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BIMETRIC SPACES, SPACE-TIME AND MATHEMATICS. METHODOLOGICAL PROBLEMS

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Abstract. The development of the geometry is inseparably linked with the history of the development of the mathematics which may be divided into seven periods. The concept of a bimetric space is introduced and studied. This notion is applied to construction of some new models of the space-time.

1. Introduction

The Universe is the entirety of space and time, all forms of the mater, energy and momentum, the physical laws that govern them. The space-time is the arena in which all physical events take place. An event is defined as a point in the space-time, a specific position in space and a specific moment in time. The space-time is any mathematical model that combines the space and the time into a single continuum. Galileo Galilei stated that the Universe is a grand book written in the language of the Mathematics. Hence the mathematical models of the Universe state the Mathematical Universe as a part of the Universe of Mathematics.

Mathematical modeling employs the tools of mathematical structures.

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The practical spirit, reality, but also the world of ideas and judgments represent the objective and effective reality, as well as the viability of the Mathematical sciences, a fact confirmed by the great variety of achievements in different domains of life and thought. It is impossible to find a domain of the human activity that does not involve Mathematics starting with the most simple problems up to the most subtle theoretical constructions. It is very possible that Pythagoras intuitively sensed this truth and affirmed that "all is a number".

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2. Mathematics and Reality

The development of the sciences and of the physics is inseparably linked with the history of the development of the mathematics which may be devided into seven periods:

- prehistoric or conceived period (the prehistoric times - until the 4th - 3rd millennium B.C);
- the period of the practical (algorithmic) mathematics (3rd millennium - the end of the 7th century B.C.);
- the period of the theoretical mathematics (7th century B.C. - 14th century A.D);
- the period of the origin of the mathematical languages (14th - 17th centuries);
- the period of the variable quantities and of the formation of the calculus (mathematical analysis) (18th century);
- the period of the mathematical structures (19th century – the middle of the 20th century);
- the period of global theories of the complex mathematical structures (the middle of the 20th century - until present).

We link the term period not to the level reached in a certain region, but to the new ideas, the methodological concepts and the mathematical apparatus elaborated during that time (see I.I. Valuţa's Elements of History of Mathematics [5]). The proposed division into periods of the development of the mathematics develops the respective Kolmogorov's concept [10].

The sixth period is the period of mathematical structures generated axiomatically, a period during which mathematical knowledge, as a part of the cultural thesaurus, becomes a motive force of the progress in all spheres of the human activity. The fifth period marks the end of the era of great scientific efforts that lasted about 2500 years. The most representative researches of this period are due to the 17th century investigations, there were obtained important results in the field of Differential Geometry, G. Monge's "Descriptive Geometry" was written and researches in the domain of Projective Geometry were initiated. Surprising results were achieved in the problem of the 5th Postulate.

The dramatic aspect of the 5th Postulate problem consists in the fact that mathematical world, in particular, and scientific world, in common, were not ready, psychologically speaking, to accept a surprising solution able to radically change the concept of a space. It is easy to notice that the possibility to find such a solution was exposed by Lambert and this chance existed even earlier due to Omar Khayyam.

The explosion determined by the existence of the distinct geometries gave birth to a huge wave of ideas that in less than a half century contributed to the creation of the contemporary mathematics fundamentals. During the years 1870 - 1910, the notion of mathematical structure was elaborated by means of axiomatic approaches, and it was based on the notion of a set. However, almost immediately there have appeared some pathological phenomena. The first has appeared because the examples of one variable functions constructed by B. Bolzano and K. Weierstrass which, being continuous, do not have tangents in any of their points, i.e. they are non-derivable. This matter is impossible to be perceived in an intuitive way, a fact that made the great French mathematician H. Poincaré to exclaim: "How could intuition mislead us to such degree?" Due to the investigations of B. Bolzano, N. I. Lobacevskii, N. Abel, A. Cauchy there were elaborated the notions of limit and continuous function. As a result S. Lie, F. Ch. Klein and H. Poincaré managed to extend in their works the notion of a group. Their investigations contributed to the development of the Lie groups and algebras, as well as of analytical manifolds. If until the 1860s Algebra was, according to J. A. Serret, "the science about analysis of the equations", by the 1900s it became the science of algebraic structures (group, ring, field, semigroup, universal algebra). This dramatic change in Algebra perception occurred, to a great extent, due to Geometry as, according to Klein's Erlangen Program, from an algebraic point of view Euclid's and Lobacevskii-Bolyai's geometries, as well as Riemann's elliptic geometry may be represented as a pair (S, G), where S is a space (a plane), and G is a group of transformations of the set S. Geometry became a theory that shapes the theory of algebraic invariants, of bilinear forms, of bilinear symmetric forms, of squared forms and of C. Hermite's forms. The works of Riemann and Poincaré develop the concept of "analysis situs" exposed by Leibniz. They are the initiators of a new mathematical domain - topology (see [3, 5, 10, 12, 13, 15]).

