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ON WEAKLY θ-PRE-I CONTINUOUS FUNCTIONS

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Abstract. In this paper, a strong form in ideal topological spaces of weak θ-pre continuity, called weak θ-pre-I continuity, is introduced. It is shown that weak θ-pre-I continuity is strictly weaker than strong θ-pre-I continuity and that it is between continuity and almost weak continuity. Also additional properties of these functions are investigated.

1. Introduction and preliminaries

Weakly θ-pre continuous functions were developed by Baker [5]. The purpose of this note is to introduce a strong form of weak θ-pre continuity, which we call weak θ-pre-I continuity. We establish that weak θ-pre-I continuity is strictly weaker than strong θ-pre-I continuity and that it is strictly between continuity and almost weak continuity. Also it is shown that weak θ-pre-I continuity is independent of weak continuity. Conditions are proved under which weak θ-pre-I continuity and weak continuity are related. Properties related to the graph of a weakly θ-pre-I continuous function are investigated. We show that the graph of a weakly θ-pre-I continuous function with a Hausdorff codomain is strongly pre-I-closed. Finally, additional properties of these functions are investigated.

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Throughout this paper, $Cl(A)$ and $Int(A)$ denote the closure and interior of A, respectively. Let (X,τ) be a topological space and let I be an ideal of subsets of X. An ideal is defined as a nonempty collection I of subsets of X satisfying the following two conditions: 1) If $A\in I$ and $B\subset A$, then $B\in I$; 2) If $A\in I$ and $B\in I$, then $A\cup B\in I$. An ideal topological space is a topological space (X,τ) with an ideal I on X and is denoted by (X,τ,I). For a subset $A\subset X$, $A^{*}(I)=\{x\in X: U\cap A\notin I \text{ for each neighbourhood } U\ of\ x\}$ is called the local function of A with respect to I and τ [13]. We simply write A^{*} instead of $A^{*}(I)$ in case there is no chance for confusion. For every ideal topological space (X,τ,I), there exists a topology $\tau^{*}(I)$, finer than τ, generated by $\beta(I,\tau)=\{U-A: U\in\tau \text{ and A} \in I\}$, but in general $\beta(I,\tau)$ is not a topology [10]. Additionally, $Cl^{*}(A)=A\cup A^{*}$ defines a Kuratowski closure operator for $\tau^{*}(I)$. We recall some known definitions.

A subset A of a topological space (X,τ) is said to be pre open [1] if $A\subset Int(Cl(A))$. The family of all pre open sets of X is denoted by $PO(X)$. The family of all pre open sets of X containing a point $x\in X$ is denoted by $PO(X,x)$.

The complement of a pre open set is called preclosed [1].The intersection of all pre closed sets containing A is called preclosure [2] of A and is denoted by $pCl(A)$.A point $x\in X$ is called a precluster point of A if $U\cap A\neq\emptyset$ for each $U\in PO(X,x)$ [8]. A subset A is pre closed if and only if $pCl(A)=A$ [8]. The preinterior [2] of A is defined by the union of all sets contained in A and is denoted by $pInt(A)$. A point $x\in X$ is called a pre-θ-cluster point of A if $pCl(U)\cap A\neq\emptyset$ for every $PO(X,x)$ [7]. The set of all pre-θ-cluster points of A is called the pre-θ-closure of A and denoted by $pCl_{\theta}(A)$. A subset A is said to be pre-θ-closed if $pCl_{\theta}(A)=A$ [7].

A subset A of an ideal topological space (X,τ,I) is said to be pre-I-open [6] (semi-I-open [11], α-I-open [11]) if $A\subset Int(Cl^{*}(A))$, $(A\subset Cl^{*}(Int(A))$, $A\subset Int(Cl^{*}(Int(A))))$. The complement of a pre-I-open set is called pre-I-closed [6]. The family of all pre-I-open sets of X is denoted by $PIO(X)$ [6]. The family of all pre-I-open sets of X containing a point $x\in X$ is denoted by $PIO(X,x)$ [6]. A point $x\in X$ is called a pre-I-cluster point [4] of A if $U\cap A\neq\emptyset$ for each $U\in PIO(X,x)$. The set of all pre-I-cluster points of A is called the pre-I-closure of A and denoted by ${}_{PI}Cl(A)$. A subset A is pre-I-closed if and only if ${}_{PI}Cl(A)=A$ [4].A point $x\in X$ is called a pre-I-θ-cluster point [3] of A if ${}_{PI}Cl(U)\cap A\neq\emptyset$ for each $U\in PIO(X,x)$. The set of all pre-I-θ-cluster points of A is called the pre-I-θ-closure of A and denoted by${}_{PI}Cl_{\theta}(A)$. A subset A is pre-I-θ-closed if and only if ${}_{PI}Cl_{\theta}(A) = A$ [3]. A point $x \in X$ is called pre-I-interior point [4] (pre-I-θ-interior point [3]) of A if there exists $U \in PIO(X, x)$ such that $x \in U \subset A$ ($x \in {}_{PI}Cl(U) \subset A$). The set of all pre-I-interior (pre-I-θ-interior) points of A is said to be pre-I-interior (pre-I-θ-interior) of A and denoted by ${}_{PI}Int(A)$[4] (${}_{PI}Int_{\theta}(A)$ [3]).

