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ON SOME WEIGHTED STATISTICAL APPROXIMATION PROPERTIES OF q-SCHURER BERNSTEIN OPERATORS

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Abstract. We investigate some weighted statistical approximation properties of Schurer-Bernstein operators in q-calculus and give an estimation of convergence in terms of Peetre's type K-functional.

1. Preliminaries

In the last decades the theory of q-calculus has developed into an interdisciplinary subject and was intensively used for the construction of various generalizations of many approximations of positive type.

A q-type of the Bernstein operators was introduced in 1987 by Stancu and later in 1997 another generalization of the classical Bernstein polynomials based on q- integer were introduced by Phillips [13]. After this, some authors studied new classes of q- generalized operators and gave approximations properties of them. In [3] O. Doğru and A. Aral constructed q- type generalization of Bleimann, Butzer and Hahn operators. T. Trif investigated Meyer-König and Zeller operators based on q integers ([14]). O.Doğru and O. Duman introduced also a new generalization of Meyer-König and Zeller operators and studied some statistical approximation properties in [5].

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A generalization of Balazs-Szabados operators based on q- integers was introduced and a Stancu type generalization of these operators is also constructed in a paper of O. Doğru. A new q-generalization of Meyer-König and Zeller type operators was constructed by Doğru and Muraru for improve the rate of convergence [7]. Recently were studied generalization of Durmeyer and Kantorovich operators based on q-integer by Gupta and Radu [11].

We remind that q- Bernstein polynomial has the following form (Philips 1997):.

$$B_n(f;q;x)=\sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right)\left[\begin{array}{c} n \\ k \end{array}\right] x^k \prod_{s=0}^{n-k-1}(1-q^s x) \tag{1}$$

where $x\in[0,1],\ \ f\in C([0,1]),\ \ 0<q<1$ and

$$[k]=\begin{cases}(1-q^k)/(1-q), & q\neq 1\\ k\hspace{6em}, & q=1\end{cases}$$

$$[k]!=\begin{cases}[k][k-1]....[1], & k=1,2,...\\ 1\hspace{6em}, & k=0\end{cases}$$

$$\left[\begin{array}{c} n \\ k \end{array}\right]=\frac{[n]!}{[k]![n-k]!}\qquad (n\geq k\geq 0)$$

Let $B_\rho(R)=\{f:R\to R\,||f(x)|\leq M_f\rho(x),\forall x\in R\}$

$$C_\rho(R)=\{f\in B_\rho(R):\text{f is continuous on R}\}$$

$$C^*_\rho(R)=\left\{f\in C_\rho(R):\exists\lim_{x\to\infty}\frac{|f(x)|}{\rho(x)}<\infty\right\}.$$

Endowed with the norm where $\|f\|_\rho:=\sup\frac{|f(x)|}{\rho(x)}$, $B_\rho(R)$ and $C_\rho(R)$ are Banach spaces.

A real function ρ is called a **weight function** if it is continuous on R and

$\lim\limits_{|x|\to\infty}\rho(x)=\infty$, $\rho(x)\geq 1$ for all $x\in R$.

2. Approximation properties of q-Schurer-Bernstein operators

Let $p \in N$ be fixed. In 1962 Schurer introduced and studied the Bernstein-Schurer operators $\tilde{B}_{m,p} : C([0,p+1]) \to C([0,1])$ defined for any $m \in N$ and any function $f \in C([0,p+1])$as follows

$$\tilde{B}_{m,p}(f;x) = \sum_{k=0}^{m+p} \left(\begin{array}{c} m+p \\ k \end{array} \right) x^k (1-x)^{m+p-k} f\left(\frac{k}{m}\right)$$

One observe that for $p = 0$, $B_{m,0}$ we obtain the operators of Bernstein B_m.

For any $m \in N$, $f \in C([0,p+1])$ and p be fixed, we construct the class of generalized q-Bernstein Schurer operators as follows

$$\tilde{B}_{m,p}(f;q;x) = \sum_{k=0}^{m+p} \left[\begin{array}{c} m+p \\ k \end{array} \right] x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) f\left(\frac{[k]}{[m]}\right) \tag{2}$$

Lemma 2.1 ([12]) For the polynomials defined above satisfy the following properties:

1. $\tilde{B}_{m,p}(e_0;q;x) = 1$
2. $\tilde{B}_{m,p}(e_1;q;x) = \frac{x[m+p]}{[m]}$
3. $\tilde{B}_{m,p}(e_2;q;x) = \frac{[m+p]}{[m]^2}\left([m+p]x^2 + x(1-x)\right)$

where we note by $e_j(x) = x^j$, $j = 0,1,2$, the test functions.

The next theorem contains the result regarding the convergence of the sequence of q-Schurer Bernstein operators, based on the well known Korovkin –Popoviciu theorem.

