"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 20 (2010), No. 2, 21 - 28

INDICATRIX OF A FINSLER VECTOR BUNDLE

MIHAI ANASTASIEI AND MANUELA GÎRŢU

Abstract. We consider a Finsler vector bundle i.e. a vector bundle $\xi : (E, p, M)$ endowed with a smooth function $F : E \to \mathbb{R}, (x, y) \mapsto F(x, y)$ that is positively homogeneous of degree 1 with respect to the variables y in fibres of ξ. Then $F(x, y) = 1$ with a fixed x defines the indicatrix of the given Finsler bundle in the fibre E_x and $F(x, y) = 1$ for every x and y is its indicatrix bundle. We show in Section 2 that the indicatrix is a totally umbilical submanifold in E_x of constant mean curvature -1. The indicatrix bundle is a submanifold of $E \setminus 0$. Assuming that ξ is endowed with a nonlinear connection compatible with F and the base M is a Riemannian manifold we define a Riemannian metric on $E \setminus 0$ and determine the normal to the indicatrix bundle.

Introduction

A nonlinear connection in a vector bundlle (v.b) $\xi = (E, p, M)$ is a distribution that is supplementary to the vertical distribution (vertical subbundle) defined by the kernel of the differential (tangent map) of p. From a nonlinear connection N a linear connection in the vertical bundle over E is easily derived. This is called the Berwald connection associated to N. The vector bundle ξ is called a Finsler vector bundle if it is endowed with a fundamental Finsler function. This determine a Riemannian metric in the vertical bundle but not a nonlinear connection.

Keywords and phrases: Finsler vector bundles, indicatrix, indicatrix bundle.
(2000)Mathematics Subject Classification:53C07,53C60.

We shall assume these two objects are compatible as it happens in Finsler geometry. This is the content of Section 1.For more details see [1]and [2]. In Section 2 we study the geometry of the indicatrix given by the equation $F(x,y)=1$ for a fixed $x\in M$ viewed as a submanifold of codimension 1 in the fibre E_x of the vector bundle ξ. We establish the Gauss and Weingarten formulae and we find that the indicatrix is totally umbilical and of mean curvature -1. A case when it is of constant curvature 1 is pointed out. In Section 3 we assume that the base manifold is a Riemannian manifold. We construct on $E\setminus 0$ a Riemannian metric of Sasaki type and determine a normal versor field to the indicatrix bundle as a submanifold of the Riemannian manifold $E\setminus 0$ .The notations and terminology are those from [3], [4] and [5]

1. Finsler vector bundles

Let $\xi=(E,p,M),\ p:E\to M$, be a vector bundle of rank m. Here M is a smooth i.e. C^∞ manifold of dimension n. The type fibre is $\mathbb{R}^m$ and E is a smooth manifold of dimension $n+m$. The projection p is a smooth submersion. Let $(U,(x^i))$ be a local chart on M and let $\varepsilon_a(x)$, $x\in U$, be a field of local sections of ξ over U. Then every section A of ξ over U takes the form $A=A^a(x)\varepsilon_a(x)$, $x\in U$, and an element $u\in p^{-1}(x):=E_x$ can be written as $u=y^a\varepsilon_a(x)$, $(y^a)\in\mathbb{R}^m$. The indices $i,j,k,...$ will range over $\{1,2,...,n\}$ and the indices $a,b,c,...$ will take their values in $\{1,2,...,m\}$. The convention on summation over repeated indices of the same kind will be used.

The local coordinates on $p^{-1}(U)$ will be (x^i,y^a) and a change of coordinates $(x^i,y^a)\to(\widetilde{x}^i,\widetilde{y}^a)$ on $U\cap\widetilde{U}\neq\emptyset$ has the form

$$\begin{aligned}&\widetilde{x}^i=\widetilde{x}^i(x^1,...,x^n),\ \operatorname{rank}\left(\frac{\partial\widetilde{x}^i}{\partial x^j}\right)=n,\\&\widetilde{y}^a=M^a_b(x)y^b,\ \operatorname{rank}(M^a_b(x))=m,\quad\forall x\in U\cap\widetilde{U}.\end{aligned}\tag{1.1}$$

On E we have the vertical distribution $u\to V_uE=\operatorname{Ker}p_{x,u}$, where p_* denotes the differential of p. This consists of vectors which are tangent to fibres and it is locally spanned by $\left(\dot\partial_a:=\dfrac{\partial}{\partial y^a}\right)$. We shall regard also the vertical distribution as a vector subbundle $VE:=\bigcup_{u\in E}V_uE\to E$ of $TE\to E$. Its sections will be called vertical vector fields of E. The tensorial algebra $\mathcal{T}(VE)=\oplus\mathcal{T}^p_q(VE),\ p,q\in\mathbb{N}$ of this subbundle will be used. Its elements will be indicated by the word "vertical".