Definition 1. *[5] A function $f : (X, \tau) \to (Y, \nu)$ is said to be weakly θ-pre continuous if $pCl_{\theta}(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every open subset V of Y.*

Definition 2. *[14] A function $f : (X, \tau) \to (Y, \nu)$ is said to be weakly continuous, if for every $x \in X$ and every neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset Cl(V)$.*

Definition 3. *[12] A function $f : (X, \tau) \to (Y, \nu)$ is said to be almost weakly continuous if $f^{-1}(V) \subset Int(Cl(f^{-1}(Cl(V)))$ for every open subset V of Y.*

Definition 4. *[9] A function $f : (X, \tau, I) \to (Y, \nu)$ is said to be strongly θ-pre-I continuous, if for every $x \in X$ and every open subset V of Y containing $f(x)$, there exists $U \in PIO(X, x)$ such that $f({}_{PI}Cl(U)) \subset V$.*

Lemma 1. *Let A and B be subsets of (X, τ, I). Then the following properties hold:*

(i) ${}_{PI}Cl_{\theta}({}_{PI}Cl_{\theta}(A)) = {}_{PI}Cl_{\theta}(A)$
(ii) $A \subset B \Rightarrow {}_{PI}Cl_{\theta}(A) \subset {}_{PI}Cl_{\theta}(B)$
(iii) ${}_{PI}Int_{\theta}(X - A) = (X - {}_{PI}Cl_{\theta}(A))$
(iv) ${}_{PI}Cl_{\theta}(X - A) = (X - {}_{PI}Int_{\theta}(A))$.

Proof. (i): Let ${}_{PI}Cl_{\theta}(A) = B$. Since ${}_{PI}Cl_{\theta}(A)$ is a pre-I-θ-closed set, B is a pre-I-θ-closed set and hence ${}_{PI}Cl_{\theta}(B) = B$, it is obvious that ${}_{PI}Cl_{\theta}({}_{PI}Cl_{\theta}(A)) = {}_{PI}Cl_{\theta}(A)$. This proves (i).

The proof of (ii) is obvious.

(iii): Let $x \in (X - {}_{PI}Cl_{\theta}(A))$. Since $x \notin {}_{PI}Cl_{\theta}(A)$, there exists $U \in PIO(X, x)$ such that ${}_{PI}Cl(U) \cap A = \emptyset$. Hence we obtain $x \in {}_{PI}Cl(U) \subset (X - A)$ and $x \in {}_{PI}Int_{\theta}(X - A)$. This shows that $(X - {}_{PI}Cl_{\theta}(A)) \subset {}_{PI}Int_{\theta}(X - A)$. Let $x \in {}_{PI}Int_{\theta}(X - A)$. Then, there exists $U \in PIO(X)$ containing x such that $x \in {}_{PI}Cl(U) \subset (X - A)$. Then ${}_{PI}Cl(U) \cap A = \emptyset$. Hence $x \notin {}_{PI}Cl_{\theta}(A)$ and $x \in$

$(X - {}_{PI}Cl_\theta(A))$. This shows that ${}_{PI}Int_\theta(X - A) \subset (X - {}_{PI}Cl_\theta(A))$. Therefore, we obtain ${}_{PI}Int_\theta(X - A) = (X - {}_{PI}Cl_\theta(A))$.

Since $X - {}_{PI}Cl_\theta(X - A) = {}_{PI}Int_\theta(X - (X - A)) = {}_{PI}Int_\theta(A)$, the statement in (iv) is obviously implied by (iii). ∎

2. Weakly θ-pre-I continuous functions and relationships with other forms of continuity

Definition 5. *A function $f : (X, \tau, I) \to (Y, \nu)$ is said to be weakly θ-pre-I continuous if ${}_{PI}Cl_\theta(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every open subset V of Y.*

The following theorem gives characterizations of weakly θ-pre-I continuous functions.

Theorem 2. *The following statements are equivalent for a function $f : (X, \tau, I) \to (Y, \nu)$:*

(a) f is weakly θ-pre-I continuous;
(b) ${}_{PI}Cl_\theta(f^{-1}(Int(Cl(B)))) \subset f^{-1}(Cl(B))$ for every $B \subset Y$;
(c) ${}_{PI}Cl_\theta(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every pre-I-open $V \subset Y$;
(d) ${}_{PI}Cl_\theta(f^{-1}(Int(Cl(V)))) \subset f^{-1}(Cl(V))$ for every open $V \subset Y$;
(e) ${}_{PI}Cl_\theta(f^{-1}(Int(F))) \subset f^{-1}(F)$ for every closed $F \subset Y$;
(f) $f^{-1}(Int(B)) \subset {}_{PI}Int_\theta(f^{-1}(Cl(Int(B))))$ for every $B \subset Y$;
(g) $f^{-1}(Int(F)) \subset {}_{PI}Int_\theta(f^{-1}(F))$ for every pre-I-closed $F \subset Y$;
(h) $f^{-1}(V) \subset {}_{PI}Int_\theta(f^{-1}(Cl(V)))$ for every open $V \subset Y$.

