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THE GEOMETRIZATION OF LAGRANGE DYNAMICAL SYSTEMS

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Abstract. A mechanical system Q generated by a Lagrangian $L(t,\ x,\ \dot{x})$ is considered, whose the evolution equations is described by the Euler-Lagrange equations (2.1.). The geometry of the dynamical system determined by Q is the geometry of a semispray whose integral curves are the evolution equations of Q. The theory is extended to Lagrangians of higher order.

1. Introduction

General theory of mechanical Lagrangian systems was realized by R.Miron [1] .

We consider a Lagrange space $L^n = (M, L(x,y), F_i(x,y))$ where $F_i(x,y)$are the external forces.

Following the Miron's theory we take the evolution equations of $\sum$

(1.1) $\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = F_i(x,y), y^i = \frac{dx^i}{dt}$.

These equations are equivalent with the system of differential equations of second order:

(1.2.) $\frac{d^2x^i}{dt^2} + 2G^i(x,\dot{x}) = \frac{1}{2}F^i(x,\dot{x})$

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where

(1.3) $$F^i = g^{ij} F_j$$

and

(1.4) $$G^i = \frac{g^{is}}{2}\left(\frac{\partial^2 L}{\partial y^s \partial x^j} y^j - \frac{\partial L}{\partial x^s}\right)$$

The system of differential equations (1.2) defines a dynamical system of second order.

The solution curves of evolution equations (1.2) are integral curves of S on $\widetilde{TM} = TM\backslash\{0\}$.

(1.5) $$S = y^i \frac{\partial}{\partial x^i} - 2(G^i - \frac{1}{4}F^i)\frac{\partial}{\partial y^i}$$

S is a semispray on the phases space $\widetilde{TM}$.

The geometry of mechanical Lagrangian systems is determined by the geometry of the pair $(\widetilde{TM}, S)$.

2. Main result

Following the Miron's theory from mechanical Lagrangian systems, we obtain some results for Lagrange dynamical systems.

The dynamical systems determined by mechanical systems are given by Euler-Lagrange equations or by differential equations of the order 2 obtained in variational problems for the Lagrangians which depend on the time t, on the material points and on their velocity.

Let us assume that an mechanical system Q generated by a Lagrangian $L(t,\ x,\ \dot{x})$ is given, in which t is time, $x = (x^i)\,,\ i = 1, \ldots, n$ is a material point and $\dot{x}^i = \frac{dx^i}{dt}$ the velocity.

The evolution of the system E is described by the Euler – Lagrange equations

(2.1) $$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0$$

These equations actually give the optimally conditions of the considered system E.

We shall note

(2.2) $$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}$$

the metric tensor determined by the Lagrangian L.

The equations (2.1) describe the evolution of the dynamical system associated to the mechanical system Q.

Developed, they give the system

(2.3) $$2g_{ij}\frac{d^2 x^i}{dt^2} = \frac{\partial L}{\partial x^i} - \left[\dot{x}^j \frac{\partial}{\partial \dot{x}^j} - \frac{\partial}{\partial t}\right]\frac{\partial L}{\partial \dot{x}^i}$$

Two cases are obtained: the metric tensor g_{ij} is non singular or the metric tensor g_{ij} is singular.

In the first case, the $\det(g_{ij}) \neq 0$, in the second $\det(g_{ij}) = 0$.

If we note $(g^{ij}) = (g_{ij})^{-1}$ we obtain in the first case:

(2.4) $\frac{d^2x^i}{dt^2} + 2G^i(x, \dot{x}) = 0$

with

(2.4′) $2G^i = g^{ij}\left[\frac{\partial^2 L}{\partial \dot{x}^j \partial \dot{x}^k}\dot{x}^k - \frac{\partial^2 L}{\partial t \partial \dot{x}^j} - \frac{\partial L}{\partial x^j}\right]$

Thus, the evolution equations of the system Q are given by a system of second order equations (2.4), (2.4′).

It occurs:

Theorem 1.1 *The operator*

(2.5) $S = \dot{x}^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial \dot{x}^i}$

is a vectorial field whose integral curves are given by the equations (2.4).

The demonstration follows the common path expressed in [1].

In the second case, the Euler – Lagrange system of equations could be reduced to a 1st order system.

For example, let us assume that x has a single coordinate x. Then (2.3) is written in the form:

(2.6) $2g_{11}\frac{d^2x}{dt^2} = \frac{\partial L}{\partial x} - \left[\dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial x} - \frac{\partial^2 L}{\partial t \partial \dot{x}}\right]$

with $g_{11} = \frac{1}{L}\frac{\partial^2 L}{\partial \dot{x}^2}$. But $\det(g_{11}) = g_{11}$

For $g_{11} \neq 0$ the dynamical system is given by the 2^{nd} order equations.

For $g_{11} = 0$ the dynamical system is given by an equation of order one:

(2.6′) $\dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial x} - \frac{\partial^2 L}{\partial t \partial \dot{x}} - \frac{\partial L}{\partial x} = 0.$

to which the condition $\frac{\partial^2 L}{\partial \dot{x}^2} = 0$ is added.

This condition leads to $\frac{\partial L}{\partial x} = A(t, x)$.