N. I. Lobacevskii ends up his book "Imaginary Geometry" in the hope that this new Geometry would serve as a foundation for a new physical theory, with new laws of dynamics. After 1870 new researches dealing with mechanics construction and physics in non-Euclidean spaces are initiated. The first attempts into this direction were made by such scholars as A. Genocchi, W. R. Boll, W. C. Clifford, H. Poincaré. By 1900 H. A. Lorentz studied a special group of transformations of the space and time. These transformations are called at present Lorentz's transformations. In 1902 for his research in the domain of Physics Lorentz was awarded the Nobel Prize. Another great scholar from this period was H. Minkowski. He was the initiator of numerous theories: the geometric theory of numbers, theory of special nets, squared forms arithmetic, theory of convex polyhedrons. In 1909 he had published his "Space and Time" in which the 4-dimensional Minkowski space was constructed with Lorentz's transformations group as isometries. These investigations formed the bases of the special theory of relativity. The theory of relativity was elaborated in 1905 by Albert Einstein and, independently, by H. Poincaré. It is considered that Poincaré did not explain in an understandable way the physical principles of this theory. Thus the creator of theory of relativity is considered A. Einstein. However, Poincaré's work "About Electrons Movement" (1905) lays at the basis of this theory. In 1905 Albert Einstein elaborated the special relativity theory and later the quantum theory of light. During 1907 - 1916 Einstein had elaborated the general theory of relativity. Of a special importance in his research was the tensor theory. The non-Euclidean geometries, and particularly the Minkowski's geometry, played a crucial role in the elaboration of the theory of relativity. A. Einstein formulates the problem of geometrization of Physics in his "Geometry and Physics". He affirms that any geometry may serve as a basis for a Physical theory. However, only in a geometry adequate to a specific Physical theory is able to represent the formulas of this very theorys laws in an elegant and simple way. Therefore the relativity theory is correlated with the Minkowski's geometry theory and Riemann's geometries (see [6]). We note that the concrete versions of the Einstein's problem were formulated and investigated in the works of T. Levi-Civita, E. Cartan, Gh. Vrănceanu, R. Miron and other mathematicians (see [11, 18]).

It is quite frequent that the following two questions occur:

1. Is Mathematics real?

2. Does the Mathematical knowledge reflect Reality?

Broadly speaking the answer to the first question is quite obvious: everything that exists or ever existed is real! It does not depend whether it was a form, a fact or a thought. The answer to the second question is not simple even for those that are familiarized with Mathematics and Philosophy. Mathematics studies domains appeared and developed due to different causes:

- There are domains that develop due to the fact that some practical problems have been solved;

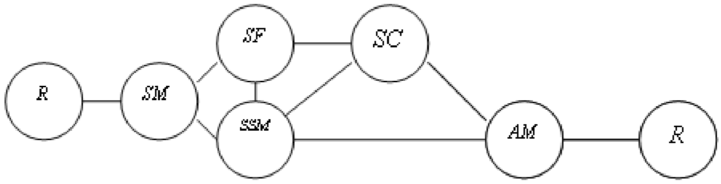
- There are domains that develop due to the inner necessities of Mathematics;

- There are fields that investigate the way in which the mathematical reasoning is constructed.

The first mathematical knowledge appeared due to some practical needs (periods one and two), and Mathematics as a science emerged due to a necessity to justify the conclusions of a mathematical character (period three). The PC is at present a reality and a practical necessity. However, computer would not have exist as we know it today, without the numbers theory and the numeric systems, Boole algebras and universal algebras, and it also owes to modern Electronics. Therefore different practical problems find solutions in the Mathematical domains. This allows us affirm that Mathematics reflect Reality. Hand in hand with this we would like to mention that each Mathematical domain develops independently from the evolution of phenomena noticed in Reality, though these very phenomena could serve as an impulse for the initiation of a research in the field of Mathematics. The place of the mathematics in the physical theories was excellent determined by the P. Dirac in 1931: "The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalize the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities."

The Universe of the Mathematics include the following components: R - the Reality, SM - Mathematical Structures, SF - Formal Systems,

SSM - Mathematical Super-Structures, SC - Computation Systems, AM - Applications of Mathematics. The relation between this components can be represented by the next diagram:



The mathematical models of the real phenomena are formal systems of some mathematical structures or mathematical super-structures. The theorems from SF, SM and SSM describe concrete properties of the real phenomena. The formal systems describe the computation systems, mathematical structures and super-structures. The mathematical structures and super-structures are defined by using the computation systems which are the special examples of mathematical structures and super-structures and produce the theorems from formal systems, mathematical structures and super-structures. There exist various computation systems: numerical calculus, geometrical computations, propositional calculus, propositional inference, etc. The mathematical structures used in physics and other domains are defined, as a rule, by computable functors, functions, relations, etc (see Computable Universe Hypothesis [16, 17]).

3. Abstract Spaces and the Real Space

The outstanding contribution of the great scholars of the old Elade to the development of the science, art and especially Mathematics is well known. Beginning with the 6th century a great contribution to the development of Mathematics was made by the Arabians. Euclidean space where the matter was freely situated in all its possible forms of existence. This was a static space geometry. The notions of the transformation, movement (isommetry) and similarity, the differential and integral calculus, the analytical geometry with its strong apparatus and methods of algebraic calculus did not manage to change the old concepts about geometrical space and the principles laying at the basis of Geometry. The emergence of the notions that led to the formation of the concept of abstract space began with the works of Gauss, Lobacevskii, Bolyai, Riemann, Taurinus, Cayley, Grasmann, Klein, Helmholtz, Frege, Pasch, Peano, Pieri and ends up with Hilberts work published in 1899.

One may question whether the notion of Euclidean space is a abstraction. It would be an absolutely appropriate question. Each word or notion are abstractions with concrete functions and content. Any notion is used according to specific characteristics, which, after a certain abstraction, may generate this very notion. For example, by the word "table" we understand an infinite totality of objects with an infinite diversity of forms. In a similar way by the word "space" we understand at present a infinity of different objects that differ in content and properties.

The Hilbert's notion of the space is conceived as a multitude of elements called "points" where some subsets are called "straight lines", some other subsets are called "plane", and these three notions satisfy certain properties comprised by axioms. As a matter of fact, in Hilberts terms "point", "straight line" and "plane" there are considered primary (indefinite) abstract elements connected by the relation of incidence. Poincaré affirmed in 1902 that axioms represent an implicit definition of the primary notions and he considered that a free agreement in the choice of primary notions and postulates exists. This is a conception that has to do with conventionality. The lack of success of the systems of axioms proposed by Peano and Pieri proves that sciences that emerged due to human practice necessities do not witness an arbitrary conventionalism and it is not made to the prejudice of an authentic creation or commodity. The choice of the primary notions, of the primary relations and axioms is made to the benefit of transparency and keeping the applicative capacities of the theory.