Proof. (a)⇒(b) Let B be any subset of Y. Since $Int(Cl(B))$ is open in Y, by (a)

$$\begin{aligned} {}_{PI}Cl_\theta\left(f^{-1}(Int(Cl(B)))\right) &\subset f^{-1}(Cl(Int(Cl(B)))) \\ &\subset f^{-1}(Cl(Cl(B))) = f^{-1}(Cl(B)) \end{aligned}$$

and hence we obtain ${}_{PI}Cl_\theta(f^{-1}(Int(Cl(B)))) \subset f^{-1}(Cl(B))$.

(b)⇒(c) Let V be any pre-I-open subset of Y. For every pre-I-open set, we can see easily that $V \subset Int(Cl^*(V)) \subset Int(Cl(V))$ and by Lemma 1,

$${}_{PI}Cl_\theta\left(f^{-1}(V)\right) \subset {}_{PI}Cl_\theta\left(f^{-1}(Int(Cl(V)))\right)$$

and by (b) we have ${}_{PI}Cl_\theta(f^{-1}(V)) \subset {}_{PI}Cl_\theta(f^{-1}(Int(Cl(V)))) \subset f^{-1}(Cl(V))$.

(c)⇒(d) Let V be any open subset of Y. Since every open set is pre-I-open [6], $Int(Cl(V))$ is an open set hence a pre-I-open set of Y.

By (c) we have

$$_{PI}Cl_\theta\left(f^{-1}(Int(Cl(V)))\right) \subset f^{-1}(Cl(Int(Cl(V)))) \\ \subset f^{-1}(Cl(Cl(V))) = f^{-1}(Cl(V)).$$

Hence we obtain $_{PI}Cl_\theta(f^{-1}(Int(Cl(V)))) \subset f^{-1}(Cl(V))$.

(d)⇒(e) Let F be any closed subset of Y. Then $Int(F)$ is an open set. We get

$$_{PI}Cl_\theta\left(f^{-1}(Int(Cl(Int(F))))\right) \subset f^{-1}(Cl(Int(F)))$$

and

$$_{PI}Cl_\theta\left(f^{-1}(Int(F))\right) \subset {}_{PI}Cl_\theta\left(f^{-1}(Int(Cl(Int(F))))\right) \\ \subset f^{-1}(Cl(Int(F))) \subset f^{-1}(Cl(F)) = f^{-1}(F).$$

Therefore $_{PI}Cl_\theta(f^{-1}(Int(F))) \subset f^{-1}(F)$.

(e)⇒(f) Let B be any subset of Y. Since $Cl(Y-B)$ is a closed set we get

$$_{PI}Cl_\theta\left(f^{-1}(Int(Cl(Y-B)))\right) \subset f^{-1}(Cl(Y-B))$$

and by Lemma 1

$$X - {}_{PI}Int_\theta\left(f^{-1}(Cl(Int(B)))\right) \subset X - f^{-1}(Int(B)).$$

Hence we obtain $f^{-1}(Int(B)) \subset {}_{PI}Int_\theta(f^{-1}(Cl(Int(B))))$.

(f)⇒(g) Let F be a pre-I-closed subset of Y. By (f) we have,

$$f^{-1}(Int(F)) \subset {}_{PI}Int_\theta\left(f^{-1}(Cl(Int(F)))\right)$$

and since F is pre-I-closed,

$$f^{-1}(Int(F)) \subset {}_{PI}Int_\theta\left(f^{-1}(Cl(Int(F)))\right) \subset {}_{PI}Int_\theta\left(f^{-1}(F)\right)$$

and $f^{-1}(Int(F)) \subset {}_{PI}Int_\theta(f^{-1}(F))$.

(g)⇒(h) Let V be any open subset of Y. Then $Cl(V)$ is closed and every closed set is pre-I-closed[6]. By (g),

$$f^{-1}(Int(Cl(V))) \subset {}_{PI}Int_\theta\left(f^{-1}(Cl(V))\right)$$

Since V is open,

$$f^{-1}(V) \subset f^{-1}(Int(Cl(V))) \subset {}_{PI}Int_\theta\left(f^{-1}(Cl(V))\right)$$

Consequently, $f^{-1}(V) \subset {}_{PI}Int_\theta(f^{-1}(Cl(V)))$.

