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GEOMETRIC ASPECTS OF THE LAGRANGIAN MECHANICAL SYSTEMS

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Abstract. In this paper, we present some aspects of the Lagrangian geometric model of the rheonomic mechanical systems and we apply them to a concrete rheonomic mechanical system: the motion of a material particle along a moving surface.

1. INTRODUCTION

We study the dynamical system of the rheonomic Lagrangian mechanical systems, whose evolution curves are given, on the phase space $TM \times R$, by Lagrange equations. Then one can associate to the considered mechanical system a vector field S on the phase space, which is named the canonical semispray. The integral curves of the canonical semispray are the evolution curves of the rheonomic mechanical system.

The article is organized as follows. In the next section we briefly recall some basic notions on rheonomic Lagrange geometry. We employ a method similar to that used in the geometrization of sclerhonomic Lagrange mechanical systems, [8], and we obtain a non-linear connection for the rheonomic Lagrangian system with external forces.

In the last section, we apply these results to a concrete rheonomic mechanical system: the constrained motion of a material particle on a time varying surface.

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2. Rheonomic Lagrangian Mechanical Systems

Let M be a smooth C^∞ manifold of finite dimension n, and (TM, π, M) be its tangent bundle.

The manifold $E = TM \times R$ is a $(2n+1)$dimensional, real manifold and the local coordinates in a chart will be denoted by (x^i, y^i, t).

The natural basis of tangent space T_uE at the point $u \in U \times (a, b)$ is given by $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t}\right)$.

$\chi(E)$ is the $C^\infty(E)$-module of (smooth) vector fields defined on E.

On the manifold E a vertical distribution V is introduced, generated by $n+1$ local vector fields $\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \ldots, \frac{\partial}{\partial y^n}, \frac{\partial}{\partial t}\right)$,

$$V : u \in E \to V_u \subset T_uE \tag{1}$$

as well as the tangent structure, [1],

$$J : \chi(E) \to \chi(E),$$

given by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}; J\left(\frac{\partial}{\partial y^i}\right) = 0; J\left(\frac{\partial}{\partial t}\right) = 0, i, j, k = 1, 2, ..., n. \tag{2}$$

The tangent structure J is globally defined on E and it is an integrable structure.

A semispray on E, [1], is a vector field $S \in \chi(E)$ which has the property $JS = C$, where $C = y^i \frac{\partial}{\partial y^i}$ is the Liouville vector field.

Locally, a semispray S has the form

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y, t)\frac{\partial}{\partial y^i} - G^0(x, y, t)\frac{\partial}{\partial t}, \tag{3}$$

where $G^i(x, y, t)$ and $G^0(x, y, t)$ are the coefficients of S.

A *non-linear connection* on E is a smooth distribution:

$$N : u \in E \to N_u \subset T_uE, \tag{4}$$

which is supplementary to the vertical distribution V:

$$T_uE = N_u \oplus V_u, \quad \forall u = (x, y, t) \in E. \tag{5}$$

We introduce the Greek indices $\alpha, \beta, \ldots$ ranging on the set $\{0, 1, 2, ..., n\}$ and $t = y^0$

The local basis adapted to the decomposition (5) is $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^\alpha}\right)$, where

(6) $$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^\alpha(x,y,t)\frac{\partial}{\partial y^\alpha}$$

and $(N_i^\alpha(x,y,t))$ are the local coefficients of the non-linear connection N on E.

The dual basis is $(\delta x^i, \delta y^\alpha)$, with

(7) $$\delta x^i = dx^i; \delta y^i = dy^i + N_j^i dx^j; \quad \delta y^0 = \delta t = dt + N_i^0 dx^i.$$

A differentiable rheonomic Lagrangian is a scalar function

$$L : TM \times R \to R$$

of the class C^∞ on the manifold $\tilde{E} = E \setminus \{(x,0,0), x \in M\}$ and continuous for all the points $(x,0,0) \in TM \times R$.

