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(E.A) PROPERTY AND ALTERING DISTANCE IN METRIC SPACES

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Abstract. In this paper a general fixed point theorem for mappings satisfying an implicit relation is proved for two weakly compatible mappings which have property (E.A), which generalize the main results from [1] and [14]. As a consequence a fixed point theorem for mappings satisfying an implicit contractive condition of integral type is obtained.

1. Introduction

Let (X, d) be a metric space and S, T be two self mappings of X. In [5], Jungck defined S and T to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

The concept was frequently used to prove existence theorems in common fixed point theory. The study of common fixed points of noncompatible mappings is also interesting.

The work along this lines has been initiated by Pant in [9], [10], [11]. Recently, Aamri and Moutawakil [1] introduced a generalization of the concept on noncompatible mappings.

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Definition 1.1 ([1])**.** Let S and T be two self mappings of a metric space (X, d). We say that S and T satisfy property $(E.A)$ if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Remark 1.2. *It is clear that two self mappings S and T of a metric space (X, d) will be noncompatible if there exists at least one sequence $\{x_n\}$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$ but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either nonzero or non existent. Therefore, two compatible self mappings of a metric space (X, d) satisfy property $(E.A)$.*

Definition 1.3 ([6])**.** Two self mappings S and T of a metric space (X, d) is said to be weakly compatible if $Su = Tu$ implies $STu = TSu$.

Two compatible mappings are weakly compatible and the converse is not true.

Remark 1.4. *It is known that the notions of weakly compatible mappings and mappings satisfying property $(E.A)$ are independent.*

Definition 1.5. Let S and T be two self mappings of a metric space (X, d). A point $x \in X$ is said to be a coincidence point of S and T if $Sx = Tx$ and the point $w = Sx = Tx$ is said to be a point of coincidence of S and T.

Lemma 1.6 ([2])**.** *Let f and g be weakly compatible self mappings on a nonempty set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

The following theorem is proved in [1].

Theorem 1.7. *Let S and T be weakly compatible mappings of a metric space (X, d) such that*

(i) T *and* S *satisfy property* $(E.A)$*;*

(ii)

$$d(Tx, Ty) < \max\{d(Sx, Sy), \frac{1}{2}[d(Sx, Tx) + d(Sy, Ty)], \frac{1}{2}[d(Sx, Ty) + d(Sy, Tx)]\}$$

for all $x \neq y \in X$;

(iii) $T(X) \subset S(X)$.

If $S(X)$ or $T(X)$ is a complete subspace of X, then T and S have a unique common fixed point.

In [12], [13], the study of fixed points for mappings satisfying implicit relations was introduced. In [14] a generalization of Theorem 1.7 for mappings satisfying a implicit relation is proved.

In [7] Khan et al. introduced the notion of altering distance.

Definition 1.8. An altering distance is a function $\psi : [0, \infty) \to [0, \infty)$ satisfying:
(ψ_1) : ψ is increasing and continuous,
(ψ_2) : $\psi(t) = 0$ if and only if $t = 0$.

Fixed point problems involving an altering distance have been studied in [7], [18], [19], [16] and in other papers.

In this paper a generalization of Theorem 2 [14], using altering distance is obtained.

In the last part of this paper, a theorem for mappings satisfying a contractive condition of integral type and property $(E.A)$ is reduced, using the method by [16], and the study of fixed points with altering distance.

2. Implicit relations

Let $\mathfrak{F}_6$ be the set of all real continuous functions $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:
(F_1) : $F(t, 0, 0, t, t, 0) > 0$, $\forall t > 0$,
(F_2) : $F(t, t, 0, 0, t, t) \geq 0$, $\forall t > 0$.

