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FIXED POINTS OF EXPANSION MAPS IN INTUITIONISTIC FUZZY METRIC SPACES

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Abstract. The purpose of this paper is to prove some fixed point and common fixed point theorems for expansion maps in intuitionistic fuzzy metric spaces. The main result has been proved for two pairs of non-surjective expansion maps on noncomplete intuitionistic fuzzy metric space through weak compatibility. Our results extend, generalize and intuitionistic fuzzify several fixed point theorems on metric spaces, Menger PM-spaces and fuzzy metric spaces.

1. Introduction

The concept of fuzzy sets was initially investigated by Zadeh [28] as a new way to represent vagueness in everyday life. As a generalization of fuzzy sets, Atanassov [3] introduced and studied the concept of intuitionistic fuzzy sets. In 2004, Park [16] introduced and discussed a notion of intuitionistic fuzzy metric spaces (briefly, IFM-spaces), which is based both on the idea of intuitionistic fuzzy sets and the concept of a fuzzy metric space given by George and Veeramani [7].

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Using the idea of intuitionistic fuzzy sets, Alaca, Turkoglu and Yildiz [2] defined the notion of IFM-space as Park [16] with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [12]. Further, they [2] proved Intuitionistic fuzzy Banach and Intuitionistic fuzzy Edelstein contraction theorems, with the different definition of Cauchy sequences and completeness than the ones given in [16]. Turkoglu, Alaca and Yildiz [25] extended the concept of compatible maps (introduced by Jungck [9] in metric spaces) to IFM-spaces. Jungck and Rhoades [10] initiated the study of weakly compatible maps in metric space and showed that every pair of compatible maps is weakly compatible but reverse is not true.

In fixed point theory, contraction map theorems have been always an active area of research since 1922 with the celebrated Banach contraction fixed point theorem [4]. Banach contraction principle also yields a fixed point theorem for a diametrically opposite class of maps, viz. expansion maps. The study of fixed point of single expansion map in a metric space is initiated by Wang, Li, Gao and Iseki [27]. Later, using expansion type conditions, several results have been proved for a pair of maps (see [17], [18], [24]) and two pairs of maps (see [11], [8], [23]) in metric spaces. The results in ([18], [24], [11], [8]) have been proved for either onto maps or surjective maps, while results in ([17], [23]) have been proved for non-surjective maps. Subsequently, there are a number of generalization of these results in different settings such as D-metric spaces [1]; probabilistic metric spaces ([6], [14], [15], [26]); 2-probabilistic metric spaces [5]; fuzzy metric spaces [20].

The main result of this paper has been proved for two pairs of non-surjective expansion maps on noncomplete IFM-space through weak compatibility. Our results extend, generalize and intuitionistic fuzzify the results of Jachymski [8, Theorem 5.2], Kang and Rhoades [11, Theorem 2.6], Kumar and Pant [14], Rhoades ([18, Theorem 1], [19]), Kumar [13, Theorem 4.1], Dimri, Pant and Kumar [6, Theorem 3.2] and Vasuki [26, Theorem 2.3].

2. Preliminaries

Definition 2.1 [21] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions:
(a) $*$ is commutative and associative,

(b) $*$ is continuous,
(c) $a * 1 = a$ for all $a \in [0, 1]$,
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.
Examples of t-norm are $a * b = min(a, b)$ and $a * b = ab$.

Definition 2.2 [21] A binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is continuous t-conorm if $\diamond$ is satisfying the following conditions:
(a) $\diamond$ is commutative and associative,
(b) $\diamond$ is continuous,
(c) $a \diamond 0 = a$ for all $a \in [0, 1]$,
(d) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.
Examples of t-conorm are $a \diamond b = max(a, b)$ and $a \diamond b = min(1, a + b)$.

