**"Vasile Alecsandri" University of Bacău**
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# ON SEMILLATICE-ORDERED SEMIGROUPS. A CONSTRUCTIVE POINT OF VIEW

DANIEL A. ROMANO

**Abstract.** Semilattice-ordered semigroup is an important algebraic structure. It is ordered semigroup under anti-order. Some basic properties of semillatice-ordered semigroups with apartness are given by constructive point of view. Let $I$ and $K$ be compatible an ideal and an anti-ideal of semilattice-ordered semigroup $S$. Constructions of compatible congruence $E(I)$ and anti-congruence $Q(K)$ on $S$, generated by $I$ and $K$ respectively, are given. Besides, we construct compatible order $\leqslant_T$ and anti-order $\theta_T$ on factor-semigroup $S/(E(I), Q(K))$. Some basic properties of such constructed semigroups are given.

## 1. Introduction and Preliminaries

Our setting is Bishop's constructive mathematics ([1], [2], [4], [7]), mathematics developed with Constructive logic (or Intuitionistic logic) - logic without 'Low of Excluded Middle' $P \vee \neg P$. We have to note that 'the crazy axiom' $\neg P \Longrightarrow (P \Longrightarrow Q)$ is included in the Constructive logic. Precisely, in Constructive logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold but the following implication $P \Longrightarrow \neg\neg P$ holds even in Minimal logic. In Constructive logic 'the Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not holds, too. It is interesting, in Constructive logic the following deduction principle $A \vee B, \neg A \vdash B$ holds, but this is impossible to prove without 'the crazy axiom'.

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The paper deals with semillatice-ordered semigroups which are examined within the restrictive framework of Bishop-style constructive mathematics. This investigation is in Bishop's constructive algebra in sense of papers [10]-[13] and books [7] and [14] (Chapter 8: Algebra). Let $(S, =, \neq)$ be a constructive set. The *diversity* relation $\neq$ ([7]) is a binary relation on $S$, which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq y \wedge y = z \Longrightarrow x \neq z\ .$$

If it satisfies the following condition

$$(\forall x, z \in S)(x \neq z \Longrightarrow (\forall y \in S)(x \neq y \vee y \neq z)),$$

it is called *apartness* (A. Heyting). For a subset $X$ of $A$ we say that it is a *strongly extensional subset* of $A$ if and only if $x \in X \Longrightarrow (\forall y \in S)(x \neq y \vee y \in X)$. Following Bridges and Vita (for example, see [5]), for subsets $X$ and $Y$ of $A$ we say that set $X$ is set-set apart from $Y$, and it is denoted by $X \bowtie Y$, if and only if $(\forall x \in X)(\forall y \in Y)(x \neq y)$. We set $x \bowtie Y$ instead of $\{x\} \bowtie Y$, and, of course, $x \neq y$ instead of $\{x\} \bowtie \{y\}$. With $S^C = \{x \in X : x \bowtie S\}$ we denote the apartness complement of $S$. So, $\bowtie$ is a relation between pairs of subsets of $A$. It is easy to see that the following hold:

(0) $\neg(X \bowtie X)$;
(1) $X \bowtie Y \Longrightarrow \neg(X = \emptyset \wedge Y = \emptyset)$;
(2) $X \bowtie Y \Longrightarrow X \cap Y = \emptyset$ ;
(3) $X \bowtie Y \wedge Z \subseteq Y \Longrightarrow X \bowtie Z$;
(3') $X \bowtie (Y \cup Z) \Longleftrightarrow X \bowtie Y \wedge X \bowtie Z$;
(4) $X \bowtie Y \Longrightarrow Y \bowtie X$.

Let $Y$ be a subset of $(S, =, \neq)$. We say that it is *detachable* if and only if $(\forall x)(x \in S \Longrightarrow x \in Y \vee x \bowtie Y)$. For a function $f : (S, =, \neq) \longrightarrow (T, =, \neq)$ we say that it is a *strongly extensional* function if and only if $(\forall a, b \in S)(f(a) \neq f(b) \Longrightarrow a \neq b)$.

For undefined notions and notations of semigroup items we referred to book [3] and articles [6] and [7] and items of Constructive Algebra we referred to books [1], [2], [4], [7] and [14] and to the author's papers [10]-[13]. Semilattice-ordered semigroups are important algebraic structures. They where studied by Martin Kuril and Libor Polka ([6], [9]). J van Plato studied in his article [8] semilattice in Constructive Algebra. In this article, in Section 2, we give a definition of anti-order relation on semilattice-ordered semigroup $S$ (with apartness and strongly extensional internal operation) and construct one on $S$ in a natural way. In Section 3 we show some examples of semilattice-ordered semigroups with diversity relation (Example 3.1, Example 3.2 and Example 3.3) and with apartness (Example 3.4). Let $I$ and $K$ be compatible an ideal and an anti-ideal of semilattice-ordered semigroup $S$. Constructions of compatible congruence $E(I)$ and anti-congruence $Q(K)$ on $S$, generated by $I$ and $K$ respectively, are given in section 4. Finishing this investigation with Section 5 we give some basic properties of semilattice-ordered semigroups with diversity.

## 2. Order and anti-order on semilattice-ordered semigroup

Following the classical definition in [6] and [9], for algebraic structure $(A, =, \neq, \cdot, \otimes)$ is called a (strong) semilattice-ordered semigroup if :

(i) $(A, =, \neq, )$ is a semigroup, where the semigroup operation is strongly extensional in the following way $(\forall a, b, c \in A)((ac \neq bc \vee ca \neq cb) \Longrightarrow a \neq b)$;

(ii)$(A, =, \neq, \otimes)$ is a semilattice, i.e. $(A, \otimes)$ is a commutative semigroup with

$$(\forall x \in A)(x \otimes x = x),$$

where the semigroup operation is strongly extensional:

$$(\forall a, b, c \in A)((a \otimes c \neq b \otimes c \ \vee \ c \otimes a \neq c \otimes b) \Longrightarrow a \neq b);$$

(iii) $(\forall a, b, c \in A)((a(b \otimes c) = ab \otimes ac) \wedge ((a \otimes b)c = ac \otimes bc))$.

For a function $f : (S, =, \neq, \cdot, \otimes) \longrightarrow (T, =, \neq, \circ, \diamond)$ we say that it is a homomorphism of semilattice-ordered semigroups if and only if hold $f(ab) = f(a) \circ f(b)$ and $f(a \otimes b) = f(a) \diamond f(b)$ for all $a$ and $b$ of $S$.

A structure $(A, =, \neq, \leqslant)$ is called an ordered semigroup ([3], [9]) if

(i) $(A, =, \neq, \cdot)$ is a semigroup, where the operation $\cdot$ is strongly extensional,

(ii) $(A, \leqslant)$ is a (partially) ordered set,

(iii) $(\forall a, b, c \in A)(a \leqslant b \Longrightarrow (ca \leqslant cb \wedge ac \leqslant bc))$.

