"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 21 (2011), No. 1, 167-174

ON THE STATISTICAL APPROXIMATION PROPERTIES OF Q-SCHURER OPERATORS

CARMEN VIOLETA MURARU AND VALER NIMINEȚ

Abstract. The results from q-Calculus theory occurs in many applications from physics, quantum theory, number theory, etc. The aim of this paper is to study some convergence properties of q-Schurer operators, in terms of statistical approximation.

1. PRELIMINARIES

We mention in the following some important achievements in this field of q-Calculus.

Lupaş introduced in 1987 a q-type of the Bernstein operators and in 1997 another generalization of the classical Bernstein polynomials based on q-integers were introduced by Phillips [9]. He has obtained the rate of convergence and Voronovskaja type asymptotic formula for the new Bernstein operators based on q−integers. After this, some authors studied new classes of q- generalized operators and gave approximations properties of them. In [3] O. Doğru and A. Aral constructed q- type generalization of Bleimann, Butzer and Hahn operators.

Keywords: linear and positive operators, q-Calculus, statistical convergence, Bohman-Korovkin type theorem, q-Schurer
(2010) Mathematics Subject Classification: 41A36

T. Trif investigated Meyer-König and Zeller operators based on q-integers ([11]). O.Doğru and O. Duman introduced also a new generalization of Meyer-Kőnig and Zeller operators and studied some statistical approximation properties in [6]. A generalization of Balazs-Szabados operators based on q-integers was introduced and a Stancu type generalization of these operators is also constructed in a paper of O. Doğru.

We remind also that uniform approximating polynomial operators in two and several variables were constructed by Stancu in 1972 ([12]).

In [4] Barbosu introduced a Stancu type generalization of two dimensional Bernstein operators based on q-integers and in a joint paper , O. Doğru and Gupta constructed a q-type generalization of Meyer-König and Zeller operators in bivariate case. ([6]) A new q-generalization of Meyer-König and Zeller type operators was constructed by Doğru and Muraru in order to improve the rate of convergence [7]. Recently were studied generalization of Durmeyer and Kantorovich operators based on q-integers by Gupta and Radu [9].

We remind that q- Bernstein polynomial has the following form (Philips 1996 [13]).

$$B_n(f;q;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right)\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1}(1-q^s x) \tag{1.1}$$

where $x \in [0,1]$, $f \in C([0,1])$, $0 < q < 1$ and

$$[k]_q = \begin{cases} (1-q^k)/(1-q), & q \neq 1 \\ k & , \ q = 1 \end{cases}$$

$$[k]_q! = \begin{cases} [k][k-1]....[1], & k = 1,2,... \\ 1 & , \ k = 0 \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \qquad (n \geq k \geq 0)$$

The q-analogue of $(x-a)^n$ is the polynomial

$$(x-a)_q^n = \begin{cases} 1 & \text{if } n=0 \\ (x-a)(x-qa)....(x-q^{n-1}a) & \text{if } n \geq 1 \end{cases}$$

As usual, we note with $C([a,b])$, the space of all real valued continuous functions defined on [a,b]. The space is endowed with usual norm $\|\cdot\|$ given by $\|f\| = \sup_{x\in[a,b]} |f(x)|$.

The sequence $p_{n,k}(x,q) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1-q^s x)$, $k = 0,1,..,n$

for $0<x<1$ and $0<q<1$ forms a normalized totally positive basis, called q-Bernstein basis.

Let $p \in N$ be fixed. In 1962 Schurer introduced and studied the Schurer operators $\tilde{B}_{m,p} : C([0,p+1]) \to C([0,1])$ defined for any $m \in N$ and any function $f \in C([0,p+1])$ as follows

$$\tilde{B}_{m,p}(f;x) = \sum_{k=0}^{m+p} \binom{m+p}{k} x^k (1-x)^{m+p-k} f\left(\frac{k}{m}\right)$$

One observes that for $p=0$, $B_{m,0}$ are the operators of Bernstein B_m.

2. APPROXIMATION PROPERTIES OF Q-SCHURER OPERATORS

In a recent paper, ([12]) we introduced the sequence of q-Schurer linear operators and gave some approximation properties of them, including an estimation of rate of convergence in the terms of first modulus of continuity .

For any $m \in N$, $f \in C([0,p+1])$, p be fixed we construct the class of generalized q-Bernstein -Schurer operators and any $x \in [0,1]$, as follows:

(2.1) $$\tilde{B}_{m,p}(f;q;x)=\sum_{k=0}^{m+p}\begin{bmatrix} m+p \\ k \end{bmatrix}_q x^k \prod_{s=0}^{m+p-k-1}(1-q^s x) f\left(\frac{[k]_q}{[m]_q}\right)$$

Lemma 2.1 The operator defined by (2.1) is linear.

