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APPROXIMATION OF FRACTALS GENERATED BY INTEGRAL OPERATORS

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Abstract. We present some results concerning fractals generated by an iterated function system which is formed using integral operators on the infinite dimensional space of continuous functions on a compact interval. We approximate the fractal via a finite approximant set and project this approximant set in two dimensions, in order to make possible the visualization of the fractal.

1. Introduction

Iterated function systems (IFS) which were introduced in the present form by John Hutchinson (see [7]) and popularized by Michael Barnsley (see [2]) are a convenient way to describe and classify deterministic fractals in the form of a deterministic definition, providing a new insight into the description of natural phenomena. See [6] too.

The problem of fractals' approximation is extremely important from the practical point of view. Numerical comparisons among approximations of a fractal set are presented in [5]. New algorithms for approximation of fractals can be found in [1] and [9].

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Within the framework of a program initiated by us, whose main aim is to study the fractals generated by IFS in infinite dimensional spaces, in [3] and [4], we presented an example of a fractal generated by Hutchinson's procedure, embedded in an infinite dimensional Banach space, together with its finite approximations. More precisely, we worked in the Banach space of real valued continuous functions on a compact interval and we approximated the attractor generated by an IFS given by a Fredholm integral equation via a finite approximant set and projected this approximant set into two dimensions.

The aim of this work, which can be viewed as a continuation of [3] and [4], is to present some results concerning fractals generated by an IFS which is formed with Hammerstein - type operators on the infinite dimensional space of real continuos functions on a compact interval. We approximate the fractal via a finite approximant set and project this approximant set into two dimensions.

2. Results

A. Let $a < b$ be real numbers. In the sequel X (respectively Y) will be the real Banach space of all continuous $f : [a, b] \to \mathbb{R}$ (respectively $K : [a, b] \times [a, b] \to \mathbb{R}$) equipped with the sup norm.

The unit ball of X is

$$S = \{f \in X \mid \|f\| \leq 1\}.$$

Equipped with the metric of X, S is a complete metric space.

Let us consider a number $h \geq 1$and $f^1, f^2, ..., f^h$ in X with

$$m = \max_{i=1}^{h} \left\|f^i\right\|$$

and $K^1, K^2, ..., K^h$ in Y with

$$c = \max_{i=1}^{h} \left\|K^i\right\|.$$

Here the letters i are indexes.

We shall also consider a continuously differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ with

$$\varphi(0) = 0,$$

$$\delta = \max\{|\varphi(x)| \mid x \in [-1, 1]\}$$

and

$$\beta = \max\{\left|\varphi^{'}(x)\right| \mid x \in [-1, 1]\}.$$

Finally, let us consider a real number λ and a strictly positive number ε such that

$$|\lambda|\,(b-a)(c+\varepsilon)\delta + m \leq 1, \tag{1}$$

and

$$r = |\lambda|\,(b-a)(c+\varepsilon)\beta < 1. \tag{2}$$

We lay stress (again) upon the fact that, in the sequel, the letters i are used as indexes.

Now it is possible to define the contractions $T^i : S \to S$, $i = 1, 2, ..., h$, given via

$$T^i(u) = v^i,$$

where

$$v^i(x) = f^i(x) + \lambda \int_a^b K^i(x,y)\varphi(u(y))dy$$

T^i are operators of Hammerstein type: a Hammerstein operator $T : X \to X$ has the form

$$T(u) = v,$$

where

$$v(x) = \int_a^b K(x,y)f(y,u(y))dy,$$

where $K : [a,b] \times [a,b] \to \mathbb{R}$ and $f : [a,b] \times [a,b] \to \mathbb{R}$ are continuous (see [8]).

More precisely, one can prove that all T^i are contractions, with contraction coefficient less than r.

In the sequel, we shall work under the following non-triviality assumptions:

a) $\lambda \neq 0$ (if $\lambda = 0$, it follows that all T^i are constant functions);

b) $K^i \neq 0$ for all $i = 1, 2, ..., h$ (if $K^i = 0$, it follows that T^i is constant);

c) $\beta \neq 0$ (if $\beta \neq 0$, it follows that all T^i are constant functions).

We know that Y includes the dense vector subspace consisting of all polynomial functions of the form $P(x,y) = \sum\limits_{s,t} a_{st}x^s y^t$. Having this in mind, we shall approximate our contractions T^i with some other contractions given by approximating kernels.

Namely, let $(\varepsilon_n)_n$ be a sequence of strictly positive numbers such that $\varepsilon_n \to 0$ and $\varepsilon_n \leq \varepsilon$ for any n. Let us consider, for any $i = 1, 2, ..., h$, a sequence $(K_n^i)_n$ of polynomial functions in Y such that, for all natural n, one has

$$\left\|K_n^i - K^i\right\| \leq \varepsilon_n,$$

for all $i = 1, 2, ..., h$.