The $d-$tensor field with the components $g_{ij}(x,y,t) = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$ is called *the fundamental* or *the metric tensor field* of the rheonomic Lagrangian $L(x,y,t)$.It is of type $(0,2)$ and symmetric.

The rheonomic Lagrangian $L(x,y,t)$ is called *regular* if $rank(g_{ij}) = n$, on $\tilde{E}$.

A rheonomic Lagrange space is a pair $RL^n = (M, L(x,y,t))$, where L is a regular rheonomic Lagrangian and its fundamental tensor g_{ij} has constant signature on $\tilde{E}$.

For a rheonomic Lagrange space $RL^n = (M,L)$ exists a non-linear connection N defined on $\tilde{E}$, whose coefficients (N_j^α) are completely determined by L, called *the canonical non-linear connection*, [8]. Its coefficients are as follows

(8) $$N_j^i = \frac{1}{4}\frac{\partial}{\partial y^j}\left[g^{ih}\left(\frac{\partial^2 L}{\partial y^h \partial x^k}y^k - \frac{\partial L}{\partial x^h}\right)\right]; N_j^0 = \frac{1}{2}\frac{\partial^2 L}{\partial t \partial y^j}.$$

A rheonomic Lagrangian mechanical system is the triplet

(9) $$\Sigma = (M, L(x,y,t), F(x,y,t)),$$

where $RL^n = (M, L(x,y,t))$ is a rheonomic Lagrange space and $F(x,y,t)$ is a vertical vector field:

(10) $$F(x,y,t) = F^i(x,y,t)\frac{\partial}{\partial y^i}.$$

The tensor $g_{ij}(x,y,t)$ of the rheonomic Lagrange space RL^n is the fundamental tensor of the mechanical system Σ.

Using the variational problem of the integral action of $L(x, y, t)$, we introduce the evolution equations of Σ by:

The evolution equations of the rheonomic Lagrangian mechanical system Σ are the following Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = F_i(x, y, t); \quad y^i = \frac{dx^i}{dt}, \tag{11}$$

where $F_i(x, y, t) = g_{ij}(x, y, t)F^j(x, y, t)$.

The Lagrange equations (11) are equivalent to the equations

$$\frac{d^2x^i}{dt^2} + 2\Gamma^i(x, y, t) = \frac{1}{2}F^i(x, y, t), \tag{12}$$

where

$$2\Gamma^i = 2G^i(x, y, t) + N_0^i(x, y, t), 2G^i = \frac{1}{2}g^{ih}\left(\frac{\partial^2 L}{\partial y^h \partial x^s}y^s - \frac{\partial L}{\partial x^h}\right), \tag{13}$$

and $N_0^i(x, y, t) = \frac{1}{2}g^{ih}\frac{\partial^2 L}{\partial t \partial y^h}$.

The equations (12) are called *the evolution equations* of the mechanical system Σ. The solutions of these equations are called *evolution curves* of the mechanical system Σ.

The vector field $\breve{S}$ given by:

$$\breve{S} = y^i\frac{\partial}{\partial x^i} - 2\breve{\Gamma}^i(x, y, t)\frac{\partial}{\partial y^i} + a\frac{\partial}{\partial t} \tag{14}$$

is a semispray on $T\tilde{M} \times R$, depending only on the rheonomic Lagrangian mechanical system Σ.

The integral curves of $\breve{S}$ are the evolution curves of Σ given by (12).

We call the semispray $\breve{S}$ *the canonical evolution semispray* of the mechanical system Σ.

We can say:

The geometry of the rheonomic Lagrangian mechanical system Σ is the geometry of the pair $(RL^n, \breve{S})$, where RL^n is a rheonomic Lagrange space and $\breve{S}$ is the evolution semispray.