The following lemma shows significance of semilattice-ordered semigroup:

**Lemma 2.1.** *If $(A, =, \neq \cdot, \otimes)$ is a semilattice-ordered semigroup and we define, for any $a$, $b$ of $A$, $a \leqslant b$ if and only if $a \otimes b = b$, it is known that the structure $(A, =, \neq \cdot, \leqslant)$ is an ordered semigroup.*

Proof: See, for example, [3]. $\square$

Since in Constructive logic the 'Law of Excluded Middle' is not valid, in Bishop's constructive algebra the following relation is also interesting: a relation symmetric to ordered relation $\leqslant$. A relation $\theta$ on $S$ is *anti-order* ([10]) on $S$ if and only if

$$\theta \subseteq \neq$$
$$(\forall x, y, z \in S)((x, z) \in \theta \Longrightarrow ((x, y) \in \theta \vee (y, z) \in \theta)),$$
$$(\forall x, y \in S)(x \neq y \Longrightarrow ((x, y) \in \theta \vee (y, x) \in \theta)), \text{ (linearity) and}$$
$$(\forall x, y, z \in S)(((xz, yz) \in \theta \Longrightarrow (x, y) \in \theta) \wedge ((zx, zy) \in \theta \Longrightarrow (x, y) \in \theta)).$$

System $(A, =, \neq, \cdot, \theta)$ is ordered semigroup under anti-order if $(A, =, \neq, \cdot)$ is a semigroup where the semigroup operation is strongly extensional, and relation $\theta$ is an anti-order relation on $(A, =, \neq, \cdot)$. In the following lemma we show that semilattice-ordered semigroup is relevant once more. This suggests that there exists interest for investigation of ordered semigroup under anti-order.

**Lemma 2.2.** *If* $(A, =, \neq \cdot, \otimes)$ *is a semilattice-ordered semigroup and we define, for any* $a$, $b$ *of* $A$,

$$a\theta b \Longleftrightarrow a \otimes b \neq a,$$

*then the structure* $(A, =, \neq \cdot, \theta)$ *is an ordered semigroup under anti-order .*

Proof: (i) Suppose that $a\theta b$, i.e. let $a \otimes b \neq a$. Then $a \otimes b \neq a \otimes a$, and thus $a \neq b$. So, the relation $\theta$ is consistent.
(ii) Let $a, b, c$ be arbitrary elements of $A$ such that $a\theta c$, i.e. such that $a \otimes c \neq a$. Then

$$a \otimes c \neq a \Longrightarrow ((a \otimes c \neq b \otimes a) \vee (b \otimes a \neq a)).$$

If $b \otimes a \neq a$, then $a\theta b$. Suppose that $a \otimes c \neq b \otimes a$. Then $a \otimes c \neq a \otimes b \otimes c$ or $a \otimes b \otimes c \neq b \otimes a$ holds. In the first case, we conclude:
$$a \otimes c \neq a \otimes b \otimes c \Longrightarrow a \neq a \otimes b \vee c \neq c$$
$$\Longrightarrow a\theta b \ .$$
In the second case, we have
$$a \otimes b \otimes c \neq b \otimes a \Longrightarrow b \otimes c \neq b$$
$$\Longrightarrow b\theta c \ .$$
Therefore, the relation $\theta$ is cotransitive.
(iii) Let $a$ and $b$ be arbitrary elements of $A$ such that $a \neq b$. Thus $a \neq a \otimes b$ or $a \otimes b \neq b$. So, we have $a \neq b \Longrightarrow a\theta b \vee b\theta a$. So, the relation $\theta$ is linear.
(iv) Let $a, b$ be arbitrary elements of semigroup $(A, =, \neq \cdot, \otimes)$ such that $ac\theta bc$. Then $ac \otimes bc \neq ac = ac \otimes ac$ implies $ac \neq ac \vee bc \neq ac$

(because the operation $\otimes$ is strongly extensional). So, we have $a \neq b$. Analogously, we conclude the implication $ca\theta cb \Longrightarrow a \neq b$.
Finally, the relation $\theta$ is an anti-order relation on semigroup $(A, =, \neq, \cdot)$ and the structure $(A, =, \neq\ \cdot, \theta)$ is a semigroup ordered under anti-order. □

**Corollary 2.1.** *If $(A, =, \neq\ \cdot, \otimes)$ is a semilattice-ordered semigroup and we define, for any $a$, $b$ of $A$,*

$$a \Theta b \Longleftrightarrow a \otimes b \neq b,$$

*then the structure $(A, =, \neq\ \cdot, \Theta)$ is an ordered semigroup under anti-order $\theta$. Except that $\Theta = \theta^{-1}$ holds.*

The following lemma shows what kind of connection is the relation $\theta$.

**Lemma 2.3.** *Let $\theta$ be an anti-order on semigroup $(S, =, \neq, \cdot)$. Then $\theta^C$ is an order on $(S, \neg \neq, \neq, \cdot)$.*

The result is known, even in more general form. The logical complement of an anti-order, which coincides in this case with the complement, is a partial order relation. (See, for instance, van Plato's paper [8].) If the ordered set is endowed with an algebraic structures, then the compability of the algebraic operations with the partial order follows from compability with the anti-order.

The following two corollaries show what kind of connection exists between relation $\leqslant$ and $\theta$.

**Corollary 2.2.** *If $(A, =, \neq, \cdot, \otimes)$ is a semilattice-ordered semigroup and we define, for any $a, b$ of $A$,*

$$a \leqslant b \Longleftrightarrow a \otimes b = b \text{ and } a\theta b \Longleftrightarrow a \otimes b \neq a,$$

*then*

$$a \leqslant b \wedge b\theta c \Longrightarrow a\theta c.$$

Proof: Let $a \leqslant b \wedge b\theta c$. Then $b\theta a \vee a\theta c$, i.e. then $b \otimes a \neq b$ and $a\theta c$. Thus, we have $a\theta c$ because $a \otimes b = b$ and $b \otimes a \neq b$ is impossible. □

**Corollary 2.3.** *If $(A, =, \neq, \cdot, \otimes)$ is a semilattice-ordered semigroup and if we define*

$$a \leqslant b \Longleftrightarrow a \otimes b = b \text{ and } b\theta a \Longleftrightarrow b \otimes a \neq b, \text{ (for any } a, b \text{ of } A)$$

*then*

$$\neg(a \leqslant b \wedge b\theta a).$$

## 3. Some examples of semilattice-ordered semigroups

**Example 3.1.** Let $S$ be a semigroup. We put for any $X \in \wp(S)$ and $Y \in \wp(S)$ $X \cdot Y = \{xy | x \in X \wedge y \in Y\}$. Then $(P(S), \cdot, \cup)$ is a semilattice-ordered semigroup. Here, as usual, $\cup$ denotes the set-theoretical union.