Following Atanassov [3] and Kramosil and Michalek [12]; Alaca, Turkoglu and Yildiz [2] have given the next definition of IFM-spaces:

Definition 2.3 [2] A 5-tuple $(X, M, N, *, \diamond)$ is said to be an IFM-space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,
(a) $M(x, y, t) + N(x, y, t) \leq 1$ for all $t > 0$;
(b) $M(x, y, 0) = 0$;
(c) $M(x, y, t) = 1$ if and only if $x = y$;
(d) $M(x, y, t) = M(y, x, t)$;
(e) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(f) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous;
(g) $\lim_{t\to\infty} M(x, y, t) = 1$;
(h) $N(x, y, 0) = 1$;
(i) $N(x, y, t) = 0$ if and only if $x = y$;
(j) $N(x, y, t) = N(y, x, t)$;
(k) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$;
(l) $N(x, y, \cdot) : [0, \infty) \to [0, 1]$ is right continuous;
(m) $\lim_{t\to\infty} N(x, y, t) = 0$.
Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 2.1 Every fuzzy metric space $(X, M, *)$ is an IFM-space of the form $(X, M, 1-M, *, \diamond)$ such that t-norm $*$ and t-conorm $\diamond$ are associated, i.e. $x \diamond y = 1 - ((1-x) * (1-y))$ for any $x, y \in X$.

Example 2.1 [16] Let (X, d) be a metric space. Define t-norm $a * b = min(a, b)$ and t-conorm $a \diamond b = max(a, b)$ and for all $x, y \in X$ and $t > 0$,
$M_d(x, y, t) = \frac{t}{t+d(x,y)}$, $N_d(x, y, t) = \frac{d(x,y)}{t+d(x,y)}$.

Then $(X, M, N, *, \diamond)$ is an IFM-space and the intuitionistic fuzzy metric (M, N) induced by the metric d is often referred to as the standard intuitionistic fuzzy metric.

Remark 2.2 In IFM-space $(X, M, N, *, \diamond)$, $M(x, y, \cdot)$ is non-decreasing and $N(x, y, \cdot)$ is non-increasing for all $x, y \in X$.

Definition 2.4 [2]. Let $(X, M, N, *, \diamond)$ be an IFM-space. Then
(i) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if for all $t > 0$ and $p > 0$,
$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$, $\lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$.
(ii) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if for all $t > 0$,
$\lim_{n\to\infty} M(x_n, x, t) = 1$, $\lim_{n\to\infty} N(x_n, x, t) = 0$.
Since $*$ and $\diamond$ are continuous, the limit is uniquely determined from (e) and (k) respectively.

Definition 2.5 [2] An IFM-space $(X, M, N, *, \diamond)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Definition 2.6 [25] Let A and S be maps from an IFM-space $(X, M, N, *, \diamond)$ into itself. The maps A and S are said to be compatible if for all $t > 0$, $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ and $\lim_{n\to\infty} N(ASx_n,\ SAx_n, t) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 2.7 [10] Two self maps A and S on a set X are said to be weakly compatible if they commute at their coincidence point.

For the proof of our results, the following lemmas are needed:

Lemma 2.1 [22] Let $(X, M, N, *, \diamond)$ be an IFM-space and $\{y_n\}$ be a sequence in X. If there exists a number $\lambda \in (0, 1)$ such that
(I) $M(y_{n+2}, y_{n+1}, \lambda t) \geq M(y_{n+1}, y_n, t)$,
(II) $N(y_{n+2}, y_{n+1}, \lambda t) \leq N(y_{n+1}, y_n, t)$
for all $t > 0$ and $n = 1, 2, 3...$ Then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 2.2 [22] Let $(X, M, N, *, \diamond)$ be an IFM-space and for all $x, y \in X, t > 0$ and if for a number $\lambda \in (0, 1)$,
$M(x, y, \lambda t) \geq M(x, y, t)$ and $N(x, y, \lambda t) \leq N(x, y, t)$,
then $x = y$.

In our results, $(X, M, N, *, \diamond)$ will denote an IFM-space with continuous t-norm $*$ and continuous t-conorm $\diamond$ defined by $a * b = min(a, b)$ and $a \diamond b = max(a, b)$ respectively for all $a, b \in [0, 1]$.

3. Results

In 2006, Alaca, Turkoglu and Yildiz [2] presented the following

Theorem A (Intuitionistic fuzzy Banach contraction theorem) Let $(X, M, N, *, \diamond)$ be a complete IFM-space. Let $A : X \to X$ be a map satisfying
$M(Ax, Ay, \lambda t) \geq M(x, y, t)$ and $N(Ax, Ay, \lambda t) \leq N(x, y, t)$,
for all $x, y \in X, 0 < \lambda < 1$. Then A has a unique fixed point.

