.P.) become the usual bottleneck transportation problem studied by L. Hammer, H. Issermann.

In this context, restrictions (4) are as follows:

$$\sum_i f_{ijk} x_{ij} + x_{N+k,j} = q_{jk} \ , \tag{6}$$

$$x_{N+k,j} \geq 0 \ , \tag{7}$$

where $x_{N+k,j}$ are compensations variables to the restrictions with impurities, and an admissible basic solution will consist of $NP + M + N - 1$ basic variables.

We will associate to the problem (T.P.) the next linear lexicographical transportation problem with impurities:

$$Lex\ \mathbf{min}\ Z = \sum_i \sum_j d_{ij} x_{ij} \tag{8}$$

with restrictions (2)-(5).

We make the above formulation partitioning the set $\gamma = M \times N$ in the subsets γ_c $(c = 1,2,...,e)$ just like H. Issermann, where γ is the Cartesian composition of the sets *{1,2,…,M}* and *{1,2,…,N}*, that are the variation sets of „i" and „j" indices and „e" represents the number of the couples (i,j) chosen to find the optimal solution.

Any of these subsets will consist of the pairs $(i, j) \in \gamma$ for which the transportation times t_{ij} have the same numerical value. The subset γ_1 contain all $(i, j) \in \gamma$ with t_{ij} the greatest value, γ_2 contain all $(i, j) \in \gamma$ with t_{ij} taking the next value as size etc. Hence, the subset γ_e contain all $(i, j) \in \gamma$ with t_{ij} taking the smallest value.

To every value x_{ij} with $(i, j) \in \gamma_c$, (c=1,2, ... ,*e*) is associated a *e*x1 unit vector and it considered the vectors $d_{ij} := e_c \ if \ (i, j) \in \gamma_c$.

3. DUALITY AND OPTIMALITY CONDITIONS

Let

$u_i, i=\overline{1,M}$, $v_j, j=\overline{1,N}$ and

t_{jk} , $j=\overline{1,N}$, $k=\overline{1,P}$

vectorial defined as follow:

$$d_{ij}-(u_i+v_j)-\sum_k t_{jk}f_{ijk}=0 \quad (9)$$

for those i,j for which x_{ij} is in the base and

$$t_{jk}=0 \quad (10)$$

for those *j* and *k* for which $x_{N+k,j}$ is in the base.

Let $\hat{U}=(\hat{u}_1,\hat{u}_2,...,\hat{u}_M;\hat{v}_1,\hat{v}_2,...,\hat{v}_N;\hat{t}_{11},\hat{t}_{12},...,\hat{t}_{NP})$ be a solution of (9) and (10). The vectors u_i, v_j and t_{jk} introduced above are the variables from the dual problem of the primary problem (L.T.P.).

The duality of the problem (L.T.P.) defined by (8), (2), (3), (5), (6), (7) can be written like this:

$$\text{(DLTP)}\ Lex\ \mathbf{max}\ \mathbf{G}=\sum_i \mathbf{v_i}a_i+\sum_j v_jb_j+\sum_j\sum_k t_{jk}q_{jk} \quad (11)$$

with restrictions

$$u_i+v_j+\sum_k f_{ijk}t_{jk}\le d_{ij} \quad (12)$$

$$t_{jk}\le 0 \quad (13)$$

and

u_i, v_j and t_{jk} without sign restrictions (14)

i=1,2,…,M , j=1,2,…,N , k=1,2,…,P.

Now, just like B. K. Haley and A. J. Smith, and also, S. Chandra and P. K. Saxena, we consider the following comparative differences:

$$\Delta_{ij}=d_{ij}-u_i-v_j-\sum_k f_{ijk}t_{jk}>0\text{, for } x_{ij}>0 \quad (15)$$

and

$$\Delta_{N+k,j}=t_{jk}=0\text{, for } x_{N+k,j}>0. \quad (16)$$

Using the theory of duality for (L.T.P.) and (DLTP) in absence of degeneration, the optimality criterions are:

$$\Delta_{ij}=d_{ij}-\hat{u}_i-\hat{v}_j-\sum_k \hat{t}_{jk}f_{ijk}\ge 0 \quad (17)$$

and

$$\Delta_{N+k,j} = \hat{t}_{jk} \geq 0 \text{, for } (i,j) \in \gamma . \qquad (18)$$

4. THE ALGORITHM

In this section is presented an algorithm to determine an admissible optimal bases solution of the problem (L.T.P.) in a finite number of iterations. The optimization algorithm is correct because I use the same structure of classical algorithms from the transportation problem. The novelty of this algorithm consists in the fact that I use the couples (i;j) instead of using the (i;j) indices in a sequential way. The steps of the algorithm are:

STEP 1.

We determine the lower bound t_e of t (according to R. S. Garfinkel and M. R. Rao) to reduce the dimensions of the vectors d_{ij} in the problem (8). Now, $t_e > t_{ij}$ for at least a pair $(i,j) \in \gamma$. Here, γ_c contains all the pairs $(i,j) \in \gamma$ with $t_e > t_{ij}$.