Let us note that, even though the set $S$ is supplied by apartness, relation $\neq_1$ on $\wp(S)$, defined in a natural way:

$$X \neq_1 Y \Longleftrightarrow (\exists x \in X)(x \bowtie Y) \vee (\exists y \in Y)(y \bowtie X)$$

is a diversity relation on $\wp(S)$ but it is not apartness in general.

For the second example we give a construction of free semigroup with apartness generated by set $(X, =, \neq)$.

**Example 3.2.** Let $(X, =, \neq)$ be a set with apartness. We form the following class $X^+$ of all strictly finite sequences of elements of $X$

$$x^+ \in X^+ \Longleftrightarrow (n_x \in \mathbf{N})(\exists f_x)(f_x : \{1, 2, ..., n_x\} \longrightarrow x^+)$$

with

$$(\forall i \in \{1, 2, ..., n_x\})(f_x(i) \in X).$$

As usual the concatenation of $x^+$ and $y^+$ is denoted by $x^+ \circ y^+$. If

$$x^+ = (f_x(1), ..., f_x(n_x)) \text{ and } y^+ = (f_y(1), ..., f_y(n_y)),$$

then

$$n_{xy} = n_x + n_y \text{ and}$$
$$i \in \{1, 2, ..., n_x\} \Longrightarrow f_{xy}(i) = f_x(i),$$
$$i = n_x + j \, (j \in \{1, 2, ..., n_y\}) \Longrightarrow f_{xy}(i) = f_y(j),$$

i.e.

$$x^+ \circ y^+ = (f_x(1), ..., f_x(n_x), f_y(1), ..., f_y(n_y)).$$

On the class $X^+$ we define

$$x^+ =_1 y^+ \Longleftrightarrow (n_x = n_y \wedge f_x = f_y) \text{ and}$$
$$x^+ \neq_1 y^+ \Longleftrightarrow (\neg(n_x = n_y) \vee f_x \neq f_y).$$

It is obvious that the relation $=_1$ is an equality relation on the class $X^+$. It is clear that the relation $\neq_1$ is consistent $\neg(x^+ \neq_1 x^+)$ and symmetric $(x^+ \neq_1 y^+ \Longrightarrow y^+ \neq_1 x^+)$. We have to prove that the relation $\neq_1$ is compatible with the equality $=_1$ and cotransitive.

$x^+ =_1 y^+ \wedge y^+ \neq_1 z^+ \Longleftrightarrow$

$(x^+ =_1 y^+ (\Longleftrightarrow (n_x = n_y \wedge f_x = f_y))) \wedge (y^+ \neq_1 z^+ (\Longleftrightarrow \neg(n_y = n_z) \wedge f_y \neq f_z))$

$\Longrightarrow \neg(n_x = n_z) \vee f_x \neq f_z$

$\Longrightarrow x^+ \neq_1 z^+$.
Let $x^+, z^+$ be arbitrary elements of $X^+$ such that $x^+ \neq_1 z^+$, and let $y^+$ be an arbitrary element of $X^+$. Then there exist natural numbers $n_x$ , $n_y$, $n_z$ and functions $f_x$, $f_y$, $f_z$ such that $\neg(n_x = n_z) \vee f_x \neq f_z$. Thus,

$$\neg(n_x = n_y) \vee \neg(n_y = n_z) \vee f_x \neq f_y \vee f_y \neq f_z.$$

Therefore, $x^+ \neq_1 y^+ \vee y^+ \neq_1 z^+$ holds.
The mapping $\circ : X^+ \times X^+ \ni (x^+, z^+) \longmapsto x^+ \circ z^+ \in X^+ \times X^+$ is an internal binary operation on the set $(X^+, =_1, \neq_1)$. Indeed: Let $(x^+, z^+)$ and $(a^+, b^+)$ be two pairs of elements of $X^+$ and let $x^+ =_1 a^+$ and $y^+ =_1 b^+$. Then $x^+ =_1 a^+ \Longleftrightarrow (n_x = n_a \wedge f_x = f_a)$ and $y^+ =_1 b^+ \Longleftrightarrow (n_y = n_b \wedge f^y = f_b)$. We have
$n_x + n_y = n_a + n_b$
and
$x^+ \circ y^+ =$
$(f_x(1), ..., f_x(n_x), f_y(1), ..., f_y(n_y)) =_1$
$(f_a(1), ..., f_a(n_a), f_b(1), ..., f_b(n_b))$
$= a^+ \circ b^+$
Let $x^+ \circ y^+ \neq_1 a^+ \circ b^+$, i.e. let $\neg(n_{xy} = n_{ab}) \vee f_{xy} \neq f_{ab}$.
Thus:
(i) If $\neg(n_{xy} = n_x + n_y = n_a + n_b = n_{ab})$, then $\neg(n_x = n_a) \vee \neg(n_y = n_b)$.
(ii) If

$$(f_x(1), ..., f_x(n_x), f_y(1), ..., f_y(n_y)) \neq_1 \\ (f_a(1), ..., f_a(n_x), f_b(1), ..., f_b(n_y)),$$

then there exists the natural number $i \in \{1, ..., n_{xy}\}$ such that $f_{xy}(i) \neq f_{ab}(i)$. If $i \in \{1, ..., n_x\}$, then $f_x(i) = f_{xy}(i) \neq f_{ab}(i) = f_a(i)$; if $i \in \{n_{x+1}, ..., n_{xy}\}$, then $i = j + n_x$ and $f_y(j) = f_{xy}(i) \neq f_{ab}(i) = f_b(j)$. So, from the both cases, we conclude that $x^+ \neq_1 a^+$ or $y^+ \neq_1 b^+$. Therefore, the operation "$\circ$" is strongly extensional.
Finally the structure $(X^+, =_1, \neq_1, \circ)$ is a semigroup with apartness. Collection $(F(X^+), =, \neq, \cdot, \cup)$, where $F(S)$ is the set of all inhabited finite detachable subsets of $S$ with the natural multiplication and joint (as in the Example 3.1.), is a semilattice-ordered semigroup with diversity.