The notion of a distance leads us to the concept of a metric space and neighborhood and, finally, to the notion of topological space. The Euclidean space, together with the differential calculus, lead us to the notions of a manifold, a topological manifold and an analytical manifold. The contemporary notion of space reflects the relations and special figures that exist in the real world, in nature and, therefore the axioms are chosen in such a way as to be supported by the argument of practice. According the theory of relativity and the quantum mechanics, a space correlated to time is an objective and universal form of existence of the matter and movement. The real space expresses the objects and real world systems coexistence order, their position, distance, size and extension. The matter, movement, the time and the space are inseparable.

Helmholtz's research was made from the point of view of physiological optics, while Einstein's research was made from the perspective of the theory of relativity. Therefore different relations that exist in reality may be reflected or modeled by the same special relations. Quantum mechanics proved that not all physical phenomena may be appropriately modeled in dimensionally finite spaces. This fact leads us to the conclusion that reality and real relations are very complicated and the existent special theories describe in a quite complex way only some of the aspects of real world. Therefore the principles that lay at the basis of any special theory reflect only some of the real space relations. The conventionalist aspects had appeared later. At different stages of a theory development it may be concluded that this theory may be exposed in a "more comfortable" way if it is based on some other primary notions and axioms. However, these changes, seen against the general background of notions and theorems, do not affect the real content of the theory. These changes that mathematicians allow themselves do in a "free" and "easy" way provoke different "philosophical" interpretations. Thus the system of axioms elaborated by Hilbert, Peano, Pieri, Weyl essentially differ from each other, but they allow us construct the same Euclid's Geometry. The basic notions of an axiomatic system define each other and the affirmations made by the axioms in an axiomatic system may be demonstrated as theorems in other axiomatic systems within the same theory. Thus, the choice of an axiomatic system is conventional only from a formal point of view. It is to be mentioned that our perception of a special structure, distance, speed, object forms and phenomena under investigation is not always precise. We quite often mix up the notions of the infinite, the unlimited, or the limited Universe, etc. In his "On the hypotheses that lie at the foundation of geometry" (1854) B. Riemann elaborates completely new principles that lay at the basis of Space Geometry. Thus he affirms that the the real space is unlimited, with the meaning that there is no border beyond which any space exits. Nevertheless, a space may be unbounded (open) or bounded (closed). A space is also infinite in content. All these notions may be analytically described with the help of topological-geometrical methods. In order to determine whether a space is bounded or unbounded one must be aware of a certain measure which would allow us establish the properties of the space.

These being said, it becomes evident that the study of the notion of a space is a problem common to Mathematics, Physics and Philosophy (see [1, 2, 3, 5, 6, 8, 9, 10, 12, 13, 14, 16]).

4. On Formal Character of Mathematics

At present all scientific domains are in a process of dynamic development getting more diversified and specialized. The science operates by means of definitions, affirmations and proofs (demonstrations). The proof represents the calculus of the truth value of an affirmation. The aim of affirmations consists in solving the different problems approached by a certain domain of science. Those problems that a certain science is concerned with form its object of research. Speaking about those fields of science that Mathematics deals with, not any notion may be defined and not any affirmation regarding mathematical notions may be demonstrated. The indefinable notions are called primary notions, or basic, or fundamental, and those impossible to demonstrate are called axioms, postulates. The axiomatic principle began already with Thales and it was exposed in a complex way by Aristotle in his remarkable Logic. By the end of the 19th century and the beginning of the 20th century G. Frege, G. Peano, D. Hilbert, B. A. W. Russel and A. N. Whitehead developed the ideas of Aristotle and G. W. Leibnitz and laid the bases of the mathematical logic. The last one perceives mathematical calculus as a system of signs that lack signification and obey a set of specific rules of composition. Therefore, formal science is characterized by the isolation of sentences content and determination of the reasoning structure that form these sentences. These "conventional" representations of the indefinable notions and of the non-demonstrable affirmations led to the formalization of mathematical sciences. This fact gave birth to different Philosophical trends. Some of them point to formalization as an indicator of a certain crisis of mathematical sciences. It goes without saying that any mathematical, physical, chemical or biological process does not represent the absolute reality, but only images of reality. However, this model represents some relations established among the elements that are copies of different real elements. More than that, these abstract models may be applied to different types of real situations. The formal systems have a special priority.

5. Spaces with the distances

By a space we understand a Tychonoff topological space. We use the terminology from [7].

Let $\mathbb{R}$ be the space of reals, $\mathbb{C}$ be the space of complex numbers $z = x + iy$, where $x, y \in \mathbb{R} \subseteq \mathbb{C}$ and $|z| = (x^2 + y^2)^{1/2}$, $I(\alpha, \beta) = [\alpha, \beta]$ for any $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$, $\mathbb{R}^\star = \mathbb{R} \cup \{\infty\}$, $\mathbb{C}^\star = \mathbb{C} \cup \{\infty\}$. If

$z = x + iy, w = u + iv\mathbb{C}$, then $z < \infty$ and $w \leq z$ provided $u \leq x$ and $v \leq y$. We consider that $w \ll z$ if $u < x$ and $v < y$.

If S is a set, then $S \times S$ is the set of all ordered pairs (x, y) with $x, y \in S$. By $\emptyset$ we denote the empty set.

A bidistance space or a space with the distances is a triplet (S, d, d_v) consisting of a non-empty set S and two functions d and d_v defined on the set $S \times S$, assuming values from $\mathbb{C}^\star$ and satisfying the following conditions:

D1. $d_v(x, y) = 0$ if and only if $x = y$.

D2. $d_v(x, y) = d_v(y, x) \leq d(x, y)$ for all $x, y \in S$.

D3. $d_v(x, y) + d_v(y, z) \geq d_v(x, z)$ for all $x, y, z \in S$.

D4. If $x, y, z \in S$ and $d_v(x, y), d_v(y, z) \in \mathbb{R}$, then $d_v(x, z) \in \mathbb{R}$.