We can construct h sequences of contractions $(T_n^i)_n$, $i = 1, 2, ..., h$, namely $T_n^i : S \to S$, $T_n^i(u) = v_n^i$, given via

$$v_n^i(x) = f^i(x) + \lambda \int_a^b K_n^i(x, y)\varphi(u(y))dy.$$

One can prove that all T_n^i are contractions with contraction coefficients less than r.

B. Before continuing, let us recall some aspects of the general theory of fractals generated by an IFS.

Let $\mathcal{K}(S)$ be the set of all non empty compact subsets of S, which becomes a complete metric space, when equipped with the Hausdorff-Pompeiu metric H, given via

$$H(A, B) = \max(d(A, B), d(B, A)),$$

where

$$d(A, B) = \sup\{dist(u, B) \mid u \in A\}$$

with

$$dist(u, B) = \inf\{\|u - b\| \mid b \in B\}.$$

(in particular, $dist(u, \{x_0\}) = \|u - x_0\|$).

Within the framework of the theory of fractals, the system of functions $(T^1, T^2, ..., T^h) = \mathcal{F}$ is called an iterated function system IFS), thus defining a term previously used. The same for $(T_n^1, T_n^2, ..., T_n^h) = \mathcal{F}_n$.

On the complete metric space $(\mathcal{K}(S), H)$, we can define the Hutchinson contractions $F : \mathcal{K}(S) \to \mathcal{K}(S)$ and $F_n : \mathcal{K}(S) \to \mathcal{K}(S)$, via

$$F(E) = \bigcup_{i=1}^{h} T^i(E) \text{ and } F_n(E) = \bigcup_{i=1}^{h} T_n^i(E)$$

with contractions coefficients less than r (see [2] and [6]).

The fixed point A of F, called the attractor of the IFS $\mathcal{F}$ (or the attractor of $\mathcal{F}$) is, generally speaking, a fractal. The attractor of $\mathcal{F}_n$ will be denoted by A_n.

Our aim in the sequel will be to approximate A with a finite set (using A_n as intermediate sets) and to project the finite approximant set into two dimensions.

C. It is known that, under special assumptions, $A_n \underset{n}{\to} A$. The next result follows this line, giving an estimation of the distance between A_n and A.

Theorem 1. *For any natural n, one has*

$$H(A, A_n) \leq \frac{\delta}{\beta} \cdot \frac{\varepsilon_n}{(1-r)(c+\varepsilon)}$$

(hence $A_n \underset{n}{\to} A$).

In order to continue, we shall consider a contraction $W : \mathcal{K}(S) \to \mathcal{K}(S)$, with attractor (i.e. fixed point) E. We can obtain E as follows (the procedure of the fixed point theorem Banach-Caccioppoli-Picard):

a) one takes an arbitrary $E_0 \in \mathcal{K}(S)$;

b) one defines the sequence $(E_n)_n$ via

$$E_n = W(E_{n-1}),$$

$n = 1, 2, ..;$

c) finally, one has

$$E = \lim_n E_n.$$

We shall work for $W = F$ or $W = F_n$ and for $E_0 = \{0\}$. In either cases we have

$$E_1 = W(E_0) = \{f^1, f^2, ..., f^h\}$$

and the set E_n has at most h^n elements, for all n.

Theorem 2. *Using previous notations and working for $W = F_n$ (arbitrary natural n) we have for any natural p:*

$$H(A_n, E_p) \leq \frac{m}{1-r} r^p$$

(the estimation does not depend upon n).

Conclusion of this part. For a given $\gamma > 0$, one can construct a finite set E_p which approximates the attractor A such that

$$H(E_p, A) < \gamma$$

as follows:

a) Write $\gamma = \gamma_1 + \gamma_2$, with $\gamma_1 > 0$, $\gamma_2 > 0$.

b) Choose a convenient natural n such that

$$\frac{\delta}{\beta} \cdot \frac{\varepsilon_n}{(1-r)(c+\varepsilon)} < \gamma_1. \tag{3}$$

c) Choose a convenient natural p such that

$$\frac{m}{1-r} r^p < \gamma_2. \tag{4}$$

For the natural n found at b), write $W = F_n$ and construct effectively E_p (with p from c))using $W = F_n$.

d) We use Theorem 1 and Theorem 2. For the number p and the set E_p which were found at c), one has (use also n found at b))

$$H(E_p, A) \leq H(E_p, A_n) + H(A_n, A) \leq$$

$$\leq \frac{m}{1-r} r^p + \frac{\delta}{\beta} \cdot \frac{\varepsilon_n}{(1-r)(c+\varepsilon)} < \gamma_1 + \gamma_2 = \gamma.$$

Remark. The intermediate set A_n appears only theoretically, generating the number n.