Integrating again we get

$$L(t, x, \dot{x}) = A(t, x)\dot{x} + B(t, x).$$

Substituting in (2.6′) we get:

$$\dot{x}\frac{\partial A}{\partial x} - \frac{\partial A}{\partial t} = \left(\frac{\partial A}{\partial x}\dot{x} + \frac{\partial B}{\partial x}\right) = 0$$

or further

$$\frac{\partial A}{\partial t} + \frac{\partial B}{\partial x} = 0.$$

Thus, the Lagrangian $L(t,\ x,\ \dot{x})$ that satisfy the condition $g_{11} = 0$ are given by $L(t,\ x,\ \dot{x}) = A(t,\ x)\,\dot{x} + B(t,\ x)$ and the Euler – Lagrange equations is reduced to

$$\frac{\partial A}{\partial t} + \frac{\partial B}{\partial x} = 0.$$

The study of mechanical systems can be done by Lagrangians of a higher order which depend on $t,\ x,\ x^{(1)}, \ldots, x^{(k)}$:

(2.7) $L\left(t,\ x,\ x^{(1)}, \ldots, x^{(k)}\right), \quad x^{(1)i} = \frac{dx^i}{dt}, \ldots, x^{(k)i} = \frac{d^k x^i}{dt^k}$

Thus, L depends on time t, on dimension x and on the accelerations x of *1, 2, . . . , k* order.

In this case, the Euler- Lagrange equations are given by:

(2.8) $\frac{\partial L}{\partial x^i} - \frac{1}{1!}\frac{d}{dt}\frac{\partial L}{\partial x^{(1)i}} + \ldots + (-1)^k \frac{1}{k!}\frac{d^k}{dt^k}\frac{\partial L}{\partial x^{(k)i}} = 0$

The difficulty of the equations (2.8) in applications consists in the fact that the equations are of the *2k* order and, although they are self adjunct, it is extremely difficult to determine vectorial space whose integral curves are given by the equations (2.8).

Assuming $\frac{\partial L}{\partial t} = 0$and applying the semispray theory from [1], we shall demonstrate that some evolution equations of (k+1) order which have a geometrical character and which give the integral curves of a vector field S determined only by the Lagrangian L can be associated to the Lagrangians of k order given by (2.1).

Indeed, let us consider the following system of differential equations

(2.9) $\frac{\partial L}{\partial x^{(k-1)i}} - \frac{d}{dt}\frac{\partial L}{\partial x^{(k)i}} = 0, \quad x^{(1)i} = \frac{dx^i}{dt}, \ldots, x^{(k)i} = \frac{1}{k!}\frac{d^k x^i}{dt^k}$

to which the Lagrangian $L\left(x,\ x^{(1)}, \ldots, x^k\right)$ of k order satisfies.

In the case of the Lagrangians L which have the fundamental tensor non singular, that is $(g_{ij}) \neq 0$ with

(2.10) $g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial x^{(k)i}\partial x^{(k)j}}$

the equations (2.9) take the equivalent form:

(2.11) $\frac{d^{k+1}x^i}{dt^{k+1}} + (k+1)!G^i\left(t,\ x,\ x^{(1)},\ \ldots, x^{(k)}\right) = 0,$
$\frac{dx^i}{dt} = x^{(1)i}, \ldots, \frac{1}{k}\frac{d^k x^i}{dt^k} = x^{(k)i}$

in which the coefficients G^i are given by

(2.11′) $(k+1)\,G^i = \frac{1}{2}g^{ij}\left[\Gamma\left(\frac{\partial L}{\partial x^{(k)j}} - \frac{\partial L}{\partial x^{(k-1)j}}\right)\right]$

Γ being the nonlinear operator

(2.11″) $\Gamma = x^{(1)i}\frac{\partial}{\partial x^i} + \ldots + kx^{(k)i}\frac{\partial}{\partial x^{(k-1)}}$

Indeed, the equation (2.11) developed, is

$2g_{ij}\frac{dx^{(k)j}}{dt^k} = \frac{\partial L}{\partial x^{(k-1)i}} - \left[x^{(1)j}\frac{\partial}{\partial x^j} + 2x^{(2)j}\frac{\partial}{\partial x^{(1)j}} + \ldots + kx^{(k)j}\frac{\partial}{\partial x^{(k-1)j}}\right]\frac{\partial L}{\partial x^{(k)i}} =$
$= \frac{\partial L}{\partial x^{(k-1)i}} - \Gamma\frac{\partial L}{\partial x^{(k)i}}$

Contracting with g^{ij} we get (2.11).

Thus the evolution equations of the systems are given by the system of differential equations of (*k+1*) order (2.11).

The following theorem holds:

Theorem 2.2. *The operator*

(2.12) $S = x^{(1)i}\frac{\partial}{\partial x^i} + 2x^{(2)i}\frac{\partial}{\partial x^{(1)i}} + \ldots + kx^{(k)i}\frac{\partial}{\partial x^{(k-1)i}} - (k+1)\,G^i\frac{\partial}{\partial x^{(k)i}}$

has the following properties:

(1) *S is a vector field determined only by the Lagrangian* $L\left(x,\ x^{(1)},\ldots,x^{(k)}\right)$

(2) *The evolution curves of S are given by the integral curves of the evolution equations* (2.11).

Proof. 1) In (1) it is shown that when G^i are the coefficients from the equations (2.11), S is given by (2.12) is a vector field on the space of the accelerations of k order, $T^{(k)}M$

2) The integral curves of S are given by the system

$$\frac{dx}{dt} = x^{(1)i},\ \frac{dx^{(1)i}}{dt} = 2x^{(2)i},\ldots,\frac{dx^{(k-1)i}}{dt} = kx^{(k)i},\ \frac{dx^{(k)i}}{dt} = -(k+1)\,G^i$$

Thus, the dynamical system defined by the equations (2.11) is characterized by the vectorial field S which governs the fundamental properties of the mechanical system described by the Lagrangian L of k order.

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