The set S is called a space, the elements of S are called points, the function d_v is called a virtual metric, the function d is called a quasi-metric, the number $d(x, y)$ is called the distance between x and y, the number $d_v(x, y)$ is called the virtual or external distance between x and y.

A bidistance space (S, d, d_v) is a bimetric space if the following conditions hold:

D5. $d(x, y) \in \mathbb{R}^\star$ for all $x, y \in X$.

D6. $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in S$.

D7. For each element $x \in S$ and any $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon) > 0$ such that $\{y \in S : d_v(x, y) < \delta\} \subseteq \{y \in S : d(x, y) < \varepsilon\}$.

If $d = d_v$ and (S, d, d_v) is a bimetric space, then $(S, d) = (S, d, d_v)$ is called a metric space. One can study the geometry of the bimetric spaces as the the geometry of the metric spaces (see [4, 7, 11, 14]).

It is obvious that it may be $d(x, y) = \infty$, or $d(x, y) \neq d(y, x)$, or $d_v(x, y) = \infty$ for some $x, y \in S$.

From Conditions D1 - D3 immediately it follows:

D8. $d_v(x, y) \geq 0$ for all $x, y \in S$.

D9. $d(x, y) \geq 0$ for all $x, y \in S$.

D10. $d(x, y) = 0$ if and only if $x = y$.

From Conditions D1 - D7 it follows:

D11. $\{y \in S : d(x, y) < r\} \subseteq \{y \in S : d_v(x, y) < r\}$ for any $r \gg 0$.

Let (S, d, d_v) be a bidistance space.

For every $x \in S$ and $r \gg 0$ the set $B(x, r) = \{y \in S : d(x, y) \ll r\}$ is called the r-ball about x, the set $B_v(x, r) = \{y \in S : d_v(x, y) \ll r\}$ is called the virtual r-ball about x, the set $S(x, r) = \{y \in S : d(x, y) = r\}$ is called the r-sphere with the center x and the set $S(x, r) = \{y \in S : d_v(x, y) = r\}$ is called the virtual r-sphere with the center x.

A subset $L \subseteq S$ is called open in S if for any point $x \in L$ there exists $r > 0$ such that $B(x, r) \subseteq L$. A set $F \subseteq S$ is called closed in the space S if its complement $S \setminus F$ is open.

The family $\mathcal{T}(d)$ of all open subsets of S is the topology of the bidistance space.

5.1. Remark. It is obvious that in a bimetric space the set $L \subseteq S$ is open in S if for any point $x \in L$ there exists $r > 0$ such that $B_v(x, r) \subseteq L$.

5.2. Remark. The restriction from D7 is very important. From that condition immediately it follows that the topology $\mathcal{T}(d)$ coincides with the topology of the metric space (S, d_v), i.e. $\mathcal{T}(d_v) = \mathcal{T}(d)$.

In general, the topology generated by a quasi-metric may be not metrizable.

5.3. Example. Let S be the space of reals, $d_v(x, y) = min\{1, |x - y|\}$, $d(x, y) = y - x$, if $x \leq y$, and $d(x, y) = 1$, if $y < x$. Then (S, d) is a non-metrizable quasi-metric space, the triplet (S, d, d_v) satisfies the conditions D1 - D4, $\mathcal{T}(d_v) \neq \mathcal{T}(d)$ and $\mathcal{T}(d_v) \subseteq \mathcal{T}(d)$. The space $(S, \mathcal{T}(d))$ is the Sorgenfrey line [7].

The following example is similar with the Example 5.3.

5.4. Example. Let S be the space of reals, $d_v(x, y) = |x - y|$, $d(x, y) = y - x$, if $x \leq y$, and $d(x, y) = +\infty$, if $y < x$. Then (S, d) is a non-metrizable quasi-metric space, the triplet (S, d, d_v) satisfies the conditions D1 - D5, $\mathcal{T}(d_v) \neq \mathcal{T}(d)$ and $\mathcal{T}(d_v) \subseteq \mathcal{T}(d)$. The space $(S, \mathcal{T}(d))$ is the Sorgenfrey line [7].

For every point $x \in S$ and every non-empty set $F \subseteq S$ the number $d(x, F) = inf\{d(x, y) : y \in F\}$ is called the distance from x to F and the number $d_v(x, F) = inf\{d_v(x, y) : y \in F\}$ is called the virtual distance from x to F. Let $d(x, \emptyset) = d_v(x, \emptyset) = \infty$.

5.5. Remark. Let (S, d, d_v) be a bimetric space. It is obvious that $d_v(x, F) \leq d(x, F)$ and $d(x, F) = d_v(x, F) = 0$ if and only if x is a point from the closure clF of the set F in S. In a bimetric space $d(x, F) = 0$ if and only if $d_v(x, F) = 0$.

The diameter and the virtual diameter of a non-empty set $L \subseteq S$ are the following numbers $diam(L) = sup\{d(x, y) : x, y \in L\}$ and $diam_v(L) = sup\{d_v(x, y) : x, y \in L\}$. We consider that $diam(\emptyset) = diam_v(\emptyset) = 0$.

A set L is said to be bounded (respectively, virtual bounded) if $diam(L) < \infty$ (respectively, $diam_v(L) < \infty$).

A sequence $\{x_n : n \in \mathbb{N} = \{1, 2, ...\}\}$ of points of S is convergent to a point x and we put $x = lim_{n\to\infty} x_n$ if $lim_{n\to\infty} d(x, x_n) = 0$.

A subset $L \subseteq S$ is compact in S if every sequence $\{x_n \in L : n \in \mathbb{N}\}$ has a convergent subsequence in S. A compact closed subset is called a compact subset. If L is a compact subset in S, then the its closure clL is compact. Every compact subset is virtual bounded.

5.6. Definition. *A subset $H \subseteq S$ is called an external black body (briefly, e-black body) of the bidistance space (S, d, d_v) if there exist $a \in S$ and $r \in \mathbb{R}$ such that:*

1. $r > 0$, $clH \setminus H \neq \emptyset$ and $H = B_v(a, r)$.