STEP 2.

We determine X^1 an admissible initial bases solution of the matrix T = $[t_{ij}]$ applying the method of K. B. Haley and A. J. Smith.

STEP 3.

From bottleneck time t of solution X^1, we determine an upper bound t_u. Let $t_u < t_{ij}$ for at least a pair $(i,j) \in \gamma$.

STEP 4.

The set $\gamma :=$ M x N is partitioned in the subsets γ_c and we determine the vectors $d_{ij} := e_c$ for all $(i,j) \in \gamma$ to obtain the cost matrix D.

STEP 5.

Using solution X^1, we recursively calculate the associated multipliers (the dual variables) u_i, v_j and t_{jk} such that

$$d_{ij} - (u_i + v_j + \sum_k t_{jk} f_{ijk}) = 0 \qquad (19)$$

for those i and j for which x_{ij} is in the bases and

$$t_{jk} = 0 , \qquad (20)$$

for those j and k for which $X_{N+k,j}$ is in the base.

Let $\hat{U} = (\hat{u}_1, \hat{u}_2, ..., \hat{u}_M ; \hat{v}_1, \hat{v}_2, ..., \hat{v}_N ; \hat{t}_{jk}, ..., \hat{t}_{NP})$ be a solution of (19) and (20).

STEP 6.

We evaluate the criterion vectors

$$\Delta_{ij} = d_{ij} - (\hat{u}_i + \hat{v}_j + \sum_k \hat{t}_{jk} f_{ijk}) \qquad (21)$$

$\Delta_{N+k,j} = \hat{t}_{jk}$ for all the others from outside the bases.(22)

STEP 7.

If all Δ_{ij} and $\Delta_{N+k,j}$ are lexicographical greater than or equal to zero, then the current bases solution is the optimal solution of the problem (L.T.P,) Go to step 10. Otherwise, go to step 8.

STEP 8.

We chose

$$\Delta_{gh} = Lex \min \{\Delta_{ij} \big| \Delta_{ij} \le 0\} \text{ or}$$

$$\Delta_{N+\bar{k},j} = Lex \min \{\Delta_{N+\bar{k},j} \big| \Delta_{N+\bar{k},j} \le 0\} \tag{23}$$

Applying the selection rule (23), the variable x_{gh} or $x_{N+\bar{k},j}$ become a bases solution of the new admissible bases solution.

STEP 9.

To change the current solution to a new admissible bases solution, we add n_{gh} or $n_{N+\bar{k},j}$ to the variable x_{gh} or $x_{N+\bar{k},j}$, and n_{rs} or $n_{N+w,s}$ to the bases variables x_{rs} or $x_{N+w,s}$.The values of n must satisfy the equations:

$$\sum_{r=1}^{N} n_{rs} = 0, s = 1,2,...,M , \tag{24}$$

$$\sum_{s=1}^{M} n_{rs} = 0, r = 1,2,...,N, \tag{25}$$

$$\sum_{w=1}^{P} f_{rsn} n_{rs} + n_{N+w,s} = 0, s = 1,2,...,M , w = 1,2,....,P , \tag{26}$$

with $n_{11} = 1$. Here $n_{rs} = 0$, if x_{rs} is not in the bases, and $n_{N+w,s} = 0$, if $x_{N+w,s}$ is not in the bases. There are NP + M + N -1 equations independent of (24), (25), (26) and NP + M + N unknowns. Moreover, the values of variables in the new admissible bases solution are given by

$$x_{rs} + n_{rs}\theta , x_{N+w,s} + n_{N+w,s}\theta .$$

Choosing a convenient value for θ, one of the variables can be bringing to zero, while the others remains positive, and we obtain a new admissible bases solution. The chosen value is:

$$\theta = \min_{\substack{n_{rs}<0, \\ n_{N+w,s}<0}} \left[-\frac{x_{rs}}{n_{rs}}, -\frac{x_{n+w,s}}{n_{N+w,s}} \right]. \tag{27}$$

Go to step 5.

STEP 10.

If $\hat{X} = (\hat{x}_{ij})$ is a optimal solution of the problem (L.T.P), then $\hat{Z} = \sum_i \sum_j (d_{ij} \hat{x}_{ij})$ and c the index of the first positive component of the optimal flux vector $\hat{Z}$. Also, $\hat{t} = t_{ij}$, with $(i, j) \in \gamma_c$ is the optimal time of the bottleneck transportation problem.

The optimal transportation solution $\hat{X} = (\hat{x}_{ij})$, also minimize the linear function $z_{\hat{c}} = \sum_{i,j} (x_{ij}), (i, j) \in \gamma_{\hat{c}}$, which represents the total repartition on which time $\hat{t}$ is asked.

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Dogaru Gheorghe
Faculty of Civile Maritime Transport
"Mircea cel Batran" Maritime Academy
Constanta Romania
gheorghe_dogaru@yahoo.com

Nistor Cristina
Faculty of Navigation and Maritime Transport
Constanta Maritime University
Constanta, Romania
nistor.cristina@gmail.com