Theorem 2.2([12])

Let $q = q_m$satisfy $0 < q_m < 1$ and $\lim\limits_{m\to\infty} q_m^p = 1$and $\lim\limits_{m\to\infty} q_m^m = a$, $a \neq 1$ .Then for any $f \in C([0,p+1])$ the next result holds

$\lim\limits_{m\to\infty} \tilde{B}_{m,p}(f;q_m) = f$ uniformly on $[0,1]$

3. Rate of convergence

We will estimate the rate of convergence the Peetre's K-functional. In 1963 J. Peetre introduced the notion, which represents another important instrument to measure the smoothness of a function.

If we approximate f by a function g with g^r in L_p, then we are interested in how small the norm of $f-g$ can be made compared to the norm of g^r. One way of making such comparisons is through the Peetre's K-functional.

Let $C^2[0,a]=\{f\in C[0,a]: f' \text{ and } f'' \text{ in } C[0,a]\}$. Then $C^2[0,a]$ is a linear normed space with the following norm:

$$\|f\|_{C^2[0,a]}=\|f\|+\|f'\|+\|f''\|$$

We define a Peetre's type K-functional as follows: $K(f;\delta)=\inf\left\{\|f-g\|+\delta\|g\|_{C^2[0,a]}\right\}$.

Theorem 3.1 Let q_m be a sequence such $0<q_m<1$ for each $m\in N$, then for all $f\in C[0,a]$, 0<a<1 we have

$$\left\|\tilde{B}_{m,p}(f;q_m;x)-f(x)\right\|\leq 2K(f;\delta_m) \tag{3}$$

with $\delta_m=\frac{a}{2}\left(1-q_m^p-\frac{[p]}{[m]}\right)+\frac{a^2}{4}\left(\frac{[m+p]}{[m]}-1\right)^2+\frac{[m+p]}{4[m]^2}$

Proof.

$$g\in C^2[0,a]; g(t)-g(x)=g'(x)(t-x)+\int_x^t g''(s)(t-s)ds$$

We conclude for all $m\in N$ that

$$\left|\tilde{B}_{m,p}(g;q_m;x)-g(x)\right|\leq\|g'\|\left|\varphi_{m,1}(x)\right|+\frac{\|g''\|}{2}\varphi_{m,2}(x)$$

where $\varphi_{m,1}(x)$ and $\varphi_{m,2}(x)$ are first and second central moment of the operators Schurer-Bernstein as follows:

$$\varphi_{m,1}(x)=\widetilde{B_{m,p}}(t-x;q_m;x)$$

$$\varphi_{m,2}(x)=\tilde{B}_{m,p}((t-x)^2;q_m;x)$$

From the Lemma 2.2 we have

$\left|\tilde{B}_{m,p}(g;q_m;x)-g(x)\right| \leq x\left(1-q_m^p-\frac{[p]}{[m]}\right)\|g'\|+$
$+\frac{1}{2}\left(x^2\left(\frac{[m+p]}{[m]}-1\right)^2+x(1-x)\frac{[m+p]}{[m]}\right)\|g''\| \leq$
$\leq\left[x\left(1-q_m^p-\frac{[p]}{[m]}\right)+\frac{1}{2}x^2\left(\frac{[m+p]}{[m]}-1\right)^2+\frac{x(1-x)}{2}\frac{[m+p]}{[m]^2}\right]\|g\|_{C^2[0,a]}$

By the linearity of operator we have

$\left|\tilde{B}_{m,p}(f;q_m;x)-f(x)\right| \leq \left|\tilde{B}_{m,p}(f;q_m;x)-\tilde{B}_{m,p}(g;q_m;x)\right|+$
$+\left|\tilde{B}_{m,p}(g;q_m;x)-g(x)\right|+$
$+\ |g(x)-f(x)| \ \leq \ \|f-g\|_{C[0,a]}\tilde{B}_{m,p}(1;q_m;x)\ +\ \|f-g\|_{C[0,a]}\ +$
$\left|\tilde{B}_{m,p}(g;q_m;x)-g(x)\right|=$
$=2\|f-g\|_{C[0,a]}+\left|\tilde{B}_{m,p}(g;q_m;x)-g(x)\right| \leq$
$\leq 2\left\{\|f-g\|+\left[\frac{a}{2}\left(1-q_m^p-\frac{[p]}{[m]}\right)+\frac{a^2}{4}\left(\frac{[m+p]}{[m]}-1\right)^2+\frac{[m+p]}{4[m]^2}\right]\|g\|_{C^2[0,a]}\right\}$

We choose

$$(4)\qquad \delta_m=\frac{a}{2}\left(1-q_m^p-\frac{[p]}{[m]}\right)+\frac{a^2}{4}\left(\frac{[m+p]}{[m]}-1\right)^2+\frac{[m+p]}{4[m]^2}$$

By taking infimum over $g \in C^2[0,a]$ on both sides and letting δ_m as in (4) we get the result from the **Theorem 3.1.**

4. Weighted statistical approximation properties

The concept of statistical convergence was introduced by Fast in [9] and recently has became an important area in approximation theory . The Turkish school have many important result in this area an we remark here the contribution of Gadjev and Orhan which proved a Bohman-Korovkin type theorem for statistical approximation.

