# A METHOD FOR THE SOLUTION OF HEAT TRANSFER PROBLEMS WITH A CHANGE OF PHASE

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**Abstract:** A method of broad applicability is presented which can be used to obtain solutions to problems involving a phase change. The solution in one of the phases is specified as a known single-phase solution; an inverse analysis then determines the solution for the other phase. Two problems are studied: The first yields the similarity solution for the planar geometry and the second gives the exact solution to a more general problem. Convergence is shown and error bounds are given. The method can accommodate convection, heat generation, variable properties, nonplanar, and multidimensional systems.

**Keywords:** heat-transfer

## 1. INTRODUCTION

Transient heat conduction problems with freezing or solidification arise in many important technical applications. Although the equations governing such problems are often easily derived, the solution of these equations has proved to be difficult even for simple problems. These difficulties arise primarily because of the unknown location of the solid-liquid interface that renders the governing equations nonlinear.

The method presented here proceeds by choosing a single-phase problem with a known exact solution $T_s(x, t)$. This solution yields a constant temperature $T_F$ along a trajectory $x_F(t)$ i.e., $T_s(x_F, t) = T_F$. We now utilize this solution to construct a solution to an "equivalent" phase-change problem. Specifically, the phase-change problem would be defined by a known fusion temperature $T_F$ that occurs on the trajectory $x_F(t)$ as given above and a known temperature profile in one of the phases; the single-phase solution $T_s(x, t)$ is assumed to equal the solution for the phase-change problem $T(x, t)$ in the region $x \geq x_F(t)$. This is consistent with the required condition

$$T(x_F(t), t) = T_s(x_F(t), t) = T_F$$

The temperature distribution in the region $0 \leq x \leq x_F(t)$ for the phase-change problem is unknown and is obtained by an inverse conduction analysis. When this solution is obtained, the phase-change problem is solved.
The first problem that is solved illustrates the method and yields Neumann's solution. In the second section, the method is used to solve a more general problem. Convergence criteria and error bounds of this solution are also presented.

## 2. ANALYSIS

Consider a solid, $0 \leq x \leq H$, that is initially uniform at the temperature $T_i$. The temperature of the wall, at $x = 0$, is specified and exceeds the fusion temperature $T_f$. Therefore, melting occurs at an unknown position given by

$x = x_F(t)$ (see Fig. 1), with a liquid region $0 \le x \le x_F(t)$, and a solid region $x_F(t) \le x \le H$. The problem formulation is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad \text{for } 0 \le x \le x_F(t), \text{liquid} \tag{1a}$$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad \text{for } x_F(t) \le x \le H, \text{solid} \tag{1b}$$

$$\mathrm{T}(0,\mathrm{t}) = \mathrm{T_w}(t) \tag{1c}$$

$$\mathrm{T}(\mathrm{H},\mathrm{t}) = \mathrm{T_H}(t) \tag{1d}$$

$$\mathrm{T}(\mathrm{x_F},\mathrm{t}) = \mathrm{T_F} \tag{1e}$$

$$-k\left.\frac{\partial T}{\partial x}\right|_L + k\left.\frac{\partial T}{\partial x}\right|_s = \frac{\rho L \cdot dx_F}{dt} \quad \text{at} \quad \mathrm{x} = \mathrm{x_F}(t) \tag{1f}$$

$$T(x,0) = T_i \tag{1g}$$

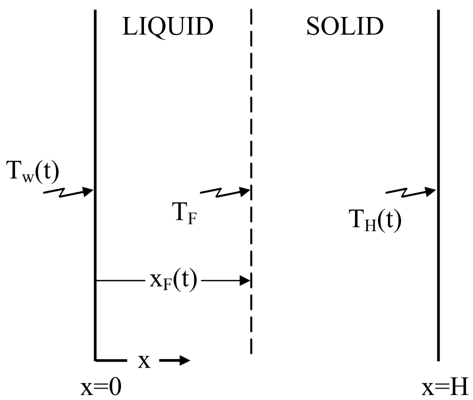


Fig. 1 Phase-change problem

We write the solution for the solid region as

$$\mathrm{T(x, t) = T_s(x, t)} \qquad \mathrm{x_F(t) \le x \le H} \tag{2}$$

Here $T_s$ may be viewed as the known solution to an appropriate single-phase problem, one that obeys the same condition at $x = H$; i.e., $T(H, t) = T_H(t)$ and $T(x, 0) = T_i$. The trajectory $x_F(t)$ is determined from $T_s(x_F(t), t) = T_F$. The task is to determine the temperature distribution in the liquid region, $0 \le x \le x_f(t)$.