The canonical non-linear connection $\breve{N}$ of the mechanical system Σ has the coefficients $(\breve{N}_j^i, \breve{N}_j^0)$:

(15)

$$\breve{N}_j^i = \frac{1}{4}\frac{\partial}{\partial y^j}\left[g^{ih}\left(\frac{\partial^2 L}{\partial y^h \partial x^k}y^k - \frac{\partial L}{\partial x^h}\right)\right] - \frac{1}{4}\frac{\partial F^i}{\partial y^j} = \frac{\partial \breve{G}^i}{\partial y^j}; \breve{N}_j^0 = \frac{1}{2}\frac{\partial^2 L}{\partial t \partial y^j},$$

with $\breve{G}^i = 2G^i(x, y, t) - \frac{1}{2}F^i(x, y, t)$.

Let us consider the adapted basis to the distributions $\breve{N}$ and V:

$$\left\{\frac{\breve{\delta}_i}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t}\right\}, \tag{16}$$

where

$$\frac{\breve{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \breve{N}^i_j(x,y,t)\frac{\partial}{\partial y^j} - \breve{N}^0_j(x,y,t)\frac{\partial}{\partial t} + \frac{1}{4}\frac{\partial F^j}{\partial y^i}\frac{\partial}{\partial y^j}. \tag{17}$$

The Lie brackets of the local vector fields from the adapted basis (16) are as the following:

$$\left[\frac{\breve{\delta}}{\delta x^j}, \frac{\breve{\delta}}{\delta x^h}\right] = \breve{R}^i_{jh}\frac{\partial}{\partial y^i} + \breve{R}^0_{jh}\frac{\partial}{\partial t}; \left[\frac{\breve{\delta}}{\delta x^j}, \frac{\partial}{\partial t}\right] = \frac{\partial \breve{N}^i_j}{\partial t}\frac{\partial}{\partial y^i} + \frac{\partial \breve{N}^0_j}{\partial t}\frac{\partial}{\partial t};$$

$$\left[\frac{\breve{\delta}}{\delta x^j}, \frac{\partial}{\partial y^h}\right] = \frac{\partial \breve{N}^i_j}{\partial y^h}\frac{\partial}{\partial y^i} + \frac{\partial \breve{N}^0_j}{\partial y^h}\frac{\partial}{\partial t}; \tag{18}$$

$$\left[\frac{\breve{\delta}}{\delta y^j}, \frac{\partial}{\partial y^h}\right] = \left[\frac{\partial}{\partial y^j}, \frac{\partial}{\partial t}\right] = \left[\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right] = 0,$$

where

$$\breve{R}^i_{jh} = \frac{\breve{\delta}\breve{N}^i_j}{\delta x^h} - \frac{\breve{\delta}\breve{N}^i_h}{\delta x^j}; \quad \breve{R}^0_{jh} = \frac{\breve{\delta}\breve{N}^0_j}{\delta x^h} - \frac{\breve{\delta}\breve{N}^0_h}{\delta x^j}. \tag{19}$$

The dual basis $\{dx^i, \breve{\delta}y^i, \breve{\delta}t\}$ is given by

$$\breve{\delta}y^i = dy^i + \breve{N}^i_j dx^j - \frac{1}{4}\frac{\partial F^i}{\partial y^j}dx^j; \quad \breve{\delta}t = dt + \breve{N}^0_i dx^i. \tag{20}$$

The canonical non-linear connection $\breve{N}$ is integrable if and only if $\breve{R}^i_{jh} = 0$ and $\breve{R}^0_{jh} = 0$.

3. Example of rheonomic mechanical system

Let us consider the rheonomic mechanical system of a constrained motion of a material particle of mass m and position vector $\overline{r} = x^1\overline{i} + x^2\overline{j} + x^3\overline{k}$, with the rheonomic constraint given by $\tilde{f}(x^1, x^2, x^3, t) = 0$. The material particle is under the action of the linear, or non-linear spring force with potential $\Pi = -U(\overline{r})$. The material particle cannot move subject to constraints without considering certain constraint reaction due to the constraint expressed by $\overline{F}_{wN} = \lambda\, grad\ \tilde{f}\,(x^1, x^2, x^3, t)$, for ideal constraint, where λ is Lagrange's multiplier.