Example 2.1.

$$F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6,$$

where $a, b, c, d, e \geq 0,\ c + d < 1$ *and* $a + d + e \leq 1$.
(F_1) : $F(t, 0, 0, t, t, 0) = t(1 - (c + d)) > 0,\ \forall t > 0.$
(F_2) : $F(t, t, 0, 0, t, t) = t(1 - (a + d + e)) \geq 0,\ \forall t > 0.$

Example 2.2.

$$F(t_1, ..., t_6) = t_1 - h \max \{t_2, t_3, ..., t_6\},$$

where $h \in (0, 1)$.
(F_1) : $F(t, 0, 0, t, t, 0) = t(1 - h) > 0,\ \forall t > 0.$
(F_2) : $F(t, t, 0, 0, t, t) = t(1 - h) \geq 0,\ \forall t > 0.$

Example 2.3.

$$F(t_1, ..., t_6) = t_1 - h \max \left\{ t_2, t_3, t_4, \frac{t_5 + t_6}{2} \right\},$$

where $h \in (0, 1)$.

(F_1) : $F(t,0,0,t,t,0) = t(1-h) > 0, \forall t > 0.$
(F_2) : $F(t,t,0,0,t,t) = t(1-h) \geq 0, \forall t > 0.$

Example 2.4.

$$F(t_1,...,t_6) = t_1 - k\max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\},$$

where $k \in (0,1]$.
(F_1) : $F(t,0,0,t,t,0) = t\left(1-\frac{k}{2}\right) > 0, \forall t > 0.$
(F_2) : $F(t,t,0,0,t,t) = t(1-k) \geq 0, \forall t > 0.$

Example 2.5.

$$F(t_1,...,t_6) = t_1 - at_2 - b\max\{t_3,t_4\} - c\max\{t_2,t_5,t_6\},$$

where a, b, $c \geq 0$, $b+c<1$ *and* $a+c \leq 1$.
(F_1) : $F(t,0,0,t,t,0) = t\,(1-(b+c)) > 0, \forall t > 0.$
(F_2) : $F(t,t,0,0,t,t) = t(1-(a+c)) \geq 0, \forall t > 0.$

Example 2.6.

$$F(t_1,...,t_6) = t_1 - at_2 - b\frac{t_5+t_6}{1+t_3+t_4},$$

where a, $b \geq 0$, $a,b \geq 0$ *and* $a+2b \leq 1$.
(F_1) : $F(t,0,0,t,t,0) = t\left(1-\frac{b}{1+t}\right) > 0, \forall t > 0.$
(F_2) : $F(t,t,0,0,t,t) = t(1-(a+2b)) \geq 0, \forall t > 0.$

Example 2.7.

$$F(t_1,...,t_6) = t_1 - at_2 - b\sqrt{t_3t_4} - c\sqrt{t_5t_6},$$

where $a,b,c \geq 0$ *and* $a+c \leq 1$.
(F_1) : $F(t,0,0,t,t,0) = t > 0, \forall t > 0.$
(F_2) : $F(t,t,0,0,t,t) = t(1-(a+c)) \geq 0, \forall t > 0.$

Example 2.8.

$$F(t_1,...,t_6) = t_1 - h\max\{t_2,t_3,t_4\} - (1-h)(at_5+bt_6),$$

where $0<h<1$, $a,b \geq 0$ *and* $a+b \leq 1$.
(F_1) : $F(t,0,0,t,t,0) = t(1-h)(1-a) > 0, \forall t > 0.$
(F_2) : $F(t,t,0,0,t,t) = t(1-h)(1-(a+b)) \geq 0, \forall t > 0.$

Example 2.9.

$$F(t_1,...,t_6) = t_1^3 - at_1^2t_2 - bt_1t_3t_4 - ct_5^2t_6 - dt_5t_6^2,$$

where $a,b,c,d \geq 0$ *and* $a+c+d \leq 1$.
(F_1) : $F(t,0,0,t,t,0) = t^3 > 0, \forall t > 0.$
(F_2) : $F(t,t,0,0,t,t) = t^3(1-(a+c+d)) \geq 0, \forall t > 0.$

Example 2.10.

$$F(t_1, ..., t_6) = t_1 - \max\{t_2, t_3, \sqrt{t_3 t_5}, \sqrt{t_4 t_6}\},$$

where $a, b, c \geq 0$ and $a + c \leq 1$.