In 1984, Wang, Li, Gao and Iseki [27] initiated the study of fixed point of single expansion map in a metric space with the following:

Theorem B Let A be a map of a complete metric space (X, d) onto itself and if there exists a constant $k > 1$ such that
$d(Ax, Ay) \geq kd(x, y)$
for all $x, y \in X$. Then A has a unique fixed point.

First, we present the main result:

Theorem 3.1 Let $(X, M, N, *, \diamond)$ be an IFM-space. Further, let A, B, S and T be self-maps of X satisfying the following conditions:
(3.1) $T(X) \subseteq A(X)$ and $S(X) \subseteq B(X)$;
(3.2) (A, S) and (B, T) are weakly compatible pairs;
(3.3) there exists a constant $k > 1$ such that
$M(Ax, By, kt) \leq M(Sx, Ty, t)$ and $N(Ax, By, kt) \geq N(Sx, Ty, t)$
for all $x, y \in X$ and $t > 0$. If one of the subspaces $A(X), B(X), S(X)$ or $T(X)$ is complete, then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. By (3.1), we define the sequences $\{x_n\}$ and $\{y_n\}$ in X such that, for all $n = 0, 1, 2...$

(3.4) $\qquad Ax_{2n+1} = Tx_{2n} = y_{2n}$ and $Bx_{2n+2} = Sx_{2n+1} = y_{2n+1}$.

By (3.3), (3.4) and Definition 2.3 [(d), (j)], we have

$$\begin{aligned} M(y_{2n}, y_{2n+1}, kt) &= M(Ax_{2n+1}, Bx_{2n+2}, kt) \\ &\leq M(Sx_{2n+1}, Tx_{2n+2}, t) = M(y_{2n+1}, y_{2n+2}, t) \end{aligned}$$

and

$$\begin{aligned} N(y_{2n}, y_{2n+1}, kt) &= N(Ax_{2n+1}, Bx_{2n+2}, kt) \\ &\geq N(Sx_{2n+1}, Tx_{2n+2}, t) = N(y_{2n+1}, y_{2n+2}, t). \end{aligned}$$

This give,

$M(y_{2n}, y_{2n+1}, kt) \leq M(y_{2n+1}, y_{2n+2}, t)$

and

$N(y_{2n}, y_{2n+1}, kt) \geq N(y_{2n+1}, y_{2n+2}, t)$.

Similarly, again by (3.3) and (3.4), we have

$M(y_{2n+1}, y_{2n+2}, kt) \leq M(y_{2n+2}, y_{2n+3}, t)$

and

$N(y_{2n+1}, y_{2n+2}, kt) \geq N(y_{2n+2}, y_{2n+3}, t)$.

Thus, for any n and $k > 1$, $t > 0$,, we have

$M(y_n, y_{n+1}, kt) \leq M(y_{n+1}, y_{n+2}, t)$

and

$N(y_n, y_{n+1}, kt) \geq N(y_{n+1}, y_{n+2}, t)$.

By Lemma 2.1, $\{y_n\}$ is a Cauchy sequence. Suppose $A(X)$ is complete, $\{y_n\}$ has a limit in $A(X)$. Call it, z. Hence, there exists a point p in X such that $Ap = z$. Consequently, the subsequences $\{Ax_{2n+1}\}$, $\{Bx_{2n}\}$, $\{Sx_{2n+1}\}$, $\{Tx_{2n}\}$ also converge to z.

By (3.3),

$M(Ap, Bx_{2n}, kt) \leq M(Sp, Tx_{2n}, t)$

and $N(Ap, Bx_{2n}, kt) \geq N(Sp, Tx_{2n}, t)$.