(h)⇒(a) Let V be any open subset of Y. Since $Int(Y-V)$ is open, we obtain

$$f^{-1}(Int(Y-V)) \subset {}_{PI}Int_\theta\left(f^{-1}(Cl(Int(Y-V)))\right)$$

and by Lemma 1 $X - f^{-1}(Cl(V)) \subset X - {}_{PI}Cl_\theta(f^{-1}(Int(Cl(V))))$. Hence we have

$$ {}_{PI}Cl_\theta\left(f^{-1}(Int(Cl(V)))\right) \subset f^{-1}(Cl(V)) $$

and

$$ \begin{aligned} {}_{PI}Cl_\theta\left(f^{-1}(V)\right) &= {}_{PI}Cl_\theta\left(f^{-1}(Int(V))\right) \\ &\subset {}_{PI}Cl_\theta\left(f^{-1}(Int(Cl(V)))\right) \subset f^{-1}(Cl(V)) \end{aligned} $$

and ${}_{PI}Cl_\theta(f^{-1}(V)) \subset f^{-1}(Cl(V))$. Thus f is weakly θ-pre-I continuous. ∎

Lemma 3. *Let (X, τ, I) be an ideal topological space. $pCl_\theta(A) \subset {}_{PI}Cl_\theta(A)$ for every subset $A \subset X$.*

Proof. Let $x \in pCl_\theta(A)$ and $U \in PIO(X)$ such that $x \in U$. Since every pre-I-open set is pre open [6], $pCl(U) \cap A \neq \emptyset$ and we know that $pCl(U) \subset {}_{PI}Cl(U)$ for every subset $U$of X, hence ${}_{PI}Cl(U) \cap A \neq \emptyset$. Therefore $x \in {}_{PI}Cl_\theta(A)$. ∎

By Lemma 3, obviously every weakly θ-pre-I continuous function is weakly θ-pre continuous. The following example shows that the converse implication does not hold.

Example 1. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$, $\nu = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $I = \{\emptyset, \{b\}\}$. The identity mapping $f : (X, \tau, I) \to (X, \nu)$ is weakly θ-pre continuous but not weakly θ-pre-I continuous.*

Theorem 4. *[9] A function $f : (X, \tau, I) \to (Y, \nu)$ is strongly θ-pre-I continuous if and only if ${}_{PI}Cl_\theta(f^{-1}(B)) \subset f^{-1}(Cl(B))$ for every subset B of Y.*

Obviously strongly θ-pre-I continuity implies weakly θ-pre-I continuity. The following example shows that the converse implication does not hold.

Example 2. *Let $X = \{a, b\}$, $\tau = \{X, \emptyset, \{a\}\}$ and $\mathbf{I} = \{\emptyset, \{\mathbf{b}\}\}$. The function $f : (X, \tau, I) \to (X, \tau)$ given by $f(a) = b$ and $f(b) = a$ is weakly θ-pre-I continuous but not strongly θ-pre-I continuous.*

Lemma 5. *If a subset A of an ideal topological space (X, τ, I) is open, then $Cl(A) = {}_{PI}Cl_\theta(A)$.*

Proof. Let A be any open subset of (X,τ,I). We must show that ${}_{PI}Cl_\theta(A) \subset Cl(A)$. Let $x \notin Cl(A)$. Then, there exists an open subset U containing x such that $U \cap A = \emptyset$. Then $U \subset (X-A)$, X-A is closed. Since every closed set is pre-I-closed, ${}_{PI}Cl(U) \subset {}_{PI}Cl(X-A) = X-A$. Hence we have ${}_{PI}Cl(U) \cap A = \emptyset$ and $x \notin {}_{PI}Cl_\theta(A)$. Thus we obtain; ${}_{PI}Cl_\theta(A) \subset Cl(A)$.
Now we must show that $Cl(A) \subset {}_{PI}Cl_\theta(A)$. Let $x \in Cl(A)$ and $x \in V \in PIO(X)$. Then $x \in V \subset Int(Cl^*(V))$ and hence $A \cap Int(Cl^*(V)) \neq \emptyset$. Since A is open

$$\begin{aligned} A \cap Int(Cl^*(V)) &\subset A \cap Int(Cl(V)) = Int(A) \cap Int(Cl(V)) \\ &= Int(A \cap Cl(V)) \subset Int(Cl(A \cap V)) \subset Cl(A \cap V). \end{aligned}$$

Therefore, we obtain $Cl(A \cap V) \neq \emptyset$ and hence $A \cap V \neq \emptyset$.

$$(A \cap V) \cup (A \cap Cl(Int(V))) = A \cap (V \cup Cl(Int(V))) = A \cap {}_PCl(V).$$

Since $A \cap V \neq \emptyset$, $A \cap pCl(V) \neq \emptyset$. By the pre-$\theta$ cluster point definition, $x \in pCl_\theta(A)$. By Lemma 3, $pCl_\theta(A) \subset {}_{PI}Cl_\theta(A)$ and hence $x \in {}_{PI}Cl_\theta(A)$. This shows that $Cl(A) = {}_{PI}Cl_\theta(A)$ for any open set A of X. ∎

Theorem 6. *If $f : (X,\tau,I) \to (Y,\nu)$ is continuous, then f is weakly θ-pre-I continuos.*

Proof. Let $V \subset Y$ be open. Then, since f is continuous $f^{-1}(V)$ is open ,

$${}_{PI}Cl_\theta\left(f^{-1}(V)\right) = Cl\left(f^{-1}(V)\right).$$

Also, $Cl(f^{-1}(V)) \subset f^{-1}(Cl(V))$. We can easily see that ${}_{PI}Cl_\theta(f^{-1}(V)) = Cl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ and hence f is weakly θ-pre-I continuous. ∎

Remark 1. *Example 2. shows that a weakly θ-pre-I continuous function is not continuous.*

Thus weak θ-pre-I continuity is strictly weaker than continuity.