Definition 1.1. A nonlinear connection N on E is a distribution $N : u \to N_u E$, $u \in E$, on E, which is supplementary to the vertical distribution on E.

We take the distribution N as being locally spanned by $\delta_k = \partial_k - N_k^a(x,y)\dot{\partial}_a$, for $\partial_k := \dfrac{\partial}{\partial x^k}$. By a change of coordinates (1.1), the condition $\delta_k = \dfrac{\partial \widetilde{x}^i}{\partial x^k}\,\widetilde{\delta}_i$ is equivalent with

$$\widetilde{N}_j^a \partial_k \widetilde{x}^j = M_b^a(x) N_k^b(x,y) - \partial_k(M_b^a(x)) y^b \tag{1.2}$$

It is important to notice that from (1.2) it follows that the set of functions $F_{bk}^a(x,y) = \dot{\partial}_b N_k^a(x,y)$ behaves under a change of coordinates (1.1) as the local coefficients of a linear connection in the vertical bundle over ξ, that is

(1.3)

$$\widetilde{F}_{bk}^a(\widetilde{x}(x), \widetilde{y}(x,y)) = M_c^a(x) \widetilde{M}_b^d(\widetilde{x}(x)) \frac{\partial x^i}{\partial \widetilde{x}^k}\, F_{di}^c(x,y) - \partial_i(M_c^a(x)) \frac{\partial x^i}{\partial \widetilde{x}^k}\, y^c,$$

where $\left(\dfrac{\partial x^i}{\partial \widetilde{x}^k}\right)$ is the inverse matrix of $\left(\dfrac{\partial \widetilde{x}^k}{\partial x^j}\right)$ and $(\widetilde{M}_b^d)$ denotes the inverse matrix of (M_c^b).

We should like to construct a linear connection D in the vertical bundle $VE \to E$. In order to do this it suffices to provide $D_{\delta_k}\dot{\partial}_a$ and $D_{\dot{\partial}_a}\dot{\partial}_b$. Using (1.3) we have the possibility

$$D_{\delta_k}\dot{\partial}_a = F_{ak}^b(x,y)\dot{\partial}_b,\ D_{\dot{\partial}_b}\dot{\partial}_c = V_{bc}^a(x,y)\dot{\partial}_a,^\circ \tag{1.4}$$

where necessarily $(V_{bc}^a(x,y))$ behave like the components of a vertical tensor field of type $(1,2)$.

In particular, we may take $V_{bc}^a = 0$ and introduce

Definition 1.2. The linear connection D in the vertical bundle $VE \to E$ given by

$$D_{\delta_k}\dot{\partial}_a = F_{ak}^b(x,y)\dot{\partial}_b, \quad D_{\dot{\partial}_a}\dot{\partial}_b = 0, \tag{1.4'}$$

is called the *Berwald connection* associated to N.

We notice that , if ξ is endowed with a linear connection of local coefficients $F_{bk}^a(x)$, then the functions

$$N_k^a(x,y) = F_{bk}^a(x,y) y^b, \tag{1.5}$$

define by setting $\delta_k = \partial_k - N_k^a(x,y)\dot{\partial}_a$ a nonlinear connection on E.

We remark that the nonlinear connection (1.5) is positively homogeneous of degree 1 in $y = (y^a)$.

Definition 1.3. A smooth function $F : E := E \setminus 0 \to \mathbb{R}$, $(x, y) \to F(x, y)$ is called a Finsler function if

(i) $F(x, y) \geq 0$,
(ii) $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$,
(iii) the matrix with the entries $g_{ab}(x, y) = \frac{1}{2}\, \dot{\partial}_a \dot{\partial}_b F^2$ is positive definite ($g_{ab}(x, y)\zeta^a\zeta^a > 0$ for $(\zeta^a) \neq 0$).

When ξ is endowed with a Finsler function F we call it a vector Finsler bundle.

The pairs (E_x, F_x) are called Minkowski spaces and F_x is called a Minkowski norm on E_x. The reason is that F_x, besides the conditions (i)–(iii) from Definition 1.3 satisfies also (see [3] p.6; (iv) $F_x(y) > 0$ whenever $y \neq 0$; (v) $F_x(y_1 + y_2) \leq F_x(y_1) + F_x(y_2)$.

Let $\xi = \tau_M = (TM, \tau, M)$ be the tangent bundle of M. If τ_M is endowed with a Finsler function F, the pair (M, F) is called a Finsler manifold. For the geometry of these manifolds we refer to [3], [5].