For non ideal constraint, with friction coefficient μ, the constraint reaction due to the constraint can be expressed by

$$\overline{F}_w = \overline{F}_{wN} + \overline{F}_{wT} = \lambda\, grad\, \tilde{f}(x^1, x^2, x^3, t) - \mu\overline{v}|\lambda\, grad\, \tilde{f}\,(x^1, x^2, x^3, t)|.$$

The material particle is under the external dumping force, linear proportional with the material particle velocity, expressed by $\overline{F}_{w,\overline{v}} = -b\overline{v}$. Let $\overline{F}(t) = X\overline{i} + Y\overline{j} + Z\overline{k}$ be the active force.

Let the generalized coordinates be: $q^1 = x^1$; $q^2 = x^2$ and because the constraint is of rheonomic nature, we consider a rheonomic coordinate $q^0 = \phi(t)$. Therefore we can express the third position coordinate $x^3 = f(q^1, q^2, q^0)$, as [4].

The kinetic energy of the material particle motion can be expressed by the two generalized coordinates and the third rheonomic coordinate in the following form:

$$E_k = \frac{1}{2}m\overline{v}^2 = \frac{1}{2}m\left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2 + \left(\dot{q}^1\frac{\partial f}{\partial q^1} + \dot{q}^2\frac{\partial f}{\partial q^2} + \dot{q}^0\frac{\partial f}{\partial q^0}\right)^2\right].$$

When the constraint is defined by $x^3 = \frac{1}{\ell}\left(q^1 - \ell\Omega t\right)^2$, we can introduce the rheonomic coordinate $q^0 = \ell\Omega t$ and the previous constrain will become as follows $x^3 = \frac{1}{\ell}\left(q^1 - q^0\right)^2$.

The kinetic energy can be written

$$E_k = \frac{1}{2}m\left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2 + \frac{4}{\ell^2}\left(\dot{q}^1 - \dot{q}^0\right)^2\left(q^1 - q^0\right)^2\right].$$

The matrix of the mass inertia moment tensor is:

$$\mathbf{A} = m\begin{pmatrix} 1 + \frac{4}{\ell^2}\left(q^1 - q^0\right)^2 & 0 & -\frac{4}{\ell^2}\left(q^1 - q^0\right)^2 \\ 0 & 1 & 0 \\ -\frac{4}{\ell^2}\left(q^1 - q^0\right)^2 & 0 & \frac{4}{\ell^2}\left(q^1 - q^0\right)^2 \end{pmatrix}.$$

The virtual work $\delta\mathbf{W}$ of the active force on the virtual displacements $\delta\vec{r} = \delta q^1\vec{i} + \delta q^2\vec{j} + \frac{2}{\ell}\left[\delta q^1\left(q^1 - q^0\right) - \delta q^0\left(q^1 - q^0\right)\right]\vec{k}$ is given by:

$$\delta\mathbf{W} = \left(\vec{F}, \delta\vec{r}\right)q^0 = \left[X + \frac{2}{\ell}Z\left(q^1 - q^0\right)\right]\delta q^1 + Y\delta q^2 - \frac{2}{\ell}Z\left(q^1 - q^0\right)\delta q^0.$$

For the active force induced by the spring:

$$\vec{F} = -c\vec{r} = -cq^1\vec{i} - cq^2\vec{j} - \left[c\frac{1}{\ell}\left(q^1 - q^0\right)^2 + mg\right]\vec{k},$$

the generalized force components are:

$$Q_1 = -cq^1 - c\frac{2}{\ell^2}\left(q^1 - q^0\right)^3 - \frac{2}{\ell}mg\left(q^1 - q^0\right); Q_2 = -cq^2;$$

$$Q_0 = c\frac{2}{\ell^2}\left(q^1 - q^0\right)^3 - \frac{2}{\ell}mg\left(q^1 - q^0\right).$$