Theorem 7. *[15] A function $f : (X,\tau) \to (Y,\nu)$ is almost weakly continuous if and only if $pCl(f^{-1}(V) \subset f^{-1}(Cl(V))$ for every open subset V of Y.*

Lemma 8. *Let (X,τ,I) be an ideal topological space. $pCl(A) \subset {}_{PI}Cl_\theta(A)$ for every subset $A \subset X$.*

Proof. It is clear that $pCl(A) \subset {}_PCl_\theta(A)$ for every subset $A \subset X$. Hence the proof is a direct consequence of Lemma 3. ∎

The following theorem is an immediate consequence of Theorem 7 and Lemma 8.

Theorem 9. *If $f : (X, \tau, I) \to (Y, \nu)$ is weakly θ-pre-I continuous, then f is almost weakly continuos.*

In the next example we see that weak θ-pre-I continuity is not implied by almost weak continuity.

Example 3. *Let $X = \{a, b, c\}$,$\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$,$\nu = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$, $I = \{\emptyset, \{b\}\}$. The identity function $f : (X, \tau, I) \to (X, \nu)$ is almost weakly continuous but not weakly θ-pre-I continuous.*

Thus weak θ-pre-I continuity is strictly between continuity and almost weak continuity.

Definition 6. *An ideal topological space (X, τ, I) is said to be pre-I-regular [3](p-I-regular [9]), if for each pre-I-closed (closed) set F and each point $x \in (X - F)$, there exist disjoint pre-I-open sets U and V such that $x \in U$ and $F \subset V$.*

Lemma 10. *An ideal topological space (X, τ, I) is pre-I-regular [3](p-I-regular [9]) if and only if for each $x \in X$ and each pre-I-open (open) set U of X containing x, there exists $V \in PIO(X, x)$ such that $x \in V \subset {}_{PI}Cl(V) \subset U$.*

Theorem 11. *[6] Let I_n be the ideal of nowhere dense sets in (X, τ). For ideal topological space (X, τ, I) and $A \subset X$; if $I = I_n$, then A is pre-I-open if and only if A is pre open.*

Lemma 12. *A space $(X, \tau, I = I_n)$ is pre-I-regular if and only if ${}_{PI}Cl_\theta(A) = pCl(A)$ for every subset A of X.*

Proof. $\Rightarrow$: Assume that (X, τ, I_n) is pre-I-regular. Let $A \subset X$ and $x \in {}_{PI}Cl_\theta(A)$. Let U be a pre open subset of X with $x \in U$. By Theorem 11, U is pre-I-open. Then there exists $W \in PIO(X, x)$ such that $x \in W \subset {}_{PI}Cl(W) \subset U$. Since $x \in {}_{PI}Cl_\theta(A)$, ${}_{PI}Cl(W) \cap A \neq \emptyset$ and hence $U \cap A \neq \emptyset$. Thus $x \in pCl(A)$. Since $pCl(A) \subset {}_{PI}Cl_\theta(A)$ for every subset $A \subset X$, we obtain ${}_{PI}Cl_\theta(A) = pCl(A)$.

$\Leftarrow$: Assume that ${}_{PI}Cl_\theta(A) = pCl(A)$ for every $A \subset X$. Let $x \in X$ and $x \notin F$, where F is pre-I-closed. Since $F = {}_{PI}Cl(F)$, $x \notin {}_{PI}Cl(F)$. Since in general, we have $pCl(F) \subset {}_{PI}Cl(F)$ for every subsets F of X, $x \notin pCl(F)$ and hence $x \notin {}_{PI}Cl_\theta(F)$. So there exists $U \in PIO(X, x)$ such that ${}_{PI}Cl(U) \cap F = \emptyset$. Thus $x \in U$ and $F \subset X -_{PI} Cl(U)$ which proves that (X, τ, I_n) is pre-I-regular. ∎

The next theorem is an immediate consequence of Theorem 9 and Lemma 12.

Theorem 13. *If (X, τ, I_n) is pre-I-regular, then a function $f : (X, \tau, I_n) \rightarrow (Y, \nu)$ is weakly θ-pre-I continuous if and only if it almost weakly continuous.*

It follows from the next example that weak θ-pre-I continuity is independent of weak continuity.

Remark 2. *Example 3. shows that a weakly continuous function is not weakly θ-pre-I continuos.*

Example 4. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$, $\nu = \{\emptyset, X, \{c\}, \{a, b\}\}$, $I = \{\emptyset, \{a\}\}$. The identity function $f : (X, \tau, I) \rightarrow (X, \nu)$ is weakly θ-pre-I continuos but not weakly continuous.*

We now investigate conditions under which weak θ-pre-I continuity is related to weak continuity.