2. There exists $b \in B(a, r)$ such that $d(b, y) < \infty$ for any $y \in H$.

3. $d(x, y) = \infty$ for all $x \in H$ and $y \in S \setminus H$.

4. H is a compact subset in S.

5. If $\{x_n \in H : n \in \mathbb{N}\}$ and $lim\ x_n \in S \setminus H$, then $lim_{n\to\infty} d(x, x_n) = lim_{n\to\infty} d(x_n, x) = +\infty$ for any $x \in H$.

If H is an e-black body, then the set H is virtual bounded, $diam(H) = +infty$ and the boundary $Fr(H) = clH \setminus H$ is non-empty.

Let $\alpha < \beta$, $I(\alpha, \beta) = [\alpha, \beta] \subseteq \mathbb{R}$ and $I = I(0, 1)$.

A curve γ with the origin $a \in S$ and the end $b \in S$ is a continuous mapping $\gamma : I(\alpha, \beta) \to S$ such that $\gamma(\alpha) = a$, $\gamma(\beta) = b$ and $a \neq \gamma(t) \neq \gamma(\tau) \neq b$ for $\alpha < t < \tau < \beta$. If $a = b$, then we say that γ is a closed loop or an orbit. Let $l(\gamma, d) = sup\{\Sigma\{|d(\gamma(t_i), \gamma(t_{i+1})| : i \leq n\} : n \in \mathbb{N}, \alpha = t_1 < ... < t_n < t_{n+1} = \beta\}$. Then $l(\gamma, d)$ is the length and $l(\gamma, d_v)$ is the virtual length of the curve γ. If $l(\gamma, d) < +\infty$, then the curve is called rectifiable. The rectifiable curve is a trajectory of some spacial object. We consider that in the moment t the object is situated in the point $\gamma(t)$ and the length $l(\gamma_t, d)$ of the curve $\gamma_t : I(\alpha, t) \to S$, where $\gamma_t(\tau) = \gamma(\tau)$ for any $\tau \in [\alpha, t]$, is the distance covered at the moment t. The limit $lim_{\tau\to 0} \tau^{-1}(l(\gamma_{t+\tau}, d) - l(\gamma_t, d))$ is the speed of the object at the moment t. The trajectory γ is real if $d(x, y)\mathbb{R}$ for all $x, y \in \gamma$.

For every curve $\gamma : I(\alpha, \beta) \to S$ we define the inverse curve $\gamma^{-1} : I(\alpha, \beta) \to S$, where $\gamma^{-1}(t) = \gamma(\beta - t + \alpha)$. Since d_v is a metric, $l(\gamma, d_v) = l(\gamma^{-1}, d_v)$. In general, $l(\gamma, d) \neq l(\gamma^{-1}, d)$.

5.7. Remark. Let $\gamma_1, \gamma_2 : I(\alpha, \beta) \to S$ be two curves for which $\gamma_1(\alpha) = \gamma_2(\alpha)$, $\gamma_1(\beta) = \gamma_2(\beta) \neq \gamma_1(\alpha)$ and $\{\gamma_1(t) : t \in I(\alpha, \beta)\} = \{\gamma_2(t); t \in I(\alpha, \beta)\}$. Then $l(\gamma_1, d) = l(\gamma_2, d)$ and $l(\gamma_1, d_v) = l(\gamma_2, d_v)$.

5.8. Remark. Let $\gamma_1 : I(\alpha, \beta) \to S$ and $\gamma_2 : I(\lambda, \theta) \to S$ be two curves. If $\alpha \leq \lambda < \theta \leq \beta$ and $\gamma_1(t) = \gamma_2(t)$ for any $t \in I(\lambda, \theta)$, then γ_2 is called a subcurve of γ_1 and $l(\gamma_1, d) \leq l(\gamma_2, d)$. If $\theta - \lambda < \beta - \alpha$, then γ_2 is called a proper subcurve of γ_1 and $l(\gamma_1, d) < l(\gamma_2, d)$.

5.9. Lemma. *Let H be an e-black body of the bimetric space (S, d, d_v), $\gamma : I(\alpha, \beta) \to S$ be a curve of S, $\gamma(\alpha) \in H$ and $\gamma(\beta) \in S \backslash H$. Then $l(\gamma, d) = l(\gamma^{-1}, d) = +\infty$.*

Proof. There exist $\lambda = min\{t : \gamma(t) \in S \setminus H\}$ and a sequence $\{t_n : n \in N\}$ such that $\lambda - t_n < 2^{-n}$ and $\alpha < t_n < t_{n+1} < \lambda$. Let $a = \gamma(\alpha)$ and $a_n = \gamma(t_n)$. Then $l(\gamma, d) \geq d(a, \gamma(\beta)) = +\infty$ and $l(\gamma^{-1}, d) \geq lim_{n\to\infty} d(a_n, a) = +\infty$. The proof is complete.

The notion of the *"absolutely black body"* is possible under some additional restrictions.

Let (S, d, d_v) be a finite-dimensional differential manifold with the distances. The family $\mathcal{T}(d)$ of open subsets of S generates the algebra $\mathcal{B}(S)$ of Borel subsets of the space S with the properties:

- $\mathcal{T}(d) \subseteq \mathcal{B}(S)$;
- if $H \in \mathcal{B}(S)$, then $S \setminus H \in \mathcal{B}(S)$;
- if $H_n \in \mathcal{B}(S)$, then $\cup\{H_n : n \in \mathbb{N}\} \in \mathcal{B}(S)$.