Lemma 2.2 ([12]) The polynomials defined above satisfy the following properties:

1. $\tilde{B}_{m,p}(e_0;q;x)=1$
2. $\tilde{B}_{m,p}(e_1;q;x)=\dfrac{x[m+p]_q}{[m]_q}$
3. $\tilde{B}_{m,p}(e_2;q;x)=\dfrac{[m+p]_q}{[m]_q^{\,2}}\left([m+p]_q x^2+x(1-x)\right)$

where we denote by $e_j(x)=x^j$, $j=0,1,2$, the test functions.

Theorem 2.3([12])

Let $q=q_m$ satisfy $0<q_m<1$, $\lim\limits_{m\to\infty} q_m=1$ and $\lim\limits_{m\to\infty} q_m^m=a, a<1$ .Then for any $f\in C([0,p+1])$ the next result holds

$$\lim_{m\to\infty}\tilde{B}_{m,p}(f;q_m)=f \quad \text{uniformly on } [0,1]$$

For estimation of convergence we obtain the next result in terms of first modulus of continuity .

Theorem 2.4 ([12])

If $f\in C([0,1+p])$ then we obtain

$$\left|\tilde{B}_{m,p}(f;q;x)-f(x)\right|\le 2\omega_f(\delta_m),$$

where $\delta_m=\dfrac{1}{\sqrt{[m]_q}}\left(p+\dfrac{1}{2\sqrt{1-q^m}}\right)$ and $\lim\limits_{n\to\infty} q_n=1$, $0<q<1$.

3. KOROVKIN TYPE STATISTICAL APPROXIMATION PROPERTIES

The concept of statistical convergence was introduced by Fast in [8] and Steinhauss [15] and recently has became an important area in approximation theory .

A sequence $x=(x_k)$ is said to be statistically convergent to a number L if for every $\varepsilon>0$

$$\delta\{k\in N:\left|x_k-L\right|\geq\varepsilon\}=0,$$

where $\delta(K)$ is the natural density of the set $K\subseteq N$. The density of subset K is defined by

$$\delta(K)=\lim_{n\to\infty}\frac{1}{n}\left|\{k\leq n,k\in K\}\right|$$

whenever the limit exists.

We denote this limit by $st-\lim_{n\to\infty}x_n=L.$

Clearly finite subsets have natural density 0.

We denote by $C_M[a.b]$ the space of all functions f which are continuous in [a,b] and bounded on the all positive axis. The next theorem of Bohman-Korovkin type due to Gadjev and Orhan contains the criterion to prove statistical convergence for a sequence of linear and positive operators.

Theorem A ([10]) If the sequence of positive linear operators

$$A_n:C_M[a,b]\to C[a,b]$$

satisfies the conditions

$$st-\lim_{n\to\infty}\left\|A_n(e_i)-e_i\right\|_{C[a,b]}=0 \quad \text{with } e_i(t)=t^i, i=0,1,2,$$

then for any function $f\in C_M[a,b]$ we have

$$st-\lim_{n\to\infty}\left\|A_n(f)-f\right\|_{C[a,b]}=0$$

Taking into account the result from Lemma 2.2 we are ready to obtain the following first main result for the operators $\tilde{B}_{m,p}$.

Theorem B

Let (q_n) be a sequence that satisfies

$$st-\lim_{n} q_n^p=1 \text{ and p a fixed natural number.}$$

Then for all $f\in C_M[0,1]$ we have

$$st-\lim_{n}\left\|B_{m,p}(f,q_n,\cdot)-f\right\|_{C[0,1]}=0$$

Proof

It is necessary to prove that

$$st-\lim_{n}\left\|B_{m.p}(e_i,q_m,\cdot)-e_i\right\|_{C[a,b]}=0 \text{ for i=0,1,2.}$$

and the proof follows from Theorem A.

From the Lemma 2.2. is clear that

$$st-\lim_{n}\left\|B_{m.p}(e_0,q_m,\cdot)-e_0\right\|_{C[0,1]}=0$$

For the second relation we have

$$\left\|B_{m,p}(e_1,q_m,x)-e_1\right\|=\left|\frac{[m+p]}{[m]}-1\right|=\left|q_m^p+\frac{[p]}{[m]}-1\right|=\left|q_m^p-1\right|+\frac{[p]}{[m]}$$

$q_m\in(0,1), q_m^p\in(0,1)\quad st-\lim_{n} q_m^p=1.$

We consider

$$A=\left\{m\in N:\left\|B_{m,p}(e_1,q_m,\cdot)-e_1\right\|\geq\varepsilon\right\}$$

$$A_1=\left\{m\in N:\left|q_m^p-1\right|\geq\frac{\varepsilon}{2}\right\}$$

$$A_2 = \left\{ m \in N : \frac{[p]}{[m]} = 1 - q_m^p \geq \frac{\varepsilon}{2} \right\}$$

$$A \subseteq A_1 \cup A_2$$

$$\left\| B_{m,p}(e_1, q_m, \cdot) - e_1 \right\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \;\Rightarrow \delta(A) \leq \delta(A_1) + \delta(A_2) = 0$$