Theorem 14. *[16] A function $f : (X, \tau) \rightarrow (Y, \nu)$ is weakly continuous if and only if $Cl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ for every open subset V of Y.*

Lemma 15. *An ideal topological space (X, τ, I) is p-I-regular if and only if ${}_{PI}Cl_\theta(A) \subset Cl(A)$ for every subset A of X.*

Proof. $\Rightarrow$ Assume that (X, τ, I) is p-I-regular. Let $A \subset X$ and let $x \in_{PI} Cl_\theta(A)$. Let U be an open subset of X with $x \in U$. Then there exists $W \in PIO(X, x)$ such that $x \in W \subset {}_{PI}Cl(W) \subset U$. Since $x \in_{PI} Cl_\theta(A)$, ${}_{PI}Cl(W) \cap A \neq \emptyset$. Hence $U \cap A \neq \emptyset$. Thus $x \in Cl(A)$.
$\Leftarrow$ Assume that ${}_{PI}Cl_\theta(A) \subset Cl(A)$ for every $A \subset X$. Let $x \in X$ and $x \notin F$, where F is closed. Then $x \notin Cl(F)$ and hence $x \notin_{PI} Cl_\theta(F)$. So, there exists $U \in PIO(X, x)$ such that ${}_{PI}Cl(U) \cap F = \emptyset$. Thus $x \in U$ and $F \subset X -_{PI} Cl(U)$ which proves that (X, τ, I) is p-I-regular. ∎

Theorem 16. *If* (X, τ, I) *is a p-I-regular space and* $f : (X, \tau, I) \to (Y, \nu)$ *is weakly continuos, then* f *is weakly* θ*-pre-I continuous.*

Proof. Let $V \subset Y$ be an open. Since (X, τ, I) is p-I-regular, by Lemma 15

$$_{PI}Cl_\theta\left(f^{-1}(V)\right) \subset Cl\left(f^{-1}(V)\right).$$

Since f is weakly continuous, $Cl(f^{-1}(V)) \subset f^{-1}(Cl(V))$. Thus

$$_{PI}Cl_\theta\left(f^{-1}(V)\right) \subset f^{-1}(Cl(V))$$

and hence f is weakly θ-pre-I continuous. ∎

Lemma 17. *Let* (X, τ, I) *be an ideal topological space. If every pre-I-open subset of* X *is* α*-I-open, then* $Cl(A) \subset_{PI} Cl_\theta(A)$ *for every subset* A *of* X*.*

Proof. Let $A \subset X$ and $x \in Cl(A)$. Let $U \in PIO(X)$ such that $x \in U$. Then U is α-I-open and

$$\begin{aligned} x &\in U \subset Int(Cl^*(Int(U))) \subset Int(U \cup Cl^*(Int(U))) \\ &\subset Int(U \cup Cl(Int(U))) = Int(pCl(U)). \end{aligned}$$

Since $x \in Cl(A)$, $Int(pCl(U)) \cap A \neq \emptyset$ and hence $pCl(U) \cap A \neq \emptyset$. Therefore $x \in pCl_\theta(A)$. Since in general $pCl_\theta(A) \subset {_{PI}Cl_\theta}(A)$,then $x \in {_{PI}Cl_\theta}(A)$. Hence $Cl(A) \subset_{PI} Cl_\theta(A)$. ∎

Theorem 18. *If* (X, τ, I) *satisfies the condition that every pre-I-open set is* α*-I-open and* $f : (X, \tau, I) \to (Y, \nu)$ *is weakly* θ*-pre-I continuous, then* f *is weakly continuous.*

Proof. Let $V \subset Y$ be open. Since f is weakly θ- pre-I continuous,

$$_{PI}Cl_\theta\left(f^{-1}(V)\right) \subset f^{-1}(Cl(V)).$$

By Lemma 17, $Cl(f^{-1}(V)) \subset_{PI} Cl_\theta(f^{-1}(V))$. Hence $Cl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ which proves that f is weakly continuous. ∎

Definition 7. *A function* $f : (X, \tau, I) \to (Y, \nu)$ *is said to be contra* θ*-pre-I continuous provided that for every open subset* V *of* Y*,* $f^{-1}(V)$ *is pre-I-*θ*–closed.*

Theorem 19. *If* $f : (X, \tau, I) \to (Y, \nu)$ *is contra* θ*-pre-I continuous, then* f *is weakly* θ*- pre-I continuous.*

Proof. Let V be an open subset of Y. Then, since $f^{-1}(V)$ is pre-I-$\theta-$closed,

$$_{PI}Cl_{\theta}\left(f^{-1}\left(V\right)\right) = f^{-1}\left(V\right) \subset f^{-1}\left(Cl\left(V\right)\right)$$

which proves that f is weakly θ- pre-I continuous. ∎

The following example shows that weak θ-pre-I continuity is not equivalent to contra θ-pre-I continuity.