D. In this subparagraph we shall "project" into two dimension the finite approximant fractal set obtained at the subparagraph C. We shall take the functions f^i to be polynomial functions and the approximant kernels K_n^i to be polynomial functions too.

In order to make computation easier, we shall work in the case $[a, b] = [0, 1]$. We shall consider the Lebesgue measure μ on $[0, 1]$ and we shall write $L^2 = L^2(\mu)$ The real Hilbert space L^2 is equipped with the usual norm

$$\left\| \tilde{f} \right\|_2 = (\int f^2 d\mu)^{\frac{1}{2}},$$

(computed for any representative f of the equivalence class $\tilde{f} \in L^2$).

We can consider the linear, continuos and injective map (embedding) $I : X \to L^2$, given by

$$I(f) = \tilde{f}$$

(in the sequel, we shall identify f and $I(f)$.

The space L^2 contains the bidimensional (closed) subspace

$$Z = \{\tilde{f} \in L^2 \mid f(x) = ax + b, a, b \in \mathbb{R}\}.$$

One can consider the orthogonal projection $P : L^2 \to L^2$, generated by Z, acting via $P(\tilde{f}) = \tilde{g} \in Z$, where $g(x) = ax + b$ for some real a and b. One knows that

$$\left\|\tilde{f} - \tilde{g}\right\|_2 = \min\{\left\|\tilde{f} - \tilde{u}\right\|_2 \mid \tilde{u} \in Z\}.$$

So, one has the linear and continuous map $P \circ I : X \to L^2$ with $(P \circ I)(X) \subset Z$.

We are interested in the projection of the attractor A, so we are interested in $P(I(A))$. One can approximate $P(I(A))$ with $P(I(E_p))$ (see the end of subparagraph C). This is because $\|P \circ I\| \leq 1$, which implies

$$H((P \circ I)(A), (P \circ I)(E_p)) \leq H(A, E_p).$$

The elements of E_p are polynomial functions, because K_n^i and f^i are polynomial functions. We must therefore project all the functions of the form $u(x) = x^n$ and use linearity.

Take a natural n and let us compute $(P \circ I)(u)$, where $u(x) = x^n$. We have $(P \circ I)(u) = \tilde{v}$, where $v(x) = ax + b$ and the real numbers a, b must be such that $\left\|\tilde{u} - \tilde{v}\right\|_2$ is minimum. This is equivalent to the fact that

$$\int_0^1 (x^n - ax - b)^2 dx$$

is minimum.

We must therefore minimize the function $t : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given via

$$t(a, b) = \frac{1}{3}a^2 + ab + b^2 - \frac{2}{n+2}a - \frac{2}{n+1}b + \frac{1}{2n+1}.$$

Using partial derivatives, we obtain the minimum point (a, b) where

$$a = \frac{6n}{(n+1)(n+2)},\ b = \frac{-2n+2}{(n+1)(n+2)}.$$

So, informally, one has

$$(P \circ I)(x^n) = \frac{6n}{(n+1)(n+2)}x + \frac{-2n+2}{(n+1)(n+2)}$$

and, for a general polynomial function:

$(P \circ I)(a_0 + a_1x + a_2x^2 + ... + a_nx^n) = (\sum_{k=0}^{n} \frac{6k}{(k+1)(k+2)} a_k)x + (\sum_{k=0}^{n} \frac{-2k+2}{(k+1)(k+2)} a_k)$.

In order to "draw the picture" of the projection of E_p, we shall proceed as follows:

a) We shall effectively construct E_p (namely $h = 2$, $p = 5$, which gives $2^5 = 32$ points for E_p; see the numerical example which follows).

b) The elements of E_p are polynomials (there are 32 polynomials of degree 4 in E_5, see the numerical example which follows). Each polynomial will be projected onto a polynomial of the form $ax + b$ according to the previous formula. We shall identify $ax + b \equiv (a, b)$. Thus, one obtains h^p (here 32) points (a, b) in the Cartesian plane, which depict the projection of E_p and an approximate the image of the projection of A.