The Finsler function F induces the Cartan nonlinear connection $\overset{\circ}{N}{}^i_j(x, y) = \gamma^i_{j0} - C^i_{jk}\gamma^j_{00}$, where $2\gamma^i_{jk} = g^{ih}(\partial_j g_{kh} + \partial_k g_{jh} - \partial_h g_{jk})$, $2C^i_{jk} = g^{ih}\dot{\partial}_h g_{jk}$, $\gamma^i_{j0} = \gamma^i_{jk}y^k$ and $\gamma^i_{00} = \gamma^i_{jk}(x, y)y^j y^k$. Of course, $g_{jk} = \dfrac{1}{2}\, \dot{\partial}_j \dot{\partial}_k F^2$ denotes the Finsler metric. This nonlinear connection is p–homogeneous of degree 1 in y.

2. On the geometry of indicatrix of F

Let ξ be a Finsler vector bundle. This means that it is endowed with a Finsler function $F : E \to \mathbb{R}$ that is positively homogeneous of degree 1 in y.

The indicatrix $I_x = \{(x, y) \in E_x \mid F(x, y) = 1\}$ in a fixed x in M is a submanifold of codimension 1 in the Riemannian manifold (E_x, g_x), where g_x in the basis $\frac{\partial}{\partial y^a}$ has the components $g_{ab}(x, y)$.

As in what follows x is fixed we shall omit it. Let $\overline{\nabla}$ be the Levi-Civita connection of g. Its Christoffel symbols are $\frac{1}{2}g^{ad}(\frac{\partial g_{db}}{\partial y^c} + \frac{\partial g_{dc}}{\partial y^b} - \frac{\partial g_{bc}}{\partial y^d}) = C^a_{bc}$ and the Riemannian curvature can be put in the form

$$S_b{}^a{}_{cd} = C^a_{ec}C^e_{db} - C^a_{eb}C^e_{cd}, \tag{2.1}$$

and verifies

$$S_{abcd}y^a = S_{abcd}y^b = \ldots = 0, \tag{2.2}$$

where $S_{abcd} = g_{ae}S_b{}^e{}_{cd}$.

The indicatrix is also described by the equations $F^2(x, y) = 1$ or $g_{ab}y^a y^b = 1$. It can be parameterized in the form

$$(2.3) \qquad y^a = y^a(u^\alpha), \quad rank(\frac{\partial y^a}{\partial u^\alpha}) = m-1, \quad \alpha = 1, 2, \ldots, m-1.$$

It follows that the vectors $B_\alpha = \frac{\partial y^a}{\partial u^\alpha}\frac{\partial}{\partial y^a}$ provide a local basis of the tangent bundle over I. We look for a vector field normal to I. By deriving with respect to u^α the identity $F^2(x, y^a(u^\alpha)) \equiv 1$ we get $\frac{\partial F^2}{\partial y^a}\frac{\partial y^a}{\partial u^\alpha} = 0, \alpha = 1, 2, \ldots, m-1$. But $\frac{\partial F^2}{\partial y^a} = 2g_{ab}y^b$. Hence $g_{ab}\frac{\partial y^a}{\partial u^\alpha}y^b = 0$. This means that the vector field $C = y^a(u^\alpha)\frac{\partial}{\partial y^a}$ is normal on B_α for every $\alpha = 1, 2, \ldots, m-1$. Moreover, it is an unitary vector field since $g_{ab}y^a(u^\alpha)y^b(u^\alpha) = 1$. This is nothing but the restriction to I of the Liouville vector field $C = y^a\frac{\partial}{\partial y^a}$.

It satisfies

Lemma 2.1. $\overline{\nabla}_X C = X$ *for every vector field* X *tangent to* E_x.

Proof. Let be $X = X^b(y)\frac{\partial}{\partial y^b}$. We have $\overline{\nabla}_X C = X^b\overline{\nabla}_{\frac{\partial}{\partial y^b}}\left(y^a\frac{\partial}{\partial y^a}\right) = X^b\left(\frac{\partial}{\partial y^b} + y^a C^c_{ab}\frac{\partial}{\partial y^c}\right) = X$, because of $C^c_{ab}y^a = 0$, q.e.d.

Let $U, V, W, Z, \ldots$ be vector field that are tangent to I. The Weingarten formula $\overline{\nabla}_U C = -AU$, where A is the Weingarten operator and Lemma 2.1 give us $A = -I$ (identity) and so the Gauss and Weigarten formulae for I take the form

$$(2.4) \qquad \overline{\nabla}_U V = \nabla_U V - g(U, V)C, \overline{\nabla}_U C = -U.$$

Here ∇ denotes the Levi-Civita connection induced by $\overline{\nabla}$ on the indicatrix I.

Therefore, we have

Theorem 2.1. *The indicatrix* I_x *in* (E_x, g_x) *is totally umbilical and of mean curvature* $H = -1$.