Example 5. *Let $X = \{a, b\}$, $\tau = \{\emptyset, X, \{a\}\}$ $I = \{\emptyset, \{b\}\}$. The identity function $f : (X, \tau, I) \to (X, \tau)$ is weakly θ-pre-I continuous but not contra θ-pre-I continuous.*

3. Properties

Definition 8. *[9] The graph $G(f)$ of a function $f : (X, \tau, I) \to (Y, \nu)$, is said to be strongly pre-I-closed provided that for every $(x, y) \in (X \times Y) - G(f)$ there exist $U \in PIO(X, x)$ and an open set $V \subset Y$ containing y such that $(x, y) \in ({}_{PI}Cl(U) \times V) \subset X \times Y - G(f)$.*

Theorem 20. *If $f : (X, \tau, I) \to (Y, \nu)$ is weakly θ-pre-I continuous and Y is Hausdorff, then the graph of f is strongly pre-I-closed.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then there exist disjoint open sets V and W in Y with $y \in V$ and $f(x) \in W$. Thus $f(x) \notin Cl(V)$ and hence $x \notin f^{-1}(Cl(V))$. Since f is weakly θ-pre-I continuous, $x \notin {}_{PI}Cl_{\theta}(f^{-1}(V))$. So there exists $U \in PIO(X, x)$ such that ${}_{PI}Cl(U) \cap f^{-1}(V) = \emptyset$. Therefore $(x, y) \in ({}_{PI}Cl(U) \times V) \subset X \times Y - G(f)$, which proves that $G(f)$ is strongly pre-I-closed. ∎

Corollary 21. *[9] If $f : (X, \tau, I) \to (Y, \nu)$ is strongly θ-pre-I continuous and Y is Hausdorff, then the graph of f is strongly pre-I-closed.*

Theorem 22. *[4] Let (X, τ, I) be an ideal topological space, A and X_0 be subsets of X. Then, the following properties hold:*

(1) If $A \in PIO(X)$ and $X_0 \in SIO(X)$, then $A \cap X_0 \in PIO(X_0)$
(2) If $A \in PIO(X_0)$ and $X_0 \in PIO(X)$, then $A \in PIO(X)$.

Theorem 23. *[4] Let (X, τ, I) be an ideal topological space, X_0 and A be subsets of X such that $A \subset X_0 \subset X$. Let ${}_{PI}Cl_{X_0}(A)$ denote the pre-I-closure of A in the subspace X_0.*

(1) If X_0 is semi-I-open in X, then ${}_{PI}Cl_{X_0}(A) \subset {}_{PI}Cl(A)$.
(2) If $A \in PIO(X_0)$ and $X_0 \in PIO(X)$, then ${}_{PI}Cl(A) \subset {}_{PI}Cl_{X_0}(A)$.

If $B \subset A \subset X$, then we shall denote the pre–I-θ-closure of B with respect to relative topology on A by ${}_{PI}Cl_{A\theta}(B)$.

Lemma 24. *Let A and B be subsets of a space X such that $B \subset A \subset X$. If A is semi-I-open, then ${}_{PI}Cl_{A\theta}(B) \subset {}_{PI}Cl_{\theta}(B)$.*

Proof. Let $x \in {}_{PI}Cl_{A\theta}(B)$ and let $U \in PIO(X,x)$. Then $x \in U \cap A$ and by Theorem 22, $U \cap A \in PIO(A)$. Since $x \in {}_{PI}Cl_{A\theta}(B)$, ${}_{PI}Cl_A(U \cap A) \cap B \neq \emptyset$. By Theorem 23, ${}_{PI}Cl_A(U \cap A) \subset {}_{PI}Cl(U \cap A)$. So ${}_{PI}Cl(U \cap A) \cap B \neq \emptyset$ and hence ${}_{PI}Cl(U) \cap B \neq \emptyset$. Thus $x \in {}_{PI}Cl_{\theta}(B)$. ∎

Theorem 25. *If $f : (X,\tau,I) \to (Y,\nu)$ is weakly θ-pre-I continuous and A is a semi-I-open subset of X, then $f/_A : (A,\tau_A,I_A) \to (Y,\nu)$ is weakly θ-pre-I continuous.*

Proof. Let V be an open subset of Y. Then, by Lemma 24, we see that

$$\begin{aligned} {}_{PI}Cl_{A\theta}\left(f/_A^{-1}(V)\right) &= {}_{PI}Cl_{A\theta}\left(f^{-1}(V) \cap A\right) \subset {}_{PI}Cl_{\theta}\left(f^{-1}(V) \cap A\right) \cap A \\ &\subset {}_{PI}Cl_{\theta}\left(f^{-1}(V)\right) \cap A \subset f^{-1}(Cl(V)) \cap A = f/_A^{-1}(Cl(V)). \end{aligned}$$

Hence $f/_A : (A,\tau_A,I_A) \to (Y,\nu)$ is weakly θ-pre-I continuous. ∎

Theorem 26. *If $f : (X,\tau,I) \to (Y,\nu)$ is weakly θ-pre-I continuous and $g : (Y,\nu) \to (Z,\sigma)$ is continuous, then $g \circ f : (X,\tau,I) \to (Z,\sigma)$ is weakly θ-pre-I continuous.*