The solution for the liquid region that will be compatible with a temperature distribution $T_s(x, t)$ in the solid region is obtained from a solution of the problem comprised of equations (1a), (1e), and (1f). This is an inverse conduction problem and the solution is

$$T(x,t)=T_s(x,t)+\frac{L}{c}\sum_{n=1}^{\infty}G_n\left(1-\frac{x}{x_F}\right)^n,\quad 0\le x\le x_F(t) \tag{3}$$

where the $G_n$ are functions only of $X_F$ (t).
This result is valid for all constant property phase-change problems for the one-dimensional Cartesian geometry. Similar results can be derived for other geometries.

To evaluate the $G_n$ in equation (3) the trajectory $x_F(t)$ is needed. As stated above, this is obtained from the solution for the solid region at $x = x_F(t)$; i.e., $T_s(x_F,t) = T_F$ yields the trajectory $x_F$ (t). We now illustrate the method.

**2.1. Single-Phase Problem: $T_{ws}$ = Const, Semi-Infinite Region**

Consider a semi-infinite solid, $x \geq 0$, that is initially uniform at the temperature $T_i$. The temperature of the wall, at $x = 0$, is instantaneously raised to a constant value $T_{ws}$, the single-phase wall temperature. The single-phase solution to this problem is given by [1]

$$T_s(x,t)=T_{ws}-(T_{ws}-T_i)\operatorname{erf}\eta \tag{4}$$

where

$$\eta = x/2\sqrt{\alpha t} \tag{5}$$

**2.2. Application of the Single-Phase Solution to a Phase-Change Problem**

The location of the phase-change inter-face is obtained from

$$T_s(x_F,t)=T_F=T_{ws}-(T_{ws}-T_i)\operatorname{erf}\eta \tag{6}$$

where

$$x_F/2\sqrt{\alpha t}=\lambda \tag{7}$$

Since $T_F$, $T_{ws}$ and $T_i$ are known constants, $\lambda$ is known and must also be a constant.
The relation above can be rearranged. Equation (6) yields

$$T_{ws}=\frac{T_F-T_i\operatorname{erf}\lambda}{\operatorname{erfc}\lambda}=T_i+\frac{T_F-T_i}{\operatorname{erfc}\lambda} \tag{8}$$

which, when substituted into equation (4) yields

$$T(x,t)=T_i+(T_F-T_i)\frac{\operatorname{erfc}\eta}{\operatorname{erfc}\lambda}\ \text{ for }\ \mathrm{x}\ge \mathrm{x}_{\mathrm{F}}(t) \tag{9}$$

The subscript s on $T_s(x,t)$ has been dropped because the relation is also applicable to the phase-change problem, i..e, equation (9) does not contain $T_{ws}$.
To obtain the solution in the region $0 \leq x \leq x_F(t)$, equation (3) is used. Substituting the trajectory $x_F(t)$ from equation (7) into equation (3) and summing the series gives

$$T(x,t)=T_s(x,t)+\frac{L}{c}\sqrt{\pi}\lambda e^{\lambda^2}\,\mathrm{erf}\lambda\left(1-\frac{\mathrm{erf}\eta}{\mathrm{erf}\lambda}\right),\ \mathrm{x}\le \mathrm{x}_\mathrm{F}(t) \tag{10}$$

where $T_s$ is the single-phase solution given by equation (4). At the wall

$$T(0,t)=T_w=T_s(0,t)+\frac{L}{c}\sqrt{\pi}\lambda e^{\lambda^2}\,\mathrm{erf}\lambda \tag{11}$$

Recall that $T_s(0,t) = T_{ws}$ is a constant. Thus the solution to the phase-change problem that has been solved is for the constant wall temperature boundary condition, $T_w$ = constant. Equation (10) may be written in a more familiar form by using equation (4) for $T_s(x,t)$ and then using equation (6). The result is

$$T(x,t)=T_w+(T_w-T_F)\frac{\mathrm{erf}\eta}{\mathrm{erf}\lambda}\ \text{ for }\ \mathrm{x}\le \mathrm{x}_\mathrm{F}(t) \tag{12}$$

Now, using equations (9) and (12) and the boundary condition equation (1*f*) yields the following relation for determining λ

$$\frac{L}{c}\sqrt{\pi}\lambda e^{\lambda^2}=\frac{T_w-T_r}{\mathrm{erf}\lambda}+\frac{T_i-T_F}{\mathrm{erfc}\lambda} \tag{13}$$

Equations (9), (12), and (13) constitute the solution to this phase-change problem. The solution to this problem was given by Neumann [1], and the present results are identical to his solution.