The potential energy can be expressed in the form:

$$\mathbf{E_p} = -\int_0^{\vec{r}}\left(\vec{F}, d\vec{r}\right) = \frac{c}{2}\vec{r}^2 = \frac{c}{2}\left[\left(q^1\right)^2 + \left(q^2\right)^2 + \frac{1}{\ell^2}\left(q^1 - q^0\right)^4\right].$$

The kinetic potential is difference of the kinetic energy and the potential energy and we can express *the Lagrangian function* in the following form:

$$L = E_k - E_p = E_k - \Pi\left(q^1, q^2, f\left(q^1, q^2, q^0\right)\right) =$$
$$= \frac{1}{2}m\left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2 + \frac{4}{\ell^2}\left[\left(\dot{q}^1 - \dot{q}^0\right)\left(q^1 - q^0\right)\right]^2\right] -$$
$$-\frac{c}{2}\left[\left(q^1\right)^2 + \left(q^2\right)^2 + \frac{1}{\ell^2}\left(q^1 - q^0\right)^4\right].$$

L is a rheonomic regular Lagrangian.

The matrix of the d-tensor field with the components $g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j}$ is the fundamental tensor field of the Lagrangian corresponding to the mechanical rheonomic system of one material particle moving along moving surface, $x^3 = \frac{1}{\ell}\left(q^1 - \ell\Omega t\right)^2$ as a rheonomic constraint:

$$\mathbf{G} = \left(g_{ij}\right)\Big|_{\downarrow i=1,2,0}^{\rightarrow j=1,2,0} = \left(\frac{1}{2}\frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j}\right)_{\downarrow i=1,2,0}^{\rightarrow j=1,2,0}$$

$$\mathbf{G} = \frac{1}{2}\mathbf{A} = \frac{1}{2}m\begin{pmatrix} 1 + \frac{4}{\ell^2}\left(q^1 - q^0\right)^2 & 0 & -\frac{4}{\ell^2}\left(q^1 - q^0\right)^2 \\ 0 & 1 & 0 \\ -\frac{4}{\ell^2}\left(q^1 - q^0\right)^2 & 0 & \frac{4}{\ell^2}\left(q^1 - q^0\right)^2 \end{pmatrix},$$

where $(q) = \left(q^1, q^2, q^0\right)$ and q^0 is the rheonomic coordinate.

We can see that the fundamental tensor field of the considered Lagrangian is half *of the mass inertia moment tensor matrix.*

The extended Lagrange system of differential equations of the second order, has the following form:

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}^1}-\frac{\partial E_k}{\partial q^1}-\frac{\partial E_p}{\partial q^1}=Q_1;\ \frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}^2}-\frac{\partial E_k}{\partial q^2}-\frac{\partial E_p}{\partial q^2}=Q_2;$$

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}^0}-\frac{\partial E_k}{\partial q^0}-\frac{\partial E_p}{\partial q^0}=Q_0+Q_{00},$$

and for $q^0=\ell\Omega t$, $\dot{q}^0=\ell\Omega$, $\ddot{q}^0=0$, it becomes:

$$\ddot{q}^1+\frac{c}{m}\frac{q^1}{\left[1+\frac{4}{\ell^2}\left(q^1-q^0\right)^2\right]}+\frac{2c}{m\ell^2}\frac{\left(q^1-q^0\right)^3}{\left[1+\frac{4}{\ell^2}\left(q^1-q^0\right)^2\right]}=$$

$$=-\frac{4}{\ell^2}\frac{\left(\dot{q}^1-\dot{q}^0\right)^2\left(q^1-q^0\right)}{\left[1+\frac{4}{\ell^2}\left(q^1-q^0\right)^2\right]}-\frac{2}{\ell}\frac{g\left(q^1-q^0\right)}{\left[1+\frac{4}{\ell^2}\left(q^1-q^0\right)^2\right]};$$

$$\ddot{q}^2+\frac{c}{m}q^2=0,$$

and from the third equation of the Lagrange system we can find the rheonomic constraint force Q_{00}

$$\frac{d}{dt}\left[-\frac{4}{\ell^2}\left(\dot{q}^1-\dot{q}^0\right)\left(q^1-q^0\right)^2\right]+\frac{4}{\ell^2}\left(\dot{q}^1-\dot{q}^0\right)^2\left(q^1-q^0\right)=$$

$$=-\frac{2c}{m\ell^2}\left(q^1-q^0\right)^3-\frac{2}{\ell}mg\left(q^1-q^0\right)+Q_{00}. \tag{21}$$