E. We conclude with a numerical example.

Take $[a, b] = [0, 1]$, $h = 2$, $K^i : [0, 1] \times [0, 1] \to \mathbb{R}$, $f^i : [0, 1] \to \mathbb{R}$ $(i = 1, 2)$, $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$K^1(x, y) = e^{xy};$$

$$K^2(x, y) = \cos xy;$$

$$f^1(x) = \frac{1}{2} - \frac{1}{2}x^4;$$

$$f^2(x) = -\frac{1}{2}x + \frac{1}{2}x^3;$$

$$\varphi(t) = t^2.$$

One has $\|K^1\| = e, \|K^2\| = 1$, hence $c = e$. Since $\|f^1\| = \frac{1}{2}$, $\|f^2\| = \frac{1}{3\sqrt{3}}$, one has $m = \frac{1}{2}$.

We have also $\delta = 1$, $\beta = 2$.

We can take $\lambda = \frac{1}{10}$, $\varepsilon = 3 - e$, hence $c + \varepsilon = 3$.

Conditions (1) and (2) are fulfilled.

Consequently, for any $u \in X$ and $x \in [0, 1]$, one has

$$T^1(u)(x) = \frac{1}{2} - \frac{1}{2}x^4 + \frac{1}{10}\int_0^1 e^{xy}(u(y))^2 dy$$

and

$$T^2(u)(x) = -\frac{1}{2}x + \frac{1}{2}x^3 + \frac{1}{10}\int_0^1 \cos(xy)(u(y))^2 dy.$$

Because

$$e^{xy} = \sum_{p=0}^{\infty} \frac{(xy)^p}{p!}$$

and

$$\cos xy = \sum_{p=0}^{\infty} (-1)^p \frac{(xy)^{2p}}{(2p)!}$$

we shall take

$$K_n^1(x,y) = 1 + \frac{xy}{1!} + \frac{(xy)^2}{2!} + ... + \frac{(xy)^{2n}}{(2n)!}$$

and

$$K_n^2(x,y) = 1 - \frac{(xy)^2}{2!} + \frac{(xy)^4}{4!} ... + (-1)^n \frac{(xy)^{2n}}{(2n)!}.$$

One can take

$$\varepsilon_n = \frac{1}{(2n+1)!} \cdot \frac{2n+2}{2n+1} < 3 - e = \varepsilon.$$

Finally, we shall take

$$\gamma = \frac{21}{200} = \frac{1}{200} + \frac{1}{10}$$

choosing $\gamma_1 = \frac{1}{200}$, $\gamma_2 = \frac{1}{10}$.

Condition (3) is fulfilled for $n = 2$:

$$\frac{\delta}{\beta} \cdot \frac{\varepsilon_2}{(1-r)(c+\varepsilon)} = \frac{5}{12} \cdot \varepsilon_2 = \frac{5}{12} \cdot \frac{1}{100} < \frac{1}{200} = \gamma_1.$$

and, because $(\varepsilon_n)_n$ is strictly decreasing, one has (3) for any $n \geq 2$.

Consequently, we take $n = 2$.

Condition (4) is fulfilled for $p = 5$:

$$\frac{m}{1-r} r^5 = \frac{243}{2500} < \frac{1}{10} = \gamma_2$$

and, because $(r^p)_p$ is strictly decreasing, one has (4) for any $p \geq 5$.

Consequently, we take $p = 5$.

Using a Java program, the authors obtained 32 points (a, b). The picture thus obtained is divided into two "clouds".

a b

-0.39006412108790817	0.6192805265549698
-0.39007454791758733	0.6192643840220823
-0.3901780302954932	0.6191037289986427
-0.39017567330511094	0.6191073864988954
-0.3911501764419598	0.6175541637284669
-0.39114818696559533	0.6175572703260482
-0.39112477729199857	0.6175938175296398
-0.3911253026899864	0.6175929966564953
-0.3988918740251305	0.601411807920607
-0.398889496944338	0.6014149183083592
-0.39886544923814427	0.6014463891856421
-0.39886599878874085	0.6014456692703021
-0.39859378045036303	0.6018022730560832
-0.3985943135334482	0.6018015648041742
-0.3986005733125027	0.6017932486985071
-0.398600432425216	0.6017934357800403
-0.05193444691813059	-0.07997563497447382
-0.05193223213444649	-0.07999264572610684
-0.05191027136257895	-0.08016190781738887
-0.05191077157182562	-0.08015805427757644
-0.051705776366793244	-0.08179146020533767
-0.051706196881663904	-0.08178818906400566
-0.05171114548444349	-0.08174970545038526
-0.05171103443993191	-0.08175056977497255
-0.05023793415218893	-0.09849583038531429
-0.050238446459930586	-0.09849251995675924
-0.05024363127769308	-0.098459024189465
-0.05024351278970853	-0.09845979038340859
-0.050302433409100705	-0.0980801341471826
-0.05030231823260451	-0.09808088747609546
-0.05030096582855058	-0.09808973286228916
-0.050300996269852224	-0.09808953386706178

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