Assume that it is defined a non-negative function $v : \mathcal{B}(S) \to \mathbb{R}^\star$ with the properties:

- $v(U) > 0$ for any non-empty open subset $U \subseteq S$;
- if $\{H_n : n \in \mathbb{N}\}$ and $H_n \cap H_m = \emptyset$ for $n < m$, then $v(\cup\{H_n : n \in \mathbb{N}\}) = \Sigma\{v(H_n) : n \in \mathbb{N}\}$;
- if $H \in \mathcal{B}(S)$ and $diam(H) < +\infty$, then $v(H) < +\infty$.

Let $p : S \to \mathbb{R}$ be a non-negative function with the properties:

- $\{x \in S : p(x) < \lambda\} \in \mathcal{B}(S)$ for any $\lambda \in \mathbb{R}$;

The Lebesgue's integral $m(U) = \int_U p(x)dv > 0$ for any non-empty open subset $U \subseteq S$.

We consider that $p(x)$ is the density of the matter of the Universe S at the point $x \in S$, $v(H)$ is the volume of the domain H and $m(H) = \int_H p(x)dv$ is the mass of the matter from the portion $H \in \mathcal{B}(S)$.

In the first, we consider that in the Universe S there exist distinct *"solid bodies"* which are in continuous movement. The state of an orbiting body at any given time is defined by the orbiting body's position and velocity with the respect to the more massive central body.

We consider the following restrictions:

R1. The limitation of the speed of the light relative to the quasi-metric d by some constant c.

R2. The limitation of the speed for some "type of the matter".

R3. For any body F with a mass m and a finite diameter it is determined the *"center of the mass"*.

R4. There exist a *"gravitational constant"* $G > 0$ and a constant $k > 1$ such that:

- the *orbital speed* for the *circular orbit* with the radius r is $v_0 = (M^2G/r(m+M))^{1/2}$, where M is the mass of the central body, m is the mass of the orbiting body and r is the distance $d(o_1, o_2)$ between the centers o_1 and o_2 of the mass of the central body and of the orbiting body;

- the escape velocity is $v_e = k \cdot v_0$.

We say that ω is a circular orbit with the radius r and the center $f \in S$ if there exists a curve $\gamma : I \to S$ such that $\gamma(0) = \gamma(1)$, $\omega = \{\gamma(t) : t \in I\}$ and $d(f, x) = r$ for any $x \in \omega$.

Suppose that Φ is a massive body with the center of the mass $f \in S$, the finite diameter $\delta = diam(\Phi)$ and the mass $M > c^2 \cdot d/G$. It is obvious that $d(f, x) \leq \delta$ for any $x \in \Phi$. Then for any body F with the mass $m < M$ we have $v_0 > c$ for $r < \delta$. In this case Φ is a "black body" and it absorbs all light that falls on it. If some mass falls on Φ, then in the result of the collapsing process the body Φ "radiate" the black-body radiation.

6. Methods of construction of bidistance spaces

Now we shall give the methods of construction of bidistance spaces with e-black bodies.

Let (S, d, d_v) be an m-dimensional differential manifold with the distances and $m \geq 2$.

Method 1. Let Γ be a non-empty set, $A = \{a_\mu : \mu \in \Gamma\}$ be a subset of S, $\{r_\mu : \mu \in \Gamma\}$ be a subset of positive reals from $\mathbb{R}$, $d(x, y) < +\infty$ for all $x, y \in B(a_\mu, r_\mu)$ and $\mu \in \Gamma$, $\bar{B}_v(a_\mu, r_\mu) = \{x \in S : d_v(a_\mu, x) \leq r_\mu\}$ be a compact subset of S for any $\mu \in \Gamma$, $\bar{B}_v(a_\mu, r_\mu) \setminus B_v(a_\mu, r_\mu) \neq \emptyset$ for any $\mu \in \Gamma$ and $d_v(a_\mu, a_\nu) > r_\mu + r_\nu$ for all distinct $\mu, \nu \in \Gamma$. Suppose that for any $\mu \in \Gamma$ there exists $b_\mu \in B_v(a_\mu, r_\mu)$ such that $d(b_\mu, x) < +\infty$ for any $x \in B_v(a_\mu, r_\mu)$.

Let $\rho(x, y)$ be the euclidean distance for $x, y \in E^m$ and $\mu(H)$ be the Lebesgue measure of the subset $H \subseteq E^m$.

We consider that the open set $H_\mu = B(a_\mu, r_\mu)$ is homeomorphic to the Euclidean space E^m and $h_\mu : H_\mu \to E^m$ is a homeomorphism of H_μ onto E^m. Let $H' = \cup\{H_\mu : \mu \in \Gamma\}$. Now we shall construct a new quasi-metric d' on S by the rules:

- if $\mu \in \Gamma$ and $x, y \in H_\mu$, then $d'(x, y) = d(x, y) + \rho(h_\mu(x), h_\mu(y))$;
- if $\mu \in \Gamma$, $x \in H_\mu$ and $y \notin H_\mu$, then $d'(x, y) = \infty$;
- if $x \in S \setminus H'$ and $y \in S$, then $d'(x, y) = d(x, y)$;

- if $p(x)$ is the initial density of the matter of the Univers S, then $p'(x) = p(x)$ for $x \in S \setminus H'$ and $p'(x) = max\{1, p(x)\}$ for $x \in H'$;
- if $v(H)$ is the initial volume in S, then $v'(L) = v(L \setminus H') + \Sigma\{\mu(h_\mu(L \cap H_\mu) : \mu \in \Gamma\}$.

6.1. Property. *(S, d', d_v) is a bidistance space. Moreover, if (S, d, d_v) is a bimetric space, then (S, d', d_v) is a bimetric space too.*

Proof. It is obvious that (S, d', d_v) satisfies the conditions D1 - D4. Fix now $a \in S$ and $\varepsilon > 0$. In this case there exists $\delta_1 > 0$ such that $\{y \in S : d_v(a, y) < \delta_1\} \subseteq \{y \in S : d(a, y) < 2^{-1}\varepsilon\}$. If $a \in S \setminus H'$, then $\delta = \delta_1$. Let $\mu \in \Gamma$ and $a \in H_\mu$. Since h_μ is a homeomorphism, there exists $\delta > 0$ such that $\delta < \delta_1$, $B_v(a, \delta) \subseteq H_\mu$ and $\rho(h_\mu(a), h_\mu(x)) < 2^{-1}\varepsilon$ for any $x \in B_v(a, \delta)$. By construction, we have $\{y \in S : d_v(a, y) < \delta\} \subseteq \{y \in S : d'(a, y) < \varepsilon\}$. The proof is complete.