**Example 3.3.** Let $(A, =, \neq)$ be a set with apartness. Then the set $Rel(A)$ of all binary relations on $A$ with the composition "$\circ$" and union $\cup$, with equality $=_2$ and diversity $\neq_2$ defined in the usual way:
$\alpha =_2 \beta \Longleftrightarrow (\forall x, y \in A)((x, y) \in \alpha \Longleftrightarrow (x, y) \in \beta)$,
$\alpha \neq_2 \beta \Longleftrightarrow$

$(\exists x, y \in A)((x, y) \in \alpha \wedge (x, y) \bowtie \beta) \vee (\exists x, y \in A)((x, y) \bowtie \alpha \wedge (x, y) \in \beta)$,
is a semilattice ordered semigroup. Let us note again that the relation $\neq_2$ is a diversity but it is not apartness in general case. The composition "$\circ$" of relations is a strongly extensional operation. Indeed: Let $\alpha, \alpha', \beta, \beta' \in Rel(A)$ such that $\beta \circ \alpha \neq_2 \beta' \circ \alpha'$. For simplicity, suppose that $(a, c) \in \beta \circ \alpha$ and $(a, c) \bowtie \beta' \circ \alpha'$. Then, there exists element $b$ of $A$ such that $(a, b) \in \alpha$ and $(b, c) \in \beta$ and for any $x$ of $A$ holds $(a, x) \bowtie \alpha'$ or $(x, c) \bowtie \beta'$. Finally, we have

$$(\exists b \in A)((a, b) \in \alpha \ \wedge \ (b, c) \in \beta \ \wedge \ ((a, b) \bowtie \alpha' \ \vee \ (b, c) \bowtie \beta')).$$

Thus, if $(\exists b \in A)((a, b) \in \alpha \ \wedge \ (a, b) \bowtie \alpha')$, then $\alpha \neq_2 \alpha'$; if $(\exists b \in A)((b, c) \in \beta \ \wedge \ (b, c) \bowtie \beta'))$, then $\beta \neq_2 \beta'$. So, the operation "$\circ$" is strongly extensional.
Let $\alpha, \alpha', \beta, \beta' \in Rel(A)$ such that $\alpha \cup \beta \neq_2 \alpha' \cup \beta'$. So, there exists an element $(a, b)$ of $A \times A$ such that $(a, b) \in \alpha \cup \beta$ and $(a, b) \bowtie \alpha' \cup \beta'$, or ... From $(a, b) \in \alpha \cup \beta$ we conclude $(a, b) \in \alpha$ or $(a, b) \in \beta$. From $(a, b) \bowtie \alpha' \cup \beta'$ we have $(a, b) \bowtie \alpha'$ and $(a, b) \bowtie \beta'$. Thus, $\alpha \neq_2 \alpha'$ or $\beta \neq_2 \beta'$. So, the operation $\cup$ is strongly extensional also.

As second we present certain basic and natural examples of semilattice-ordered semigroups.

**Example 3.4.** Let $(A, =_A, \neq_A, \otimes)$ be a semilattice with strongly extensional operation $\otimes$. We mark with $End_{\neq}(A, \otimes)$ the set of all strongly extensional endomorphisms of semilattice $(A, \otimes)$ with joint and operation of composition as in Example 3.3. The inclusion $End_{\neq}(A, \otimes) \subseteq End(A, \otimes)$ holds. Equality $=$ and apartness $\neq$ on $End_{\neq}(A, \otimes)$ we define in the usual way:

$$\alpha = \beta \Longleftrightarrow (\forall x \in A)(\alpha(x) =_A \beta(x)),$$
$$\alpha \neq \beta \Longleftrightarrow (\exists x \in A)(\alpha(x) \neq_A \beta(x)).$$

The first, composition "$\circ$" is strongly extensional. Indeed: Let $\alpha$, $\alpha'$, $\beta$, $\beta'$ be endomorphisms such that $\beta \circ \alpha \neq \beta' \circ \alpha'$. It means that there exist element $a$ of $A$ with $(\beta \circ \alpha)(a) =_A (\beta' \circ \alpha')(a)$, i.e. with $\beta(\alpha(a)) \neq_A \beta'(\alpha'(a))$. Then, $\beta \neq \beta'$ and $\alpha \neq \alpha'$. The second, joint operation "$\diamond$" is strongly extensional also. Indeed: Let $\alpha \diamond \beta \neq \alpha' \diamond \beta?'$, i.e. let be there exists element $a$ of $A$ such that $(\alpha \diamond \beta)(a) \neq_A (\alpha' \diamond \beta')(a)$. Then $\alpha(a) \otimes \beta(a) \neq_A \alpha'(a) \otimes \beta'(a)$. Thus, $\alpha(a) \neq_A \alpha'(a)$ or $\beta(a) \neq_A \beta'(a)$, because the operation $\otimes$ is strongly extensional. Last means that $\alpha \neq \alpha'$ or $\beta \neq \beta'$. So, set $(End_{\neq}(A, =_A, \neq_A, \otimes), =, \neq, \circ, \diamond)$ is a (strong) semilattice ordered semigroup with apartness.

## 4. Ideals and anti-ideals of semilattice-ordered semigroup

Let $(S, =, \neq, \cdot, \otimes)$ be a semilattice-ordered semigroup. A subset $J$ of $S$ is its ideal if
(i) $a \in J \wedge b \in S \wedge b \leqslant a \Longrightarrow b \in J$,
(ii) $a \in J \wedge b \in J \Longrightarrow a \otimes b \in J$.

**Example 4.1.** (1) If $a \in S$, then the set $(a] = \{x \in S : x \leqslant a\}$ is an ideal of $S$ called a principal ideal generated by $a$.
(2) If $J$ is an ideal of $S$ and $a \in S$, then subsets $(a : J) = \{x \in S : ax \in J\}$ and $(J : a) = \{x \in S : xa \in J\}$ are ideals of $S$ called left and right quotient of ideal $J$ by element $a$.

**Lemma 4.1.** *An ideal $J$ of a semilattice-ordered semigroup $S$ defines a relation $E(J)$ on the set $S$ by*
$$(a, b) \in E(J) \Longleftrightarrow (\forall x, y \in S^1)(xay \in J \Longleftrightarrow xby \in J).$$
*This relation $E(J)$ is a congruence of $(S, =, \neq, \cdot, \otimes)$.*

Proof immediately follows from definitions. □

Let $(S, =, \neq, \cdot, \otimes)$ be a semilattice-ordered semigroup. A subset $K$ of $S$ is its *anti-ideal* if and only if
(1) $a \otimes b \in K \Longrightarrow a \in K \vee b \in K$,
(2) $b \in K \Longrightarrow a\theta b \vee a \in K$.

**Remark 4.1.** **A**. *Any anti-ideal of semilattice ordered semigroup is a strongly extensional subset of $S$.* Let us note if $K$ is an anti-ideal of $S$, then $K$ is strongly extensional. Indeed: If $b \in K$ then by (2) of definition, $a\theta b \vee a \in K$. Since the relation $\theta$ is consistent, then follows $a \neq b$ or $a \in K$. So, the set $K$ is a strongly extensional subset of $S$.
**B**. *If $K$ is an anti-ideal of a semilattice-ordered semigroup $S$, then $(\forall a, b \in S)(a \in K \Longrightarrow a \otimes b \in K)$ and $(\forall a, b \in S)(a \in K \wedge a \leqslant b \Longrightarrow b \in K)$.* Let $a \in K$ and $b$ be an arbitrary element of $S$. Then, by (2) of definition of anti-ideal, we have $(a \otimes b)\theta a$ or $a \otimes b \in K$. Since $\neg((a \otimes b) \otimes a \neq a \otimes b)$, we have to $a \otimes b \in K$. The second implication immediately follows from the first.

