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STOCHASTIC ORDERING OF SUMS OF TWO INDEPENDENT EXPONENTIAL RANDOM VARIABLES

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Abstract. In this paper, we formulate a necessary and sufficient condition for ranking in the *convex order* the sums of two independent exponential random variables. This characterization is added to a series of recent results in the field.

1. INTRODUCTION

The stochastic comparisons of positive random variables play an important role in many fields of applied probabilities, especially in reliability theory. On the other hand, the lifetimes of the components of a Markov system in reliability are exponential random variables. Many recent studies refer to the comparison, in various stochastic orders, of the convolutions of exponential random variables. We limit here the discussion to the case of two random variables. Our study focuses on the comparison in the *convex order*.

Let X and Y be two positive random variables. Recall that X is said to be greater that Y in the *convex order* (denoted by $X \geq_{cx} Y$) if $E[\varphi(X)] \geq E[\varphi(Y)]$, for all convex functions $\varphi : [0, \infty) \to$ R.

If $\overline{F}$ and $\overline{G}$ are the survival functions of X and Y, respectively, then $X \geq_{cx} Y$ in and only if $\int_x^\infty \overline{F}(u)du \geq \int_x^\infty \overline{G}(u)du$, for all $x \geq 0$. Note that a comprehensive treatment of stochastic orders can be found in [4].

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Let X be an exponential random variable with parameter (hazard rate) $a > 0$. The survival function of X is $\overline{F}(x) = P(X > x) = e^{-ax}$, for all $x \geq 0$. Assume that (X_1, X_2) and (Y_1, Y_2) are two pairs of heterogeneous independent exponential random variables. Our purpose is to characterize the inequality

$$X_1 + X_2 \geq_{cx} Y_1 + Y_2 .$$

In addition, we include the obtained characterization in a nice unitary review of the topic.

2. MAIN RESULTS

We shall formulate a necessary and sufficient condition for the comparison in convex order of the exponential convolutions having two terms. In fact, we point out a Schur convexity property of these convolutions. To express this, we need a technical lemma.

Lemma 2.1. *The symmetrical continuous differentiable function* $\Psi_x : (0,\infty)^2 \to$ R, *defined by*

$$\Psi_x(s,t) = \begin{cases} \dfrac{s^2 e^{-\frac{x}{s}} - t^2 e^{-\frac{x}{t}}}{s-t}, & s \neq t \\ (2t+x)e^{-\frac{x}{t}}, & s = t \end{cases},$$

is Schur convex, for all $x > 0$.

Proof. Given an arbitrary (fixed) $x > 0,$ it suffices to show (see [3]) that

$$\text{(1)} \qquad \Delta_x(s,t) := (s-t)\left(\frac{\partial \Psi_x}{\partial s}(s,t) - \frac{\partial \Psi_x}{\partial t}(s,t)\right) \geq 0, \text{ for all } s,t > 0.$$

Consider *s, t>0*. Without loss of generality, we can assume that *s>t*. We have

$$(s-t)\Delta_x(s,t) = (g_x^{'}(s) + g_x^{'}(t))(s-t) - 2(g_x(s) - g_x(t)),$$

where $g_x : (0,\infty) \to$ R, $g_x(u) = u^2 e^{\frac{x}{u}}$, for $u > 0.$ Denote $h_{x,t} : [t,\infty) \to$ R the function defined by $h_{x,t}(u) = (g_x^{'}(u) + g_x^{'}(t))(u-t) - 2(g_x(u) - g_x(t)).$ We have $h_{x,t}(t) = 0$ and $h_{x,t}^{'}(u) = g_x^{''}(u)(u-t) + g_x^{'}(t) - g_x^{'}(u),$ for $u \geq t.$

Since $g_x^{(3)}(u) = \dfrac{x^3}{u^4} e^{-\frac{x}{u}} > 0,$ $\forall u > 0$, the first derivative of g_x is convex. Hence $h_{x,t}^{'}$ is positive on $(t,\infty).$ As follows $\Delta_x(s,t) = \dfrac{h_{x,t}(s)}{s-t} > 0.$

□

Thus, the relation (1) is proved.

Next, we formulate the main result of the paper.

Theorem 2.1. *Let* (X_1, X_2) *and* (Y_1, Y_2) *be two pairs of independent exponential random variables with corresponding parameters (a,b) and* $(a^{'}, b^{'})$, *respectively. The following equivalence holds:*

$$X_1 + X_2 \geq_{cx} Y_1 + Y_2 \iff \min\{a,b\} \leq \min\{a^{'}, b^{'}\} \text{ and } \frac{1}{a} + \frac{1}{b} \geq \frac{1}{a^{'}} + \frac{1}{b^{'}}$$

Proof. Without loss of generality, we can assume that $a \leq b$ and $a^{'} \leq b^{'}$. Denote $\overline{F}$ and $\overline{G}$ the survival functions of $X_1 + X_2$ and $Y_1 + Y_2$, respectively. We easily obtain

$$(2) \qquad \int_x^\infty \overline{F}(u)du = \Psi_x\left(\tfrac{1}{a}, \tfrac{1}{b}\right) \text{ and } \int_x^\infty \overline{G}(u)du = \Psi_x\left(\tfrac{1}{a^{'}}, \tfrac{1}{b^{'}}\right), \text{ where } x>0.$$

Assume that $X_1 + X_2 \geq_{cx} Y_1 + Y_2$, i.e. $\int_x^\infty \overline{F}(t)dt \geq \int_x^\infty \overline{G}(t)dt$, for $x \geq 0$.

Hence $\frac{1}{a} + \frac{1}{b} = E\left(X_1 + X_2\right) = \int_0^\infty \overline{F}(t)dt \geq \int_0^\infty \overline{G}(t)dt = E\left(Y_1 + Y_2\right) = \frac{1}{a^{'}} + \frac{1}{b^{'}}$.