(F_1) : $F(t, 0, 0, t, t, 0) = t > 0$, $\forall t > 0$.

(F_2) : $F(t, t, 0, 0, t, t) = 0$, $\forall t > 0$.

The following results which generalize Theorem 1.7 are proved in [14].

Theorem 2.11. *Let T and S be two weakly compatible self mappings of a metric space (X, d) such that*

(i) T *and* S *satisfy property* $(E.A)$;

(ii)

$$F(d(Tx, Ty), d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\} < 0$$

for each $x, y \in X$ and $F \in \mathfrak{F}_6$,

(iii) $T(X) \subset S(X)$.

If $S(X)$ or $T(X)$ is a complete subspace of X, then T and S have a unique common fixed point.

Corollary 2.12 ([14])**.** *Let T and S be noncompatible weakly compatible mappings such that the inequality (ii) of Theorem 2.11 is satisfied for each $x, y \in X$ and $T(X) \subset S(X)$.*

If $S(X)$ or $T(X)$ is a complete subspace of X, then T and S have a unique common fixed point.

3. Fixed point theorems

Theorem 3.1. *Let S and T be two self mappings of the metric space (X, d) satisfying the following inequality*

$$\begin{gathered}F(\psi(d(Tx, Ty)), \psi(d(Sx, Sy)), \psi(d(Sx, Tx)),\\ \psi(d(Sy, Ty)), \psi(d(Sx, Ty)), \psi(d(Sy, Tx))) < 0\end{gathered} \tag{3.1}$$

for all $x \neq y \in X$, where ψ is an altering distance and F satisfy property (F_2). If S and T have a point of coincidence, then this point is the unique point of coincidence.

Proof. Suppose that T and S have two distinct points of coincidence $u = Ta = Sa$ and $v = Tb = Sb$. By (3.1) we have successively

$$\begin{gathered}F(\psi(d(Ta, Tb)), \psi(d(Sa, Sb)), \psi(d(Sa, Ta)),\\ \psi(d(Sb, Tb)), \psi(d(Sa, Tb)), \psi(d(Sb, Ta))) < 0\end{gathered},$$

$$F(\psi(d(v, v)), \psi(d(u, v)), 0, 0, \psi(d(u, v)), \psi(d(u, v))) < 0 ,$$

a contradiction. Hence $u = v$. □

Theorem 3.2. *Let S and T be two weakly compatible self mappings of the metric space (X, d) such that:*

(1) S and T satisfy property $(E.A)$;

(2) S and T satisfy inequality (3.1), for all $x \neq y \in X$, where ψ is an altering distance and $F \in \mathfrak{F}_6$.

(3) $T(X) \subset S(X)$.

If $S(X)$ or $T(X)$ is a complete subspace of X, then T and S have a unique common fixed point.

Proof. Since T and S satisfy property $(E.A)$, there exists in X a sequence $\{x_n\}$ satisfying

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in X$.

Suppose that $S(X)$ is complete. Then $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$. Also, $\lim_{n\to\infty} Tx_n = Sa$. By (3.1) we have for $x = x_n$ and $y = a$

$$F(\psi(d(Tx_n, Ta)), \psi(d(Sx_n, Sa)), \psi(d(Sx_n, Tx_n)),$$
$$\psi(d(Sa, Ta)), \psi(d(Sx_n, Ta)), \psi(d(Sa, Tx_n))) < 0.$$

Letting n tend to infinity we obtain

$$F(\psi(d(Sa, Ta)), 0, 0, \psi(d(Sa, Ta)), \psi(d(Sa, Ta)), 0) < 0,$$

a contradiction of (F_2) if $\psi(d(Sa, Ta)) > 0$, hence $\psi(d(Sa, Ta)) = 0$ which implies $d(Sa, Ta) = 0$, i.e. $Sa = Ta$. Then $w = Sa = Ta$ is a point of coincidence of S and T. By Theorem 3.1, w is the unique point of coincidence of T and S.