Letting $n \to \infty$; we get
$M(z, z, kt) \le M(Sp, z, t)$ and $N(z, z, kt) \ge N(Sp, z, t)$,
that is, $1 \le M(Sp, z, t)$ and $0 \ge N(Sp, z, t)$ for all $t > 0$, which give $M(Sp, z, t) = 1$ and $N(Sp, z, t) = 0$. Thus, it follows that $Sp = z$. Therefore, $Ap = Sp = z$.
The weak compatibility of (A, S) implies $ASp = SAp$, that is, $Az = Sz$. But $S(X) \subseteq B(X)$, so there exists $q \in X$ such that $Bq = Sp = Ap$. Again by (3.3), we have
$M(Ap, Bq, kt) \le M(Sp, Tq, t)$ and $N(Ap, Bq, kt) \ge N(Sp, Tq, t)$,
that is, $1 \le M(Sp, Tq, t)$ and $0 \ge N(Sp, Tq, t)$ for all $t > 0$, which give $M(Sp, Tq, t) = 1$ and $N(Sp, Tq, t) = 0$. It follows that $Sp = Tq$. Hence, we have $z = Ap = Sp = Bq = Tq$. The weak compatibility of (B, T) implies $BTq = TBq$, that is, $Bz = Tz$.
Now we claim that $Az = z$. By (3.3), we get
$M(Az, z, kt) = M(Az, Bx_{2n}, kt) \le M(Sz, Tx_{2n}, t) = M(Az, z, t)$
and
$N(Az, z, kt) = N(Az, Bx_{2n}, kt) \ge N(Sz, Tx_{2n}, t) = N(Az, z, t)$.
By Lemma 2.2, it follows that $Az = z$. Thus, $Az = Sz = z$. Similarly, it can be proved that $Bz = Tz = z$. And hence, $Az = Sz = Bz = Tz = z$.
A similar argument holds by taking one of subspaces $B(X), S(X)$ or $T(X)$ complete. Finally, the uniqueness of z as a common fixed point of A, B, S and T is obvious from (3.3) and Lemma 2.2.

Remark 3.1 Theorem 3.1 is intuitionistic fuzzy version of the results of Jachymski [8, Theorem 5.2], Kang and Rhoades [11, Theorem 2.6] and Kumar and Pant [14]. Theorem 3.1 also extend the results in [8], [11] and [14] to non-surjective maps as well as the results in [11] and [14] to weakly compatible maps.

By setting $B = A$ and $T = S$ in Theorem 3.1, the following result is obtained:

Corollary 3.1 Let $(X, M, N, *, \diamond)$ be an IFM-space. Further, let A and S be self-maps of X satisfying the following conditions:
(3.5) $S(X) \subseteq A(X)$;
(3.6) (A, S) is a weakly compatible pair;
(3.7) there exists a constant $k > 1$ such that

$M(Ax, Ay, kt) \leq M(Sx, Sy, t)$ and $N(Ax, Ay, kt) \geq N(Sx, Sy, t)$ for all $x, y \in X$ and $t > 0$. If one of the subspaces $A(X)$ or $S(X)$ is complete, then A and S have a unique common fixed point in X.

Proof. Let $x_0 \in X$. By (3.5), we define the sequence $\{y_n\}$ in X such that, for all $n = 0, 1, 2...$

(3.8) $\quad Ax_{n+1} = Sx_n = y_n.$

By (3.7) and (3.8), we have

$$M(y_n, y_{n+1}, kt) = M(Ax_{n+1}, Ax_{n+2}, kt) \\ \leq M(Sx_{n+1}, Tx_{n+2}, t) = M(y_{n+1}, y_{n+2}, t)$$

and

$$N(y_n, y_{n+1}, kt) = N(Ax_{n+1}, Ax_{n+2}, kt) \\ \geq N(Sx_{n+1}, Tx_{n+2}, t) = N(y_{n+1}, y_{n+2}, t).$$

By Lemma 2.1, $\{y_n\}$ is a Cauchy sequence. Suppose $A(X)$ is complete, $\{y_n\}$ has a limit in $A(X)$. Call it, z. Hence, there exists a point p in X such that $Ap = z$. Consequently, the subsequences $\{Ax_{n+1}\}$, and $\{Sx_n\}$ also converge to z. Now applying the same technique as in Theorem 3.1, proof is obvious.

Remark 3.2 Corollary 3.1 is intuitionistic fuzzy version of the results of Rhoades [19] and Kumar [13, Theorem 4.1], Vasuki [26, Theorem 2.3]. Also, it extend the results in [19], [26, Theorem 2.3] to weakly compatible maps.