From our assumption, $a \leq b$ and $a^{'} \leq b^{'}$. If $a > a^{'}$, then

$$\lim_{x\to\infty} \frac{\int_x^\infty \overline{F}(t)dt}{\int_x^\infty \overline{G}(t)dt} = \lim_{x\to\infty} \frac{\Psi_x\left(\frac{1}{a}, \frac{1}{b}\right)}{\Psi_x\left(\frac{1}{a^{'}}, \frac{1}{b^{'}}\right)} = 0. \text{ But we have } \frac{\int_x^\infty \overline{F}(t)dt}{\int_x^\infty \overline{G}(t)dt} \geq 1, \text{ for all } x>0.$$

Contradiction. Therefore, $a \leq a^{'}$.

Conversely, let us assume that $a \leq a^{'}$ and $\frac{1}{a} + \frac{1}{b} \geq \frac{1}{a^{'}} + \frac{1}{b^{'}}$.

Denote $c = \left(\frac{1}{a^{'}} + \frac{1}{b^{'}} - \frac{1}{a}\right)^{-1}$. We have $\frac{1}{a} + \frac{1}{c} = \frac{1}{a^{'}} + \frac{1}{b^{'}}$ and $0 < a \leq b \leq c$.

Let Z be an exponential random variable with parameter c, such that Z is independent on X_1. Denote $\overline{H}$ the survival function of $X_1 + Z$. Since the vector $\left(\frac{1}{a}, \frac{1}{c}\right)$ majorizes the vector $\left(\frac{1}{a^{'}}, \frac{1}{b^{'}}\right)$, denoted by $\left(\frac{1}{a}, \frac{1}{c}\right) \succ \left(\frac{1}{a^{'}}, \frac{1}{b^{'}}\right)$, and Ψ_x is Schur convex, for all *x>0* (Lemma 2.1),

$$\int_x^\infty \overline{H}(u)du = \Psi_x\left(\tfrac{1}{a}, \tfrac{1}{c}\right) \geq \Psi_x\left(\tfrac{1}{a^{'}}, \tfrac{1}{b^{'}}\right) = \int_x^\infty \overline{G}(u)du, \quad \forall x > 0.$$

Consequently

(3) $$X_1 + Z \geq_{cx} Y_1 + Y_2 .$$

On the other hand, $b \leq c$ ensures $X_2 \geq_{cx} Z$. Then, using a closure property (see [4], Theorem 3.A.12), we obtain

(4) $$X_1 + X_2 \geq_{cx} X_1 + Z$$

From (3) and (4) it follows that $X_1 + X_2 \geq_{cx} Y_1 + Y_2$. □

Theorem 2.1 supplements some recent advances in the topic. We propose now an unitary consideration of these outcomes. For *x,y>0*, let us consider the elementary means: *A(x, y)* - the arithmetic mean, G(x, y) - the geometric mean, and *H(x, y)* - the harmonic mean.

Theorem 2.2. *Let* (X_1, X_2) *and* (Y_1, Y_2) *be two pairs of independent exponential random variables with corresponding parameters (a,b) and* (a', b'), *respectively.*

(a) *The following statements are equivalent:*

(a_1) $\min\{a,b\} \leq \min\{a',b'\}$ *and* $A(a,b) \leq A(a',b')$.

(a_2) $X_1 + X_2$ *is greater that* $Y_1 + Y_2$ *in the likelihood ratio order (denoted by* $X_1 + X_2 \geq_{lr} Y_1 + Y_2$*).*

(b) *The following statements are equivalent:*

(b_1) $\min\{a,b\} \leq \min\{a',b'\}$ *and* $G(a,b) \leq G(a',b')$.

(b_2) $X_1 + X_2$ *is greater that* $Y_1 + Y_2$ *in the hazard rate order (denoted by* $X_1 + X_2 \geq_{hr} Y_1 + Y_2$*).*

(b_3) $X_1 + X_2$ *is greater that* $Y_1 + Y_2$ *in the usual stochastic order (denoted by* $X_1 + X_2 \geq_{st} Y_1 + Y_2$*).*

(b_4) $X_1 + X_2$ *is greater that* $Y_1 + Y_2$ *in the dispersive order (denoted by* $X_1 + X_2 \geq_{disp} Y_1 + Y_2$*).*

(c) *The following statements are equivalent:*

(c_1) $\min\{a,b\} \leq \min\{a',b'\}$ *and* $H(a,b) \leq H(a',b')$.

(c_2) $X_1 + X_2$ *is greater that* $Y_1 + Y_2$ *in the mean residual life order (denoted by* $X_1 + X_2 \geq_{mrl} Y_1 + Y_2$*).*

(c_3) $X_1 + X_2$ *is greater that* $Y_1 + Y_2$ *in the convex order (denoted by* $X_1 + X_2 \geq_{cx} Y_1 + Y_2$*).*

The equivalence $(a_1) \Leftrightarrow (a_2)$ can be found in [6]. The relations $(b_1) \Leftrightarrow (b_2) \Leftrightarrow (b_3)$ have been obtained in [1]. $(b_1) \Leftrightarrow (b_4)$ is recently proved in [7] (in the proof, some results of [2] are used). Then, $(c_1) \Leftrightarrow (c_2)$ is taken from [5]. Finally, $(c_1) \Leftrightarrow (c_3)$ is given by Theorem 2.1.

There are many important extensions of these basic equivalences. To express these generalizations, some specific weak majorization type orders are used: *increasing weak majorization* (for ***(a)***), *p larger order* (for ***(b)***), and *reciprocal majorization* (for ***(c)***).

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