**Example 4.2.** (1) If $a \in S$, then the set $B(a) = \{x \in S : a\theta x\}$ is an anti-ideal of $S$. Indeed. Let $x \otimes y \in B(a)$, i.e. let $a\theta(x \otimes y)$. Then $a \otimes (x \otimes y) \neq a$. Thus, from $(a \otimes x) \otimes (a \otimes y) = a \otimes (x \otimes y) \neq a \otimes a$ follows $a \otimes x \neq a$ or $a \otimes y \neq a$. Therefore, $x \in B(a)$ or $y \in B(a)$. If $x \in B(a)$, i,e, if $a\theta x$, then $a\theta y$ or $y\theta x$. Thus $y\theta x \vee y \in B(a)$.

(2) If $K$ is an anti-ideal of $S$ and $a \in S$, then subsets $[a : K] = \{y \in S : ay \in K\}$ and $[K : a] = \{y \in S : ya \in K\}$ are anti-ideals of $S$. We give proof for set [a:K]:

$$\begin{aligned} x \otimes y \in [a : K] &\Longleftrightarrow a(x \otimes y) \in K \\ &\Longleftrightarrow ax \otimes ay \in K \qquad\qquad y \in [a : K] \Longleftrightarrow ay \in K \\ &\Longrightarrow ax \in K \vee ay \in K \qquad\qquad \Longrightarrow xa\theta ya \vee xa \in K \\ &\Longleftrightarrow x \in [a : K] \vee y \in [a : K] \Longrightarrow (x\theta y \vee x \in [a : K]). \end{aligned}$$

Proof for set $[K : a]$ is analogous.

In the following proposition we show what kind of connection exists between ideals and anti-ideals:

**Proposition 4.1.** *Let $K$ be an anti-ideal of semilattice-ordered semigroup $(S, =, \neq, \cdot, \otimes)$. Then sets $\neg K$ and $K^C$ are ideals of $S$ and $\neg K = K^C$ holds.*

Proof: (i) Let $a \in \neg K$ and $b \leqslant a$. Suppose that $b \in K$. Then, by part (2) of definition of anti-ideal, we have $a\theta b \vee a \in K$. It is impossible by hypothesis and by Corollary 2.3. So, $b \in \neg K$.
Let $a \in \neg K$ and $b \in \neg K$ Then from $a \otimes b \in K$ we conclude that $a \in K$ or $b \in K$. Since it is impossible, we have $a \otimes b \in K$.
(ii) Since $\neg K \supseteq K^C$ holds we have to prove only the inclusion $\neg K \subseteq K^C$. Let $a$ be an element of $\neg K$ and let $b$ be an arbitrary element of $K$. From (2)of the definition of anti-ideal, we have $a\theta b$ or $a \in K$. So, we have $a\theta b$, i.e. $a \otimes b \neq a = a \otimes a$ by hypothesis $\neg(a \in K)$. Since the operation '$\otimes$' is a strongly extensional function, we have $b \neq a$. Thus, $a \bowtie K$. Therefore, $\neg K = K^C$. □

**Note**. Another proof that $K^C$ is an ideal of $S$ is the following: Let $a \in K^C$ and $b \leqslant a$, and let $t$ be an arbitrary element of $K$. Then $t \neq b$ or $b \in K$. Since, from $b \in K$ follows $a \in K$, by Remark B. It is a contradiction. So, we have $b \bowtie K$.
Let $a \in K^C$ and $b \in K^C$ and let $t$ be an arbitrary element of $K$. Then $t \neq a \otimes b$ or $a \otimes b \in K$. Since the second case is impossible, we conclude that $a \otimes b \in K^C$.

For an ideal $J$ of $S$ and an anti-ideal $K$ of $S$ we say that they are compatible if and only if $J \subseteq \neg K$. For example, if ideal $I$ and anti-ideal $K$ are compatible, then ideal $(a : I)$ and anti-ideal $[a : K]$ are compatible also. Indeed. If $x \in (a : I)$, i.e. if $ax \in I$, then $\neg(ax \in K)$. Thus $\neg(x \in [a : K])$.

**Proposition 4.2.** *Let $K$ be an anti-ideal of a semilattice-ordered semigroup $(S, =, \neq, \cdot, \otimes)$. Then the relation $Q(K)$, defined by*

$$(a,b) \in Q(K) \Longleftrightarrow (\exists x, y \in S^1)((xay \in K \wedge xby \bowtie K) \vee (xby \in K \wedge xay \bowtie K)),$$

*is anti-congruence on $S$. If ideal $I$ and anti-ideal $K$ are compatible, then $E(I)$ and $Q(K)$ are compatible too.*

Proof: (i) If $(a,b) \in Q(K)$, then there exist elements $x, y \in S^1)$ such that $(xay \in K \wedge xby \bowtie K)$ or such that $(xby \in K \wedge xay \bowtie K)$. If $xay \in K \wedge xby \bowtie K$, we have $xay \neq xby$ and $a \neq b$. Analogously we conclude the implication $(xby \in K \wedge xay \bowtie K) \Longrightarrow a \neq b$. So, the relation $Q(K)$ is consistent.

(ii) It is clear that the relation $Q(K)$ is symmetric.

(iii) Let $a, b, c$ be arbitrary elements of $S$ such that $(a,c) \in Q(K)$. Suppose that $(\exists x, y \in S^1)((xay \in K \wedge xcy \bowtie K)$. Let $u$ be an arbitrary element of $K$. Then, by strongly extensionality of $K$, we conclude: $u \neq xby \vee xby \in K$. If $u \neq xby$ and $xay \in K$, then $(a,b) \in Q(K)$. If $xby \in K$ and $xcy \bowtie K$, then $(b,c) \in Q(K)$.

(iv) Let $a, b, c$ be arbitrary elements of $S$ such that $(ac, bc) \in Q(K)$. Suppose that $(\exists x, y \in S^1)((xacy \in K \wedge xbcy \bowtie K)$. Then $(\exists x, cy \in S^1)(xa(cy) \in K \wedge xb(cy) \bowtie K)$. So, $(a,b) \in Q(K)$. Similarly, we prove the implication $(ca, cb) \in Q(K) \Longrightarrow (a,b) \in Q(K)$ Therefore, the relation $Q(K)$ is a coequality relation on $S$ compatible with the operation "$\cdot$".

(v) Suppose that $a, b, c$ are arbitrary elements of $S$ such that $(a \otimes c, b \otimes c) \in Q(K)$. Then there exist elements $x, y \in S^1$ such that $(x(a \otimes c)y \in K \wedge x(b \otimes c)y \bowtie K)$ or $(x(a \otimes c)y \bowtie K \wedge x(b \otimes c)y \in K))$. If $x(a \otimes c)y \in K \wedge x(b \otimes c)y \bowtie K)$, then $(xay \otimes xcy \in K \wedge xby \otimes xcy \bowtie K)$ and $(xay \in K \vee xcy \in K) \wedge (xby \bowtie K \wedge xcy \bowtie K)$. Thus, $(xay \in K \wedge xby \bowtie K)$ because the following $(xcy \in K \wedge xcy \bowtie K)$ is impossible. Therefore, $(a,b) \in Q(K)$.