Let S, R be the curvature tensor field of $\overline{\nabla}$ and ∇, respectively. With the notations $R(W, Z, U, V) = g(R(U, V)Z, W)$, $S(W, Z, U, V) = g(S(U, V)Z, W)$ the Gauss equation for I looks as follows:

(2.5)

$$S(W, Z, U, V) = R(W, Z, U, V) + g(U, Z)g(V, W) - g(V, Z)g(U, W).$$

It takes the equivalent form

$$(2.5') \qquad S(U, V)Z = R(U, V)Z + g(Z, U)V - g(Z, V)U,$$

which says that the normal component of $S(U, V)Z$ vanishes. As $S(U, V)C = 0$, we have no other integrability conditions. We write the Gauss equation (2.5') for $V = B_\alpha$, take the inner product of both members by B_β and sum up over $\alpha, \beta = 1, 2, \ldots, m-1$. We get

$$Ric(U,Z) = \overline{Ric}(U,Z) - (n-2)g(U,Z), \tag{2.6}$$

where Ric is the Ricci tensor of ∇ and $\overline{Ric}$ is the Ricci tensor of $\overline{\nabla}$. From (2.6) it immediately follows

$$Sc = \overline{Sc} - (n-1)(n-2), \tag{2.7}$$

where $\overline{Sc}$ and Sc is the scalar curvature of $\overline{\nabla}$ and ∇, respectively.

Coming back to the Gauss equation, $R(W,Z,U,V) = S(W,Z,U,V) + g(V,Z)g(U,W) - g(U,Z)g(V,W)$ taking $W = U = X$, $Z = V = Y$ and dividing the both members by $g(X,X)g(Y,Y) - g^2(X,Y)$ one obtains

$$K(X,Y) = \overline{K}(X,Y) + 1, \tag{2.8}$$

where K and $\overline{K}$ mean respectively sectional curvatures of 2-plan (X,Y). We have

Theorem 2.2. *The indicatrix I is of constant sectional curvature 1 if and only if S vanishes.*

Proof. If $K = 1$, we get $\overline{K} = 0$. It is well known that $\overline{K}$ determines S in such a way that $\overline{K} = 0$ implies $S = 0$. The converse is obvious.

3. Normal of the indicatrix bundle

The set $IB = (x,y) \in E \setminus 0, F(x,y) = 1$ is a $(2n-1)$-dimensional submanifold of $E \setminus 0$. We call it the indicatrix bundle of the vector bundle ξ, extending a term used in Finsler geometry.

We assume that the base M is a Riemannian manifold with the Riemannian metric of local coefficients h_{ij}. Then we may consider a Riemannian metric of Sasaki type on $E \setminus 0$ defined in the adapted basis as follows : $G = h_{ij}dx^i dx^j + g_{ab}\delta y^a \delta y^b$. Moreover, we assume that ξ is endowed with a nonlinear connection that is compatible with F i.e. the condition $\delta_i F = 0$, holds. We are interested to find the unit normal vector field to IB.

Let be

$$x^i = x^i(u^\alpha), y^i = y^i(u^\alpha), \alpha = 1, 2, ..., 2n-1 \tag{3.1}$$

a parametrization of the submanifold IB. The local vector fields $\dfrac{\partial}{\partial u^\alpha}$ that form a basis of the tangent space to IB can be put in the form

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial x^i}{\partial u^\alpha}\delta_i + \left(\frac{\partial y^i}{\partial u^\alpha} + N^i_j(x,y)\frac{\partial x^j}{\partial u^\alpha}\right)\dot{\partial}_i. \tag{3.2}$$

If one derives the identity $F(x(u^\alpha), y(u^\alpha)) \equiv 1$ with respect to u^α, one obtains

$$(\delta_i F)\frac{\partial x^i}{\partial u^\alpha} + (\dot{\partial}_i F)\left(\frac{\partial y^i}{\partial u^\alpha} + N^i_j\frac{\partial x^j}{\partial u^\alpha}\right) \equiv 0. \tag{3.3}$$

On using (3.2) and (3.3) we see that the vector field C is normal to IB since

$$G(\frac{\partial}{\partial u^\alpha}, y^a\dot{\partial}_a) = (g_{ab}y^b)\left(\frac{\partial y^a}{\partial u^\alpha} + N^a_j(x(u), y(u))\frac{\partial x^j}{\partial u^\alpha}\right) = 0\ . \tag{3.4}$$

for every $\alpha = 1, 2, ..., 2n-1$.

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Mihai Anastasiei
Faculty of Mathematics,
University "Al.I.Cuza" Iaşi
700506, Iaşi, Romania
and
Mathematics Institute "O.Mayer"
Romanian Academy Iaşi Branch
700506, Iaşi, Romania
email : anastas@uaic.ro

Manuela Gîrţu
"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Department of Mathematics and Informatics
Calea Mărăşeşti 157, Bacău 600115, ROMANIA
email: girtum@yahoo.com