### 2.3. More General Problem: $x_F = bt^m$

We now consider a more general phase-change problem governed by equations (1a-g), with an interface trajectory: $x_F = bt^m$ with $T(x,0) = T_F$, $T(H,t) = T_F$. The solution is given by

$$T(x,t)=T_F+(L/c)F(x,t)\ \text{ for }\ x\le x_F(t) \tag{14}$$

with

$$F(x^*,\tau)=\sum_{n=1}^{\infty}(1-x^*/x^*{}_F)^n G_n=\sum_{n=1}^{\infty}(-x^*)^n H_n \tag{15}$$

where $x^*=x/l,\quad \tau=\alpha t/l^2,\quad x_F^*=x_F/l,\quad l=[ba^{-m}]^{1/(1-2m)}$

For n even:

$$H_n=(1/n!)\tau^{-n/2}\sum_{i=1}^{\infty}\frac{\tau^{i(2m-1)}}{(2i)!}\prod_{j=0}^{(i-1+n/2)}[2im-j] \tag{16a}$$

for n odd:

$$H_n=(1/n!)\tau^{-n/2}\sum_{i=1}^{\infty}\frac{\tau^{(i-1/2)(2m-1)}}{(2i-1)!}\prod_{j=0}^{(i-3/2+n/2)}[(2i-1)m-j] \tag{16b}$$

Direct substitution into the governing equations (1a-g) confirms the above solution. Note that m = 1/2 gives Neumann's solution and m = 1 gives Stefan's solution [2, 3].

**2.4. Convergence: m > 1/2**

The convergence and error bounds for this solution are now considered for m > 1/2. We first determine the convergence of $H_n$ for n even. Define

$$H_n = \sum_{i=1}^{\infty} C_{i,n} \quad \text{and} \quad R_{i,n} = \frac{C_{i,n}}{C_{i-1,n}} \tag{17}$$

For the series to converge, $R_{i,n} < 1$ as $I \to \infty$ for fixed values of n. $C_{i,n}$ may be written as

$$C_{i,n} = \tau^{i(2m-1)} \frac{\Gamma(2im+1)}{\Gamma(2i+1)\Gamma(2mi-i-[n/2]+1)} \frac{1}{n!} \tau^{-n/2} \tag{18}$$

where $\Gamma$ is the gamma function. Using $\lim_{p\to\infty} \Gamma(x+p+1) = \Gamma(p+1)p^x$ one obtains for large *I*

$$R_{i,n} = \frac{(2m)^{2m} \tau^{2m-1}}{4i(2m-1)^{2m-1}} \tag{19}$$

Thus, as $i \to \infty$, $R_{i,n} \to 0$ showing convergence of equation (16a). Analysis of $H_n$, n odd, also yields rapid convergence as $i \to \infty$. To determine the convergence of the series given in equation (15) the quantity $S_{i,n}$ is defined according to

$$S_{i,n} = \frac{C_{i,n}}{C_{i,n-2}} \tag{20}$$

For convergence, $S_{i,n} < 1$ as $n \to \infty$ for fixed values of i. Equation (18) gives

$$S_{i,n} = \frac{\tau^{-1}}{n(n-1)}[2im - i - (n/2) + 1] \tag{21}$$

Thus, $\lim_{n\to\infty} S_{i,n}$ for fixed i. Identical results are obtained for odd values of n.

**2.5. Error Bound for Wall Temperature, m > 1/2**

The error bound for the wall temperature is related directly to the error bound on $H_o$. From our definitions we have that

$$H_o = \sum_{i=1}^{\infty} C_{i,0} \tag{22}$$

The estimate for $H_o$ is given by

$$\tilde{H}_o = \sum_{i=1}^{p} C_{i,0} \tag{23}$$

We define $R_{p,0} = C_{p,0}/C_{p-1,0}$) and assume that $R_{p,0}$ decreases monotonically as p increases. The error is then given

$$H_o - \widetilde{H}_o = \sum_{i=p+1}^{\infty} C_{i,0} \leq \sum_{i=p+1}^{\infty} C_{p,0}(R_{p,0})^{i-p} \tag{24}$$

Also note that for m > 1 /2 all $C_{i,0} \geq 0$. Thus

$$0 \leq H_o - \widetilde{H}_o \leq C_{p,0} \frac{R_{p,0}}{1 - R_{p,0}} \tag{25}$$

using five terms in the series $\widetilde{H}_o$ for p = 0 gives upper and lower bounds for $H_o$ for several values of τ and m. Wall temperature converge ratio, $R_{i,0}$ for m = 1, 2, 4 and 10 show that $R_{p,0}$ decreases monotonically as p increases, thus showing the foregoing analysis to be valid over a broad range of values of rand m.

**2.6. Solution for m = 1/3**

For the case m = 1/3, 0 < τ < 0.8, Chow and Woo [4] give approximate results for the dimensionless wall temperature and heat flux. Their results correspond to the quantities $H_0(\tau)$ and $H_1(\tau)$. The maximum difference between the results of [4] and the present results is 0.1 percent.

**4. CONCLUSIONS**

A method has been presented witch solves phase-change problems. The solution in one of the phases is specified as a known single-phase solution. An inverse analysis then determines a power-series solution for the other phase. The method can accommodate planar, cylindrical, and spherical geometries and convection, heat generation, and differing liquid and solid properties. With more difficulty, the method can include thermal varying properties and multidimensional systems, and can be implemented numerically.

**References**

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