Proof. Let V be an open set in Z. Then

$$\begin{aligned} {}_{PI}Cl_{\theta}\left((g \circ f)^{-1}(V)\right) &= {}_{PI}Cl_{\theta}\left(f^{-1}\left(g^{-1}(V)\right)\right) \subset f^{-1}\left(Cl\left(g^{-1}(V)\right)\right) \\ &\subset f^{-1}(g^{-1}(Cl(V)) = (gof)^{-1}(Cl(V)) \end{aligned}$$

which proves that gof is weakly θ-pre-I continuous. ∎

Definition 9. *A function $f : (X,\tau,I) \to (Y,\nu,I_1)$ is said to be pre-I-θ–closed provided that for every pre-I-θ–closed subset V of X, $f(V)$ is pre-I-θ–closed in Y.*

Theorem 27. *A function $f:(X,\tau,I)\to(Y,\nu,I_1)$ is pre-I-θ–closed function if and only if ${}_{PI}Cl_\theta(f(A))\subset f({}_{PI}Cl_\theta(A))$ for every subset $A\subset X$.*

Proof. $\Rightarrow$ Let A be any subset of (X,τ,I). Since $A\subseteq_{PI} Cl_\theta(A)$ for every subset $A\subset X$, $f(A)\subseteq f({}_{PI}Cl_\theta(A))$ and ${}_{PI}Cl_\theta(f(A))\subseteq_{PI} Cl_\theta(f({}_{PI}Cl_\theta(A)))$. Since f is pre-I-θ–closed function, $f({}_{PI}Cl_\theta(A))$ pre-I-θ–closed. Therefore,
${}_{PI}Cl_\theta(f({}_{PI}Cl_\theta(A)))=f({}_{PI}Cl_\theta(A))$ and hence ${}_{PI}Cl_\theta(f(A))\subseteq f({}_{PI}Cl_\theta(A))$.
$\Leftarrow$ Let K be a pre-I-θ–closed subset of (X,τ,I). Then $K=_{PI} Cl_\theta(K)$ and hence ${}_{PI}Cl_\theta(f(K))\subseteq f({}_{PI}Cl_\theta(K))=f(K)$. Since $A\subseteq_{PI} Cl_\theta(A)$ for every subset $A\subset X$, $f(K)\subset_{PI} Cl_\theta(f(K))$. This shows that ${}_{PI}Cl_\theta(f(K))=f(K)$. ∎

Theorem 28. *Let $f:(X,\tau,I)\to(Y,\nu,I_1)$ and $g:(Y,\nu,I_1)\to(Z,\sigma)$ be functions. If $g\circ f:(X,\tau,I)\to(Z,\sigma)$ is weakly θ-pre-I continuous and f is surjective, pre -I-θ closed, then g is weakly θ-pre-I continuous.*

Proof. Let V be an open subset of Z. Since $g\circ f:(X,\tau,I)\to(Z,\sigma)$ is weakly θ-pre-I continuous,${}_{PI}Cl_\theta(f^{-1}(g^{-1}(V)))\subset f^{-1}(g^{-1}(Cl(V)))$ and hence

$$f\left({}_{PI}Cl_\theta\left(f^{-1}\left(g^{-1}(V)\right)\right)\right)\subset g^{-1}(Cl(V)).$$

Since f is pre-I-θ closed, $f({}_{PI}Cl_\theta(f^{-1}(g^{-1}(V))))$ is pre-I-θ closed,

$${}_{PI}Cl_\theta\left(f\left(f^{-1}\left(g^{-1}(V)\right)\right)\right)\subset f\left({}_{PI}Cl_\theta\left(f^{-1}\left(g^{-1}(V)\right)\right)\right)$$

and hence ${}_{PI}Cl_\theta(f(f^{-1}(g^{-1}(V))))\subset g^{-1}(Cl(V))$. Finally, since f is surjective, it follows that ${}_{PI}Cl_\theta((g^{-1}(V)))\subset g^{-1}(Cl(V))$, which proves that g is weakly θ-pre-I continuous. ∎

Theorem 29. *Let $f:(X,\tau,I)\to(Y,\nu)$ and $g:(Y,\nu)\to(Z,\sigma)$ be functions. If $g\circ f:(X,\tau,I)\to(Z,\sigma)$ is weakly θ-pre-I continuous and g is a clopen injection, then f is weakly θ-pre-I continuous.*

Proof. Let V be an open subset of Y. Since g is open and $g\circ f$ is weakly θ-pre-I continuous, we have

$${}_{PI}Cl_\theta\left(f^{-1}(V)\right)\subset_{PI} Cl_\theta\left(f^{-1}\left(g^{-1}(g(V))\right)\right)\subset f^{-1}\left(g^{-1}(Cl(g(V)))\right).$$

Furthermore, since g is closed and injective,

$$f^{-1}\left(g^{-1}(Cl(g(V)))\right)\subset f^{-1}\left(g^{-1}(g(Cl(V)))\right)=f^{-1}(Cl(V)).$$

Hence ${}_{PI}Cl_{\theta}\left(f^{-1}\left(V\right)\right) \subset f^{-1}\left(Cl\left(V\right)\right)$, which proves that f is weakly θ-pre-I continuous. ∎

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