6.2. Property. *For any $\mu \in \Gamma$ the domain H_μ is an e-black body of the bidistance space (S, d', d_v).*

Proof. It is obvious that the domain H_μ satisfies the conditions 1 - 4 from the Definition 5.6. Let $\{x_n \in H_\mu : n \in \mathbb{N}\}$ and $lim_{n\to\infty} x_n \in S \setminus H_\mu$. Then $lim_{n\to\infty} d'(x, x_n) = lim_{n\to\infty} d'(x_n, x) \geq lim_{n\to\infty} \rho(h_\mu(x), h_\mu(x_n)) = +\infty$ for any $x \in H_\mu$. The proof is complete.

6.3. Remark. If H is an e-black body of the bidistance space (S, d, d_v), then H is an e-black body of the bidistance space (S, d', d_v) too. The family $\{H_\mu : \mu \in \Gamma\}$ of e-black bodies of the bidistance space (S, d', d_v) is discrete.

6.4. Remark. If H is a black body of the bidistance space (S, d, d_v) and $H \subseteq S \setminus H'$ or $clH \subseteq H_\mu$ for some $\mu \in \Gamma$, then H is a black body of the bidistance space (S, d', d_v) too.

6.5. Remark. Let H be an e-black body of the bidistance space (S, d, d_v), the speed of the light relative to the quasi-metric d is bounded by the constant $c > 2$. Let $\gamma : I \to S$ be a curve with the origin $a \in H$, the end $b \in S \setminus H$ and $\gamma(t) \in H$ for any $t < 1$. If $v_e(t)$ is the speed of the light relative to the metric d_v in the point $\gamma(t)$, then $lim_{t\to 1} v_e(t) = 0$.

Method 2. Let Γ be a non-empty set, $\{\Gamma_n : n \in \mathbb{N}\}$ be a sequence of non-empty subsets of Γ, $A_n = \{a_{n\mu} : \mu \in \Gamma\}$ and $B_n = \{b_{n\mu} : \mu \in \Gamma\}$ be subsets of S, $\{r_{n\mu} : n \in \mathbb{N}, \mu \in \Gamma\}$ be a subset of positive reals from $\mathbb{R}$ with the next properties:
- $\Gamma = \Gamma_1$ and $\Gamma_{n+1} \subseteq \Gamma_n$ for any $n \in \mathbb{N}$;
- $\cap\{\Gamma_n : n \in \mathbb{N}\} = \emptyset$;

- if $1 \leq k \leq n$ and $\mu \in \Gamma_n$, then $b_{k\mu} \in B_v(a_{k\mu}, r_{k\mu}) \setminus B_v(a_{n\mu}, r_{n\mu})$;
- $d(b_{n\mu}, x) < +\infty$ for all $n \in \mathbb{N}$, $\mu \in \Gamma_n$ and $x \in B_v(a_{n\mu}, r_{n\mu})$;
- $\bar{B}_v(a_{1\mu}, r_{1\mu}) = \{x \in S : d_v(a_\mu, x) \leq r_{1\mu}\}$ is a compact subset of S;
- $d_v(a_{1\mu}, a_{1\nu}) > r_{1\mu} + r_{1\nu}$ for all distinct $\mu, \nu \in \Gamma$;
- for all $n \in \mathbb{N}$ and $\mu \in \Gamma_n$ there exists a homeomorphism $h_{n\mu}$ of $B_v(a_{n\mu}, r_{n\mu})$ onto the Euclidean space E^m.

We put $H_{n\mu} = B_v(a_{n\mu}, r_{n\mu})$, if $\mu \in \Gamma_n$, and $H_{n\mu} = \emptyset$, if $\mu \in \Gamma \setminus \Gamma_n$.

As in the Method 1 we construct:

- the quasi-metric $d_1 = d'$, the density $p_1(x) = p'(x)$ and the volume $v_1(L) = v'(L)$ for the bidistance space (S, d, d_v) and $n = 1$;
- the quasi-metric $d_2 = d'_1$, the density $p_2(x) = p'_1(x)$ and the volume $v_2(L) = v'_1(L)$ for the bidistance space (S, d_1, d_v) and $n = 2$;
- the quasi-metric $d_n = d'_{n-1}$, the density $p_n(x) = p'_{n-1}(x)$ and the volume $v_n(L) = v'_{n-1}(L)$ for the bidistance space (S, d_{n-1}, d_v) and $n \geq 3$.

Now we put $d''(x, y) = lim_{n\to\infty} d_n(x, y)$, $p''(x) = lim_{n\to\infty} p_n(x)$ and $v''(L) = lim_{n\to\infty} v_n(L)$ for all $x, y \in S$ and $L \subseteq S$.

From the Properties 6.1 and 6.2 it follows.

6.6. Property. *(S, d'', d_v) is a bidistance space.*

6.7. Property. *For all $n \in \mathbb{N}$ and $\mu \in \Gamma$ the domain $H_{n\mu}$ is an e-black body of the bidistance space (S, d'', d_v).*

6.8. Remark. If H is a black body of the bidistance space (S, d, d_v) and $H \subseteq S \setminus \cup\{H_{1\mu} : \mu \in \Gamma\}$ or $clH \subseteq H_{n\mu} \setminus H_{(n+1)\mu}$ for some $\mu \in \Gamma$ and $n \in \mathbb{N}$, then H is an black body of the bidistance space (S, d'', d_v) too.