(vi) Suppose that $(a,b) \in E(I)$ and $(b,c) \in Q(K)$. Then there exists elements $x, y \in S^1$ such that $(xay \in I \Longleftrightarrow xby \in I)$ and there exist elements $u, v \in S^1$ such that $(ubv \in K \wedge ucv \bowtie K)$ or $(ucv \in K \wedge ubv \bowtie K)$. Thus $(\exists u, v \in S^1)((ucv \in K \wedge uav \bowtie K))$ because $I \subseteq \neg K$. Therefore, $(a,c) \in Q(K)$. $\square$

## 5. Main properties

In this section we show some properties of semillatice-ordered semigroups with apartness. In Theorem 5.1 we describe factor-semigroup

$$S/(E(J), Q(K)) = \{aE(J) : a \in S\},$$

where $aE(J) = \{x \in S : (a,x) \in E(I)\}$. In Proposition 5.1 and Proposition 5.2 we give another description of relations $\leqslant_T$ and $\Theta_T$ on semigroup $S/(E(J), Q(K))$. Let $\varphi : S \longrightarrow T$ be a epimorphism of semilattice-ordered semigroups and let $J$ and $L$ be compatible an ideal and an anti-ideal in $T$. Then $I = \varphi^{-1}(J)$ is an ideal and $K = \varphi^{-1}(L)$ is an anti-ideal of $S$. In case of surjective $\varphi$, there exists a homomorphism $\psi : T/(E(J), Q(L)) \longrightarrow S/(E(I), Q(K))$ such that $\pi_S = \psi \circ \pi_T \circ \varphi$.

**Theorem 5.1.** *Let $J$ and $K$ be compatible an ideal and an anti-ideal of a semilattice-ordered semigroup $(S, =, \neq, \cdot, \otimes)$. Then the factor-structure*

$$S/(E(J), Q(K)) = \{aE(J) : a \in S\}$$

*with equality and coequality defined by*

$$aE(J) =_1 bE(J) \Longleftrightarrow (a,b) \in E(J),$$
$$aE(J) \neq_1 bE(J) \Longleftrightarrow (a,b) \in Q(K),$$

*and internal operations by*

$$aE(J) \circ bE(J) = abE(J),\ aE(J) \diamond bE(J) = (a \otimes b)E(J)$$

*is a semilattice-ordered semigroup. Besides, the mapping*

$$\pi_S : S \ni a \longmapsto aE(J) \in S/(E(J), Q(K))$$

*is a strongly extensional surjective homomorphism.*

Proof: (1) The function "$\circ$" is well-defined and strongly extensional. Indeed.

$(aE(J) =_1 xE(J) \wedge bE(J) =_1 yE(J)) \Longleftrightarrow (a,x) \in E(J) \wedge (b,y) \in E(J)$

$\Longrightarrow (ab, xb) \in E(J) \wedge (xb, xy) \in E(J)$

$\Longrightarrow (ab, xz) \in E(J)$

$\Longrightarrow abE(J) =_1 xyE(J);$

$abE(J) \neq_1 xyE(J) \Longleftrightarrow (ab, xy) \in Q(K)$

$\Longrightarrow ((ab, xb) \in Q(K) \vee (xb, xy) \in Q(K))$

$\Longrightarrow ((a,x) \in Q(K) \vee (b,y) \in Q(K))$

$\Longrightarrow (aE(J) \neq_1 xE(J) \vee bE(J) \neq_1 yE(J)).$

(2) The function "$\diamond$" is well-defined and strongly extensional. Indeed.

$(aE(J) =_1 xE(J) \wedge bE(J) =_1 yE(J)) \Longleftrightarrow ((a,x) \in E(J) \wedge (b,y) \in E(J))$

$\Longleftrightarrow (\forall u, v \in S^1)(uav \in J \Longleftrightarrow uxv \in J) \wedge (\forall u, v \in S^1)(ubv \in J \Longleftrightarrow uyv \in J)$

$\Longrightarrow (\forall u, v \in S^1)(u(a \otimes b)v \in J \Longleftrightarrow u(x \otimes y)v \in J)$
$\Longrightarrow (a \otimes b, x \otimes y) \in E(J)$
$\Longrightarrow (a \otimes b)E(J) =_1 (x \otimes y)E(J)$
$\Longleftrightarrow aE(J) \diamond bE(J) =_1 xE(J) \diamond yE(J)$.
$aE(J) \diamond bE(J) \neq_1 xE(J) \diamond yE(J) \Longleftrightarrow (a \otimes b)E(J) \neq_1 (x \otimes y)E(J)$
$\Longleftrightarrow (a \otimes b, x \otimes y) \in Q(K)$
$\Longleftrightarrow (\exists u, v \in S^1)((u(a \otimes b)v \in K \wedge u(x \otimes y)v) \bowtie K) \vee$
$(u(a \otimes b)v \bowtie K \wedge u(x \otimes y)v \in K))$
$\Longrightarrow (\exists u, v \in S^1)((uav \in K \vee ubv \in K) \wedge (uxv \bowtie K \wedge uyv \bowtie K)) \vee$
$((uav \bowtie K) \wedge (ubv \bowtie K) \wedge (uxv \in K \wedge uyv \in K))$
$\Longrightarrow (a, x) \in Q(K) \vee (b, y) \in Q(K)$
$\Longleftrightarrow aE(J) \neq_1 xE(J) \vee bE(J) \neq_1 yE(J)$ .
(3) It is clear that the operation "$\diamond$" on $S/(E(J), Q(K))$ is commutative because the operation "$\otimes$" is commutative and $aE(J) \diamond aE(J) =_1 (a \otimes a)E(J) =_1 aE(J)$ holds.
(4) It is clear that $\pi_S$ is a surjective function. Let $a$, $b$ be elements of $S$ such that $\pi_S(a) \neq_1 \pi_S(b)$, i.e. such that $aE(J) \neq_1 bE(J)$. Then $(a, b) \in Q(K)$. Thus, $a \neq b$. So, the mapping $\pi_S$ is strongly extensional. □

**Proposition 5.1.** *Let $I$ be an ideal of a semilattice-ordered semigroup $S$. Then*

$$aE(I) \leqslant_T bE(I) \Longleftrightarrow (\forall x, y \in S^1)(xby \in I \Longrightarrow xay \in I).$$