The theoretical form of the Lagrange equations is:

$$\ddot{q}^i+2G^i(q,\dot{q},t)+N_0^i(q,\dot{q},t)=\frac{1}{2}Q^i(q,\dot{q},t);\quad \dot{q}^i=\frac{dq^i}{dt},$$

with the coefficients

$$2G^i(q,\dot{q},t)=$$

$$=mg^{i1}\left[\left(\dot{q}^1\right)^2\frac{\partial^2 f}{\partial\left(q^1\right)^2}+\left(\dot{q}^2\right)^2\frac{\partial^2 f}{\partial\left(q^2\right)^2}+2\frac{\partial^2 f}{\partial q^1\partial q^2}\dot{q}^1\dot{q}^2+\frac{\partial^2 f}{\partial q^2\partial t}\dot{q}^2\right]\frac{\partial f}{\partial q^1}-$$

$$-mg^{i1}\frac{\partial^2 f}{\partial q^1\partial t}\frac{\partial f}{\partial t}-mg^{i1}\dot{q}^2\frac{\partial^2 f}{\partial q^1\partial t}\frac{\partial f}{\partial q^2}+mg^{i2}\left[\left(\dot{q}^1\right)^2\frac{\partial^2 f}{\partial\left(q^1\right)^2}+\left(\dot{q}^2\right)^2\frac{\partial^2 f}{\partial\left(q^2\right)^2}+\right.$$

$$\left.+2\frac{\partial^2 f}{\partial q^1\partial q^2}\dot{q}^1\dot{q}^2+\frac{\partial^2 f}{\partial q^1\partial t}\dot{q}^1\right]\frac{\partial f}{\partial q^2}-mg^{i2}\frac{\partial^2 f}{\partial q^2\partial t}\frac{\partial f}{\partial t}-mg^{i2}\dot{q}^1\frac{\partial^2 f}{\partial q^2\partial t}\frac{\partial f}{\partial q^1}+$$

$$+g^{i1}\frac{\partial\Pi}{\partial q^1}+g^{i2}\frac{\partial\Pi}{\partial q^2}+\left(g^{i1}\frac{\partial f}{\partial q^1}+g^{i2}\frac{\partial f}{\partial q^2}\right)\frac{\partial\Pi}{\partial f};\quad i=1,2.$$

and

$$N_0^i(q,\dot q,t)=\frac{1}{2}g^{ih}\frac{\partial^2 L}{\partial q^h\partial t}=mg^{i1}\left(2\frac{\partial f}{\partial q^1}\frac{\partial^2 f}{\partial q^1\partial t}\dot q^1+\dot q^2\frac{\partial^2 f}{\partial q^1\partial t}\frac{\partial f}{\partial q^2}+\right.$$

$$+\dot q^2\frac{\partial^2 f}{\partial q^2\partial t}\frac{\partial f}{\partial q^1}+$$

$$\left.+\frac{\partial^2 f}{\partial q^1\partial t}\frac{\partial f}{\partial t}+\frac{\partial^2 f}{\partial t^2}\frac{\partial f}{\partial q^1}\right)+mg^{i2}\left(2\frac{\partial f}{\partial q^2}\frac{\partial^2 f}{\partial q^2\partial t}\dot q^2+\dot q^1\frac{\partial^2 f}{\partial q^1\partial t}\frac{\partial f}{\partial q^2}+\right.$$

$$\left.+\dot q^1\frac{\partial^2 f}{\partial q^2\partial t}\frac{\partial f}{\partial q^1}+\frac{\partial^2 f}{\partial q^2\partial t}\frac{\partial f}{\partial t}+\frac{\partial^2 f}{\partial t^2}\frac{\partial f}{\partial q^2}\right);\quad i=1,2.$$