By setting $S = T = I_X$ (the identity) in Theorem 3.1 and assuming that X is complete, the maps A and B are surjective maps; the following result is obtained:

Corollary 3.2 Let $(X, M, N, *, \diamond)$ be a complete IFM-space. Further, let A and B be surjective self-map of X. There exists a constant $k > 1$ such that

(3.9) $\quad M(Ax, By, kt) \leq M(x, y, t)$ and $N(Ax, By, kt) \geq N(x, y, t)$

for all $x, y \in X$ and $t > 0$. Then A and B have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Since A is surjective, there exists a point $x_1 \in A^{-1}x_0$. Since B is surjective, there exists a point

$x_2 \in B^{-1}x_1$. Continuing in this manner, we have a sequence $\{x_n\}$ with $x_{2n+1} \in A^{-1}x_{2n}$, $x_{2n+2} \in B^{-1}x_{2n+1}$ for all $n = 0, 1, 2...$ Now we have two cases as follows:

Case I: When $x_{2n+1} = x_{2n}$ for some n. By (3.9), we have
$M(x_{2n+1}, x_{2n}, kt) = M(Bx_{2n+2}, Ax_{2n+1}, kt) \leq M(x_{2n+2}, x_{2n+1}, t)$
and
$N(x_{2n+1}, x_{2n}, kt) = N(Bx_{2n+2}, Ax_{2n+1}, kt) \geq N(x_{2n+2}, x_{2n+1}, t)$,
that is, $1 \leq M(x_{2n+2}, x_{2n+1}, t)$ and $0 \geq N(x_{2n+2}, x_{2n+1}, t)$ for all $t > 0$, which give $M(x_{2n+2}, x_{2n+1}, t) = 1$ and $N(x_{2n+2}, x_{2n+1}, t) = 0$. The condition $x_{2n+1} = x_{2n}$ implies that x_{2n} is a fixed point of A. Since, also $x_{2n+2} = x_{2n+1}, x_{2n}$ is a fixed point of B. Similarly, $x_{2n+2} = x_{2n+1}$ leads to x_{2n+1} being a common fixed point of A and B.

Case II: When $x_{2n+1} \neq x_{2n}$ for some n. By (3.9),
$M(x_{2n}, x_{2n+1}, kt) = M(Ax_{2n+1}, Bx_{2n+2}, kt) \leq M(x_{2n+1}, x_{2n+2}, t)$
and
$N(x_{2n}, x_{2n+1}, kt) = N(Ax_{2n+1}, Bx_{2n+2}, kt) \geq N(x_{2n+1}, x_{2n+2}, t)$.
Similarly, when $x_{2n+1} \neq x_{2n+2}$ for some n, we have
$M(x_{2n+1}, x_{2n+2}, kt) \leq M(x_{2n+2}, x_{2n+3}, t)$
and $N(x_{2n+1}, x_{2n+2}, kt) \geq N(x_{2n+2}, x_{2n+3}, t)$.
Thus, for any n and $k > 1$, $t > 0$, we have
$M(x_n, x_{n+1}, kt) \leq M(x_{n+1}, x_{n+2}, t)$
and $N(x_n, x_{n+1}, kt) \geq N(x_{n+1}, x_{n+2}, t)$.
Therefore, in view of Lemma 2.1, $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ has a limit $z \in X$. As A and B are surjective, there exist $u, v \in X$ satisfying $u \in A^{-1}z$ and $v \in B^{-1}z$. Now, by (3.9), we get
$M(x_{2n}, z, kt) = M(Ax_{2n+1}, Bv, kt) \leq M(x_{2n+1}, v, t)$
and $N(x_{2n}, z, kt) = N(Ax_{2n+1}, Bv, kt) \geq N(x_{2n+1}, v, t)$.
Letting $n \to \infty$; we get
$1 \leq M(z, v, t)$ and $0 \geq N(z, v, t)$ for all $t > 0$, which give $M(z, v, t) = 1$ and $N(z, v, t) = 0$. Therefore, $z = v$.
In the similar pattern, taking $x = u$ and $y = x_{2n+2}$ in (3.9), and therefore proceeding as above, we obtain $z = u$. Therefore, $z = u = v$ which immediately implies $Az = Bz = z$ and so z is a common fixed point of A and B. Also, the uniqueness of z as a common fixed point

of A and B is obvious from (3.9).