Proof: (1) Let $aE(I) \leqslant_T bE(I)$, i.e. let $aE(I) \diamond bE(I) =_1 bE(b)$. Then $(a \otimes b)E(I) =_1 bE(I)$, i.e. then $(\forall x, y \in S^1)(x(ab)y \in I \Longleftrightarrow xby \in I)$. Since $xay \otimes xby \in I$ and $xay \leqslant xay \otimes xby$, follows $xay \in I$. So, we have $(\forall x, y \in S^1)(xay \in I \Longleftarrow xby \in I)$.
(2) Let $(\forall x, y \in S^1)(xay \in I \Longleftarrow xby \in I)$ holds and, of course, $(\forall x, y \in S^1)(xay \otimes xby \in I \Longleftarrow xby \in I)$. Opposite, the implication $xay \otimes xby \in I \Longrightarrow xby \in I$ $(x, y \in S^1)$ holds by definition of ideal. So, we have $(\forall x, y \in S^1)(xay \otimes xby \in I \Longleftrightarrow xby \in I)$. So, $aE(I) \leqslant_T bE(I) \Longleftrightarrow (\forall x, y \in S^1)(xby \in I \Longrightarrow xay \in I)$. □

**Proposition 5.2.** *Let $I$ and $K$ be compatible an ideal and anti-ideal of a semilattice-ordered semigroup $S$. Then*

$$aE(I)\Theta_T bE(I) \Longleftrightarrow (\exists x, y \in S^1)(xby \in K \wedge xay \bowtie K).$$

Proof:
$aE(I) \diamond bE(I) \neq_T aE(I) \Longleftrightarrow ((a \otimes b)E(I), aE(I)) \in Q(K)$
$\Longleftrightarrow (\exists u, v \in S^1)((u(a \otimes b)v \in K \wedge uav \bowtie K) \vee (uav \in K \wedge u(ab)v \bowtie K))$

$\Longleftrightarrow (\exists u, v \in S^1)(uav \otimes ubv \in K \wedge uav \bowtie K) \vee (uav \in K \wedge uav \otimes ubv \bowtie K))$
$\Longrightarrow (\exists u, v \in S^1)(((uav \in K \vee ubv \in K) \wedge uav \bowtie K) \vee$
$(uav \in K \wedge uav \bowtie K \wedge ubv \bowtie K))$
$\Longrightarrow (\exists u, v \in S^1)(ubv \in K \wedge uav \bowtie K)$
$\Longleftrightarrow aE(I)\Theta_T bE(I)$.
Opposite, let $aE(I)\Theta_T bE(I)$. Then $(\exists u, v \in S^1)(ubv \in K uav \wedge \bowtie K)$. Thus

$$(\exists u, v \in S^1)((uav \otimes ubv)\theta ubv \vee uav \otimes ubv \in K) \wedge uav \bowtie K),$$

i.e.

$$(\exists u, v \in S^1)((uav \otimes ubv) \otimes ubv \neq (uav \otimes ubv) \vee uav \otimes ubv \in K) \wedge uav \bowtie K).$$

Therefore $(\exists u, v \in S^1)(uav \otimes ubv \in K \wedge uav \bowtie K)$ because

$$(uav \otimes ubv) \otimes ubv \neq (uav \otimes ubv)$$

is impossible. So, we have $(\exists u, v \in S^1)(u(a \otimes b)v \in K \wedge uav \bowtie K)$. This means that $((a \otimes b)E(I), aE(I)) \in Q(K)$, i.e. that $aE(I) \diamond bE(I) \neq_T aE(I)$. $\square$

**Lemma 5.1.** *Let $f : S \longrightarrow T$ be a strongly extensional homomorphism of semilattice-ordered semigroups. Then $Kerf$ and $Antikerf = \{(x, y) \in S \times S : f(x) \neq f(y)\}$ are compatible congruence and anti-congruence on $S$ and there exists homomorphism $g : S/(Kerf, Antikerf) \longrightarrow T$ such that $f = g \circ \pi_S$, where $\pi_S : S \longrightarrow S/(Kerf, Antikerf)$ the canonical strongly extensional epimorphism.*

Proof: Compound of this assertion is well-known. $\square$

**Theorem 5.2.** *Let $(I, K)$ and $(J, L)$ be compatible pairs of ideal and anti-ideal of a semilattice-ordered semigroup $S$. Then:*
*(1) $E(I \cap J) \supseteq E(I) \cap E(J)$ and $Q(K \cup L) \subseteq Q(K) \cup Q(L)$.*
*(2) There a homomorphism $S/(E(I \cap J), Q(K \cup L)) \longrightarrow S/E(I) \times S/E(J)$.*
*(3) $E((a : I)) \supseteq E(I)$ and $Q([a : K]) \subseteq Q(K)$, and*
*(4) There a homomorphism $S/(E((a : I)), Q([a : K])) \longrightarrow S/(E(I), Q(K))$.*

Proof: (1) The inclusion $E(I \cap J) \supseteq E(I) \cap E(J)$ follows from the definitions of $E(I)$, $E(J)$ and $E(IJ)$ and the inclusion $Q(K \cup L) \subseteq Q(K) \cup Q(L)$ follows from definitions of $Q(K)$, $K(L)$ and $Q(K \cup L)$.
(2) If we define $f : S \longrightarrow S/E(I) \times S/E(J)$ by $f(x) = (xE(I), xE(J))$, then $Kerf = E(I) \cap E(J)$ and $Antikerf = Q(K) \cup Q(L)$. Therefore,

by Lemma 5.1, there exists homomorphism $g : S/(E(I)\cap E(J), Q(K)\cup Q(L)) \longrightarrow S/E(I) \times S/E(J)$. Since $E(I \cap J) \supseteq E(I) \cap E(J)$ and $Q(K \cup L) \subseteq Q(K) \cup Q(L)$, there exists homomorphism $h : S/(E(I \cap J), Q(K \cup L)) \longrightarrow S/E(I) \times S/E(J)$.
(3) Let $(x, y)$ be an arbitrary element of $E(I)$, i.e. let

$$(\forall u, v \in S^1)(uxv \in I \Longleftrightarrow uyv \in I).$$

Thus,

$$(\forall u, v \in S^1)(auxv \in I \Longleftrightarrow auyv \in I).$$

So,

$$(\forall u, v \in S^1)(uxv \in (a : I) \Longleftrightarrow uyv \in (a : I)),$$

i.e. $(x, y) \in E((a : I))$. Therefore, $E(I) \subseteq E((a : I))$.
Let $(x, y)$ be an arbitrary element of $Q([a : K])$, i.e. let

$$(\exists u, v \in S^1)((uxv \in [a : K] \wedge uyv \wedge [a : K]) \vee (uxv \bowtie [a : K] \wedge uyv \in [a : K])).$$

Then

$$(\exists u, v \in S^1)((auxv \in K uyv \bowtie [a : K]) \vee (uxv \bowtie [a : K] \wedge auyv \in K)).$$

For simplicity, suppose that $auxv \in K \wedge uyv \bowtie [a : K]$ and let $t$ be an arbitrary element of $K$. Then $t \neq auxv$ or $auxv \in K$. Since the second case $auxv \in K(\Longleftrightarrow uxv \in [a : K])$ is impossible, we have to $auxv \bowtie K$. Thus, there exists elements $au, v \in S^1$ such that $(au)xv \in K \wedge (au)xv \bowtie K$. So, $(x, y) \in Q(K)$. Therefore $Q([a : K]) \subseteq Q(K)$.
It is clear that there exists homomorphism $g : S/(E((a : I)), Q([a : K])) \longrightarrow S/(E(I), Q(K))$. □