So

$$st - \lim_n \left\| B_{m,p}(e_1, q_m, \cdot) - e_1 \right\| = 0$$

$$\left\| B_{m,p}(e_2, q_m, \cdot) - e_2 \right\|_{C[0,1]} \leq \left| \frac{[m+p]^2}{[m]^2} - \frac{[m+p]}{[m]^2} - 1 \right| + \frac{[m+p]}{[m]^2} \leq$$

$$\leq \left| \frac{[m+p]^2}{[m]^2} - 1 \right| + \frac{[m+p]}{[m]^2} + \frac{[m+p]}{[m]^2} = \left| \frac{[m+p]^2}{[m]^2} - 1 \right| + \frac{2[m+p]}{[m]^2}$$

We use in the above inequality that $\left| |x| - |y| \right| \leq |x + y| \leq |x| + |y|, \forall x, y \in R$.

We set

$$A' = \left\{ m \in N : \left\| B_{m,p}(e_2, q_m, \cdot) - e_2 \right\| \geq \varepsilon \right\}$$

$$A'_1 = \left\{ m \in N : | \frac{[m+p]^2}{[m]^2} - 1 \geq \frac{\varepsilon}{2} \right\}$$

$$A'_2 = \left\{ m \in N : \left| \frac{2[m+p]}{[m]^2} \geq \frac{\varepsilon}{2} \right. \right\}$$

$$A' \subseteq A'_1 + A'_2 \Rightarrow \delta(A') \leq \delta(A'_1) + \delta(A'_2) = 0$$

$$st - \lim_n \left\| B_{m,p}(e_2, q_m, \cdot) - e_2 \right\|_{C[0,1]} = 0$$

REFERENCES

[1] Agratini O., **On statistical approximation in spaces of continuous functions**, Positivity, 13(2009), 735-743;

[2] Altomare F., Campiti M., **Korovkin-type approximation theory and applications** , Berlin, Walter de Gruyter, 1994
[3] A. Aral, O. Dogru, **Bleimann, Butzer, and Hahn Operators Based on the q-Integers**, Journal of Inequalities and Applications, vol. 2007,
[4] Barbosu D., **Some generalized bivariate Bernstein operators**, Math. Notes (Miskolc) **1**,3-10 (2000)
[5] Doğru O., Duman O., **Statistical approximation of Meyer-Kőnig and Zeller operators based on q-integers,** Publ. Math. Debrecen 68, 199-214 (2006)
[6] Doğru O., Gupta V., **Korovkin-type approximation properties of bivariate q-Meyer-Kőnig and Zeller operators** , Calcolo 43, 51—63 (2006)
[7] Doğru O., Muraru C.V., **Statistical approximations by a Stancu type bivariate generalization of Meyer-Konig and Zeller type operators** , Mathematical and Computer Modelling , vol. 48, issues 5-6, September 2008, 961-968
[8] Fast, H., Sur la convergence statistique, Colloq. Math., 2 (1951), 241-244.
[9] Goodman T.N.T., Oruç H., Phillips G.M., **Convexity and generalized Bernstein polynomials**, Proc. Edinburgh Math. Soc. **42(2)**, 179-190 (1999)
[10] Gadjiev, A.D., Orhan, C., **Some approximation theorems via statistical convergence**, Rocky Mountain J. Math., 32 (2002), 129-138.
[11] Gupta V., Radu C., **Statistical approximation properties of q-Baskakov-Kantorovich operators**, Central European Journal of Mathematics, 7(4), 2009, 809-818
[12] Muraru C.V., **Note on q-Bernstein-Schurer operators**, Studia UBB Mathematica, Vol. LVI (2), 2011, pp.1-11
[13] Philips, G.M., **On generalized Bernstein polynomials**, in: D.F. Griffits, G.A. Watson (Eds.), 263-269 (1996)
[14] Trif T., **Meyer-Kőnig and Zeller operators based on q-integers**, Rev. Anal. Numer. Theor. Approx., **29**, 221-229 (2000)
[15] Stancu D.D., **A new class of uniform approximation polynomial operators in two and several variables** , In Alexits G., Stechkin S.B., (eds), Proceedings of the conference on constructive theoru of functions , Budapest , Akadémiai Kiadó, 443-455 (1972)
[16] Steinhaus, H., **Sur la convergence ordinarie et la convergence asymptotique**, Colloq. Math., 2 (1951), 73-74.

"Vasile Alecsandri" University of Bacău
Department of Mathematics and Informatics
Faculty of Sciences,
Email: carmen_7419@yahoo.com