One obtain

$$2G^1(q,\dot q,t)+N_0^1(q,\dot q,t)=\frac{4}{\ell^2}\frac{\left(\dot q^1-\dot q^0\right)^2\left(q^1-q^0\right)}{\left[1+\frac{4}{\ell^2}\left(q^1-q^0\right)^2\right]};$$

$$2G^2(q,\dot q,t)+N_0^2(q,\dot q,t)=0.$$

We define the following functions

$$2\overset{0}{\Gamma^i}=2G^i(q,\dot q,t)+N_0^i(q,\dot q,t)-\frac{1}{2}Q^i(q,\dot q,t).$$

So, $\overset{0}{S}=y^i\frac{\partial}{\partial x^i}-2\overset{0}{\Gamma^i}(x,y,t)\frac{\partial}{\partial y^i}+\frac{\partial}{\partial t}$ is *the evolution semispray of the mechanical system* $\overset{0}{\Sigma}$.

The integral curves of $\overset{0}{S}$ are *the evolution curves of the mechanical system* $\overset{0}{\Sigma}$.

The canonical non-linear connection $\overset{0}{N}$ of mechanical system, $\overset{0}{\Sigma}$, depending only on the rheonomic Lagrangian mechanical system, has the coefficients $(\overset{0}{N^i_j},\overset{0}{N^0_j})$ given by

$$\overset{0}{N^i_1}(q,\dot q,t)=\frac{\partial G^i(q,\dot q,t)}{\partial \dot q^1}-\frac{1}{4}\frac{\partial F^i(q,\dot q,t)}{\partial \dot q^1}=$$

$$=mg^{i1}\left(2\frac{\partial f}{\partial q^1}\frac{\partial^2 f}{\partial (q^1)^2}\dot q^1+2\dot q^2\frac{\partial^2 f}{\partial q^1\partial q^2}\frac{\partial f}{\partial q^1}\right)+mg^{i2}\left(2\dot q^1\frac{\partial^2 f}{\partial (q^1)^2}\frac{\partial f}{\partial q^2}+\right.$$

$$\left.+2\dot q^2\frac{\partial^2 f}{\partial q^1\partial q^2}\frac{\partial f}{\partial q^2}+\frac{\partial^2 f}{\partial q^1\partial t}\frac{\partial f}{\partial q^2}-\frac{\partial f}{\partial q^1}\frac{\partial^2 f}{\partial q^2\partial t}\right)-\frac{1}{2}\frac{\partial Q^i}{\partial \dot q^1};$$

$$\overset{0}{N^i_2}(q,\dot q,t)=\frac{\partial G^i(q,\dot q,t)}{\partial \dot q^2}-\frac{1}{4}\frac{\partial F^i(q,\dot q,t)}{\partial \dot q^2}=$$

$$= mg^{i1}\left(2\frac{\partial f}{\partial q^1}\frac{\partial^2 f}{\partial q^1\partial q^2}\dot{q}^1 + 2\dot{q}^2\frac{\partial^2 f}{\partial (q^2)^2}\frac{\partial f}{\partial q^1} + \frac{\partial f}{\partial q^1}\frac{\partial^2 f}{\partial q^2\partial t} - \frac{\partial^2 f}{\partial q^1\partial t}\frac{\partial f}{\partial q^2}\right) +$$

$$+mg^{i2}\left(2\dot{q}^1\frac{\partial^2 f}{\partial q^1\partial q^2}\frac{\partial f}{\partial q^2} + 2\dot{q}^2\frac{\partial^2 f}{\partial (q^2)^2}\frac{\partial f}{\partial q^2}\right) - \frac{1}{2}\frac{\partial Q^i}{\partial \dot{q}^2}, \quad i = 1, 2.$$

It is visible that for considered rheonomic mechanical system is easy to obtain geometrical description and the dynamical properties.

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