6.9. Remark. The construction from the Method 2 can be applied in the conditions $\Gamma' = \cap\{\Gamma_n : n \in \mathbb{N}\} \neq \emptyset$ and $\cap\{B_v(a_{n\mu}, r_{n\mu}) : n \in \mathbb{N}\}$ for any $\mu \in \Gamma'$.

7. On discreteness and continuity

The following questions are still open:

1. Is the real space discrete or continuous?
2. What is the set (class) of all possible Universes?
3. Which mathematical structures are isomorphic to real Universe?
4. What is the dimension of the space-time?

The next example illustrate that the continuity and multi-dimensionality of the space-time is an effect of the perpetual motion of the matter in the time and the time is an one-dimensional ordered continuum isomorphic to the space of real numbers.

The next examples do to the concept of multiverse and parallel universes [13, 14]. Moreover, these examples are in close connection with the wave-particle duality and the Heisenberg uncertainty principle.

7.1. Example. Let $S' = S_1 \times T$ be the observable space-time, where T is the space of the time, and P be some non-empty space. Consider some mapping (as a rule continuous) $\pi : S' \to P$. We identify tne space S' with the subspace $S = \{(x, \pi(x)) : x \in S\}$ of the set $S \times P$. Let $S' \subseteq U \subseteq S \times P$ and on U we admit some topology $\mathcal{T}$ for which S is a subspace of $(U, \mathcal{T})$. In this case, $U \setminus S$ is the unobservable part of Universe U. It is possible that $dimS > dimU$. If $p \in P$ and $\pi(S) = \{p\}$, then $U \cap (S' \times \{q\}$ are parallel worlds.

7.2. Example. Let $\delta > 0$, $S(\delta) = \{(x, y, z, t) : x, y, z, t$ *are real numbers*, t is time, $d_v((x, y, z, t), (u, v, s, w)) = ((xu)^2 + (yv)^2 + (zs)^2 + (tw)^2)^{1/2}$, $d((x, y, z, t), (u, v, s, w)) = dv((x, y, t), (u, v, w))$, if $(x, y, u, t) = (u, v, s, w)$ or $t \neq w$, and $d((x, y, z, t), (u, v, s, w)) = \delta + d_v((x, y, z, t), (u, v, s, w))$, if $t = w$ and $(x, y, u, t) \neq (u, v, s, w)$.

By construction, $(S(\delta), d, d_v)$ is a bidistance space. On $S(\delta)$ we consider the topology generated by the distance d. This space is separable, Hausdorff, non-regular and non-metrizable.

We put $S(t) = \{(x, y, z, t)\}$ is the real space in the moment t. The space-time $S(\delta)$ is not discrete, and the real space $S(t)$ is discrete for any moment t, and the space $S(\delta)$ is continuous in the time. The influence between points of $(S(t)$ are in the space $S(\delta)$ in time. For any two distinct points $p, q \in S(t)$ and any $\epsilon > 0$ in $S(\delta)$ there exists a curve γ with the endpoints p, q such that $l(\gamma, d_v) = l(\gamma, d) < d_v(p, q) + \varepsilon$. We mention that $l(\gamma, d_v) = l(\gamma, d)$ for any curve γ of the space $S(\delta)$. Therefore the length of trajectories is not influenced by the distance d.

7.3. Example. Let (M, d') be a metric space and for any two points $x, y \in M$ and each $\epsilon > 0$ there exists a curve γ with the endpoints x, y such that $d'(x, y) \leq l(\gamma, d') < d'(x, y) + \epsilon$. One can consider that M is a smouts manifold with the Riemann's distance. Denote by $\mathbb{R}$ the space of reals as the space of the time. We put $S = M \times \mathbb{R}$ and $(d_v((x, t), (y, \tau)^2 = d(x, y)^2 + (t - \tau)^2$. Let $\delta > 0$, $S(\delta) = S$, $S(t) = S \times \{t\}$ is the real space-time in the moment t, $d((x, t), (y, w)) = dv((x, t), (y, w))$, if $(x, t) = (y, w)$ or $t \neq w$, and $d((x, t), (y, w)) = \delta + d_v((x, t), (y, w))$, if $t = w$ and $(x, t) \neq (y, w)$. For any two distinct points $p, q \in S(t)$ and any $\epsilon > 0$ in $S(\delta)$ there exists a curve γ with the endpoints p, q such that $l(\gamma, d_v) = l(\gamma, d) < d_v(p, q) + \varepsilon$. We mention that $l(\gamma, d_v) = l(\gamma, d)$ for any curve γ of the space $S(\delta)$. Therefore the length of trajectories is not influenced by the distance d.

7.4. Remark. If in Examples 7.2 and 7.3 δ is insignificant, then the difference $d_v - d$ is insignificant too. If the real space-time is of the that kind, then it is possible that there exists *a distance effect* which permits to determine the existence of δ and of the distance d.

7.5. Remark. One can consider the bidistance spaces (S, d, d_v), where d and d_v are distance functions with the properties D1 and D2. In this case in Example 7.2 the distance function d_v may to be the Minkowski metric.

Different unusual and original examples were constructed by many scholars (see [4, 6, 8, 9, 11, 16, 17, 7]). Still, who determines the real space structure? The answer would be that it is determined by the matter in all its forms. A space is not only a place where the matter is situated. The big quantity of the matter comprised even in a small spatial domains determines the geometrical properties of the space. The black bodies appear as a result of accumulation of enormous quantities of a substance in a certain bounded domain of the space. The "weight" of the substance situated in the close neighborhood of the point makes the space to bend and in this way there appear so-called the curvature of the space. The Lobacevskii-Bolyai's space curvature has a negative curvature. The Euclidean's space curvature is zero, and that of Riemann's elliptic space is positive. If a space is closed (compact), then its curvature is positive. The space with a negative curvature is open. The matter is in an eternal process of movement. Depending on the density variation $p(x, t)$ the distance $d(x, y)$ varies in the time too. Therefore the geometrical-topological structure of space is changing.

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