A sequence $x=(x_k)$is said to be statistically convergent to a number L if for every $\varepsilon>0$

$$\delta\{k \in N : |x_k-L| \geq \varepsilon\}=0,$$

where $\delta(K)$ is the natural density of the set $K \subseteq N$. The density of subset K is defined by

$\delta(K)=\lim_{n\to\infty}\frac{1}{n}\{the \text{ number } k \leq n, k \in K\}$ whenever the limit exist.

We denote this limit by $st-\lim_{n\to\infty} x_n = L$.

Clearly finite subsets have natural density 0.

In this section using a Korovkin type theorem proved in [8] we present the weighted statistical approximation of q-Schurer Bernstein operators.

We recall the concept of A-statistical convergence. Let $A = (a_{in})$ be a non negative regular summability matrix. A sequence $\{x_n\}$is said to be A-statistically convergent to a number L if for every $\varepsilon > 0$, $\lim\limits_{j} \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0$. We denote this limit by $st_A - \lim\limits_{n} x_n = L$.

For $A := C_1$, the Cesaro matrix of order one, A-statistical convergence reduces to statistical convergence. We will use the next result due to Duman and Orhan

Theorem 4.1([8]) Let $A = (a_{in})$ be a nonnegative regular summability matrix and let $\{L_n\}$be a sequence of positive operators from C_ρ into $B_\rho(R)$ where ρ_1 and ρ_2 satisfy

$$\lim_{|x| \to \infty} \frac{\rho_1(x)}{\rho_2(x)} = 0 \tag{5}$$

Then $st_A - \lim\limits_{n} \|L_n f - f\|_{\rho_2} = 0$ for all $f \in C_{\rho_1}(R)$ if only if $st_A - \lim\limits_{n} \|L_n F_\nu - F_\nu\|_{\rho_1} = 0$ for all $\nu = 0, 1, 2$.

Where $F_\nu = \frac{x^\nu \rho_1(x)}{1+x^2}$, $\nu = 0, 1, 2$.

We consider the weight functions $\rho_1(x) = 1 + x^2, \rho_2(x) = 1 + x^{2\alpha}$, $\alpha > 1$.

Further on, we consider a sequence $(q_m)_m, q_m \in (0, 1)$ such that

$$st - \lim_{m} q_m = 1 \tag{6}$$

From the (6) we obtain also that $st - \lim\limits_{m} q_m^p = 1$, for p a fixed natural number.

Theorem 4.2 Let $(q_n)_n$ be a sequence satisfying (6). Then for all non-decreasing $f \in C_{\rho_0}(R_+)$ we have

$$st - \lim \left\| \tilde{B}_{m,p}(f; q_n; \cdot) - f \right\|_{\rho_\alpha} = 0, \alpha > 0$$

Proof

It is clear that

$$st - \lim_{m} \left\| \tilde{B}_{m,p}(e_0; q_m; \cdot) - e_0 \right\|_{\rho_0} = 0. \tag{7}$$

Based on Lemma 2.1 we have

$$\left\|\tilde{B}_{m,p}(e_1;q_m;x)-e_1(x)\right\|_{\rho_0}=\sup_{x\in R_+}\frac{\left|\frac{x[m+p]}{[m]}-x\right|}{1+x^2}\le\|e_1\|_{\rho_0}\left|\frac{[m+p]}{[m]}-1\right|$$

Taking into account that $st-\lim_m q_m=1$ and $st-\lim_m\left|\frac{[m+p]}{[m]}-1\right|=0$ the following take place

$$st-\lim_m\left\|\tilde{B}_{m,p}(e_1;q_m;\cdot)-e_1\right\|_{\rho_0}=0 \tag{8}$$

Using the last relation from Lemma 2.1 we obtain

$$\frac{\left|\tilde{B}_{m,p}(e_2;q_m;x)-e_2(x)\right|}{1+x^2}\le\|e_2\|_{\rho_0}\left|\frac{[m+p]^2}{[m]^2}-\frac{[m+p]}{[m]^2}-1\right|+\|e_1\|_{\rho_0}\frac{[m+p]}{[m]^2}$$

From the next relation

$$st-\lim_m\left|\frac{[m+p]^2}{[m]^2}-\frac{[m+p]}{[m]^2}-1\right|=0$$

we have consequently

$$st-\lim_m\left\|\tilde{B}_{m,p}(e_2;q_m;\cdot)-e_2\right\|_{\rho_0}=0. \tag{9}$$

Finally, using (7), (8), (9) the proof follows from Theorem 3.1 by choosing $A=C_1$, the Cesaro matrix of order one and $\rho_0(x)=1+x^2$, $\rho_2(x)=1+x^{2+\alpha}$, $x\in R_+,\alpha>0$.

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