Remark 3.3 Corollary 3.2 is intuitionistic fuzzy version of the result of Rhoades [18, Theorem 1].

By setting $B = A$ in Corollary 3.2, we have intuitionistic fuzzy version of the result of Wang, Li, Gao and Iseki [27] as follows:

Corollary 3.3 Let $(X, M, N, *, \diamond)$ be a complete IFM-space. Further, let A be a map of X onto itself. There exists a constant $k > 1$ such that
(3.10) $M(Ax, Ay, kt) \leq M(x, y, t)$ and $N(Ax, Ay, kt) \geq N(x, y, t)$
for all $x, y \in X$ and $t > 0$. Then A has a unique fixed point in X.

Proof. Let $x_0 \in X$. Since A is onto, there exists an element x_1 satisfying $x_1 \in A^{-1}x_0$. In the same way, we can take $x_n \in A^{-1}x_{n-1}$ for $n = 2, 3, 4...$ If $x_m = x_{m-1}$ for some m, then x_m is a fixed point of A. Without loss of generality, we can suppose $x_n \neq x_{n-1}$ for every n. Then by (3.10), for all $t > 0$ and $k > 1$, we have
$M(x_{n-1}, x_n, kt) = M(Ax_n, Ax_{n+1}, kt) \leq M(x_n, x_{n+1}, t)$
and $N(x_{n-1}, x_n, kt) = N(Ax_n, Ax_{n+1}, kt) \geq N(x_n, x_{n+1}, t)$.
Therefore, by Lemma 2.1, $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ has a limit $z \in X$. Now, the proof follows as in Corollary 3.2.

Thoerem 3.2 Let $(X, M, N, *, \diamond)$ be an IFM-space. Further, let A and S be self-maps of X satisfying (3.5), (3.6) and
(3.11) there exists a constant $k > 1$ such that
$(M(Ax, Ay, kt))^2 \leq M(Ax, Sx, t)M(Ay, Sy, t)$
and
$(N(Ax, Ay, kt))^2 \geq N(Ax, Sx, t)N(Ay, Sy, t)$
for all $x, y \in X$ and $t > 0$. If one of the subspaces $A(X)$ or $S(X)$ is complete, then A and S have a unique common fixed point in X.

Proof. Let $x_0 \in X$. By (3.5), we define the sequence $\{y_n\}$ in X as $Ax_{n+1} = Sx_n = y_n$ for all $n = 0, 1, 2...$
By (3.11), for all $t > 0$ and $k > 1$, we get
$(M(y_n, y_{n+1}, kt))^2 = (M(Ax_{n+1}, Ax_{n+2}, kt))^2$

$$\leq M(Ax_{n+1}, Sx_{n+1}, t)M(Ax_{n+2}, Sx_{n+2}, t)$$
$$\leq M(y_n, y_{n+1}, t)M(y_{n+1}, y_{n+2}, t)$$

and

$$(N(y_n, y_{n+1}, kt))^2 = (N(Ax_{n+1}, Ax_{n+2}, kt))^2$$
$$\geq N(Ax_{n+1}, Sx_{n+1}, t)N(Ax_{n+2}, Sx_{n+2}, t)$$
$$\geq N(y_n, y_{n+1}, t)N(y_{n+1}, y_{n+2}, t).$$

This give

$(M(y_n, y_{n+1}, kt))^2 \leq M(y_n, y_{n+1}, t)M(y_{n+1}, y_{n+2}, t)$

and

$(N(y_n, y_{n+1}, kt))^2 \geq N(y_n, y_{n+1}, t)N(y_{n+1}, y_{n+2}, t).$

Thus it follows that

$M(y_n, y_{n+1}, kt) \leq M(y_{n+1}, y_{n+2}, t)$

and

$N(y_n, y_{n+1}, kt) \geq N(y_{n+1}, y_{n+2}, t).$

In view of Lemma 2.1, $\{y_n\}$ is a Cauchy sequence. Now, the proof follows as in Theorem 3.1.

Remark 3.4 Theorem 3.2 is intuitionistic fuzzy version of the result of Dimri, Pant and Kumar [6, Theorem 3.2].

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