**Theorem 5.3.** *Let $\varphi : S \longrightarrow T$ be a homomorphism of semilattice-ordered semigroups and let $J$ and $L$ be compatible an ideal and an anti-ideal in $T$. Then*
*(i) $I = \varphi^{-1}(J)$ is an ideal and $K = \varphi^{-1}(L)$ is an anti-ideal of $S$;*
*(ii) In case of surjective $\varphi$, there exists a homomorphism $\psi : T/(E(J), Q(L)) \longrightarrow S/(E(I), Q(K))$ such that $\pi_S = \psi \circ \pi_T \circ \varphi$.*

Proof: (1) It is easy to see that $\varphi^{-1}(J)$ and $\varphi^{-1}(L)$ are compatible an ideal and an anti-ideal of $S$.
(2) Define $\psi : bE(J) \longmapsto aE(I)$ where $\varphi(a) = b$, $a \in S$, $b \in T$. This assignment is really a mapping. Let $\varphi(a) = b$ and $\varphi(a') = b'$. Then:
$bE(J) = b'E(J) \Longleftrightarrow (b, b') \in E(J)(\Longleftrightarrow (\forall x, y \in T^1)(xby \in J \Longleftrightarrow xb'y \in J))$
$\Longrightarrow |x = \varphi(u), y = \varphi(v), b = \varphi(a), b' = \varphi(a')$

$\Longrightarrow (\forall u, v \in S^1)(\varphi(u)\varphi(a)\varphi(v) \in J \Longleftrightarrow \varphi(u)\varphi(a')\varphi(v) \in J)$
$\Longleftrightarrow (\forall u, v \in S^1)(uav \in \varphi^{-1}(J) \Longleftrightarrow ua'v \in \varphi^{-1}(J))$
$\Longleftrightarrow (a, a') \in E(I)$
$\Longleftrightarrow aE(I) =_1 a'E(I)$.

Let $aE(I)$, $a'E(I)$ be arbitrary elements of $S/(E(I), Q(K)$ such that $aE(I) \neq_1 a'E(I)$. Then there exist elements $u, v \in S^1$ such that $(uav \in K \wedge ub'v \bowtie K)$ or $(uav \bowtie K \wedge ub'v \in K)$. Suppose that $\varphi(a) = b$, $\varphi(a') = b'$, $\varphi(u) = x$ and $\varphi(v) = y$ and suppose $(\exists u, v \in S^1)((uav \in \varphi^{-1}(L) \wedge ua'v \bowtie \varphi^{-1}(L))$. Then there exists element $x = \varphi(u)$ and $y = \varphi(v)$ of $im\varphi = T$ such that $\varphi(u)\varphi(a)\varphi(v) \in L$. Let $t$ be an arbitrary element of $L$. Then $t \neq \varphi(ua'v)$ or $\varphi(u)\varphi(a')\varphi(v) \in L$. In the second case we should have $ua'v \in \varphi^{-1}(L)$ which is impossible. So, $\varphi(u)\varphi(a')\varphi(v) \bowtie L$. Therefore, there exist elements $x = \varphi(u)$ and $y = \varphi(v)$ such that $xby \in L$ and $xb'y \bowtie L$. Thus $(b, b') \in Q(L)$. Finally, the mapping $\psi$ is strongly extensional.

Let $bE(J)$ and $b'E(J)$ be arbitrary elements of $T/(E(J), Q(L))$. Then there exist elements $a$ and $a'$ of $S$ such that $\psi : bE(J) \longmapsto aE(I)$ and $\psi : b'E(J) \longmapsto a'E(I)$. Since $\varphi(aa') = \varphi(a)\varphi(a') = bb'$ and $(a \otimes a') = \varphi(a) \otimes \varphi(a') = b \otimes b'$ we conclude that $\psi$ is a homomorphism.

(3) The equality $\pi_S = \psi \circ \pi_T \circ \varphi$ immediately follows from definitions of homomorphisms $\pi_S : S \longrightarrow S/(E(I), Q(K))$, $\psi$, $\pi_T : T \longrightarrow T/(E(J), Q(L))$ and $\varphi$. $\square$

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## References

[1] E. Bishop, **Foundations of Constructive Analysis**, McGraw-Hill, New York, 1967.

[2] E. Bishop and D.S. Bridges, **Constructive Analysis**, Grundlehren der mathematischen Wissenschaften 279, Springer, Berlin, 1985.

[3] S. Bogdanović and M. Ćirić: **Semigroups**; Prosveta, Nis 1993. (In Serbian)

[4] D.S. Bridges and F. Richman, **Varieties of Constructive Mathematics**, London Mathematical Society Lecture Notes 97, Cambridge University Press, Cambridge, 1987

[5] D. S. Bridges and L. S. Vita, **Techniques of Constructive Analysis**, Springer, New York, 2006.

[6] Martin Kuril and Libor Polak: **On Varieties of Semilattice-ordered Semigroups**; Semigroup Forum, 71(1) (2005), 27-48

[7] R. Mines, F. Richman and W. Ruitenburg: **A Course of Constructive Algebra**; Springer-Verlag, New York 1988.

[8] J. von Plato, **Positive lattices**, In: **Reuniting the Antipodes-Constructive and Nonstandard Views of the Continuum** (P. Schuster, U. Berger, H. Osswald eds.), Kluwer, Dordrecht, 2001, 185-197

[9] L. Polak: **A Classification of Rational Languages by Semilattice-ordered Monoids**; Archivum Mathematicum (Brno), 40(2000), 395-406.

[10] D.A.Romano: **Semivaluations on Heyting Fields**; Kragujevac Journal of Mathematics, 20(1998), 24-40.

[11] D. A. Romano: **A Left Compatible Coequality Relation on Semigroup with Apartness;** Novi Sad J. Math, 29(2)(1999), 221-234.

[12] D. A. Romano: **Some Relations and Subsets Generated by Principal Consistent Subset of Semigroup with Apartness**; Univ. Beograd. Publ. Elektotehn. Fak. Ser. Math, 13(2002), 7-25.

[13] D.A.Romano: **An Isomorphism Theorem for Anti-ordered Sets**; Filomat, 22(1)(2008), 145-160

[14] A. S. Troelstra and D. van Dalen: **Constructivism in Mathematics, An Introduction, Volume II**; North - Holland, Amsterdam 1988.

Author's addresses

Faculty of Mechanical Engineering, 78000 Banja Luka,
71, Vojvoda Stepa Stepanovic Street, Bosnia and Herzegovina

Faculty of Education, 76300 Bijeljina,
Semberskih ratara Street, b.b., Bosnia and Herzegovina

e-mail: bato49@hotmail.com