By Lemma 1.6, w is the unique common fixed point of S and T. The proof is similar when $T(X)$ is a complete subsequence of (X, d), since $T(X) \subset S(X)$. □

Remark 3.3. *If $\psi(t) = t$ we obtain Theorem 2.11.*

Corollary 3.4. *Let T and S be noncompatible weakly compatible mappings satisfying the inequality (3.1) for all $x \neq y \in X$, where ψ is an altering distance and $T(X) \subset S(X)$. If $S(X)$ or $T(X)$ is a complete subspace of X, then T and S have a unique common fixed point.*

4. Applications

In [4], Branciari established the following theorem which opened the way to the study of mappings satisfying a contractive condition of integral type.

Theorem 4.1 ([4]). *Let (X,d) be a complete metric space, $c \in (0,1)$ and $f : X \to X$ a mapping such that for each $x, y \in X$*

$$\int_0^{d(fx,fy)} h(t)dt \le c \int_0^{d(x,y)} h(t)dt \tag{4.1}$$

where $h(t) : [0,\infty) \to [0,\infty)$ is a Lebesgue measurable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,\infty)$, such that for $\varepsilon > 0$, for $\int_0^\varepsilon h(t)dt > 0$.

Then f has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = z$.

Theorem 4.1 has been generalized in several papers, e.g. it has been extended to a pair of compatible mappings in [7].

Theorem 4.2 ([8]). *Let f and g be compatible self mappings of a complete metric space (X,d), with g continuous satisfying the following conditions:*

(1) $f(X) \subset g(X)$,

(2)

$$\int_0^{d(fx,gy)} h(t)dt \le c \int_0^{d(x,y)} h(t)dt\ ,$$

for some $c \in (0,1)$, whenever $x, y \in X$ and $h(t) : [0,\infty) \to [0,\infty)$ satisfies the assumptions from Theorem 4.1.

Then f and g have a unique fixed point.

Some fixed point results for mappings satisfying contractive conditions of integral type are obtained in [3], [15], [16], [17] and in other papers.

Lemma 4.3 ([16]). *Let $h(t) : [0,\infty) \to [0,\infty)$ as in Theorem 4.1. Then, $\psi(t) = \int_0^t h(t)dt$ is an altering distance.*

Theorem 4.4. *Let S and T be two weakly compatible self mappings of the metric space (X,d) such that:*

(1) S *and* T *satisfies property* $(E.A)$;

(2)

$$\begin{aligned} &F\Big(\textstyle\int_0^{d(Tx,Ty)} h(t)dt, \int_0^{d(Sx,Sy)} h(t)dt, \int_0^{d(Sx,Tx)} h(t)dt, \\ &\textstyle\int_0^{d(Sy,Ty)} h(t)dt, \int_0^{d(Sx,Ty)} h(t)dt, \int_0^{d(Sy,Tx)} h(t)dt\Big) < 0 \end{aligned} \tag{4.2}$$

for all $x \neq y \in X$, $F \in \mathfrak{F}_6$ and $h(t)$ is as in Theorem 4.1.

(3) $T(X) \subset S(X)$.

If $S(X)$ or $T(X)$ is a complete subspace of X, then S and T have a unique common fixed point.

Proof. By Lemma 4.3,

$$\psi(t) = \int_0^t h(t)dt$$

is an altering distance. By (4.2) we obtain

$$F\left(\psi(d(Tx,Ty)), \psi(d(Sx,Sy)), \psi(d(Sx,Tx)), \psi(d(Sy,Ty)), \psi(d(Sx,Ty)), \psi(d(Sy,Tx))\right) < 0$$

which is inequality (3.1). Hence, the condition of Theorem 3.2 are satisfied and S and T have a unique common fixed point. □

Remark 4.5. *If $h(t) = 1$, then by Theorem 4.1 we obtain Theorem 2.11.*

Corollary 4.6. *Let S and T be noncompatible weakly compatible mappings satisfying inequality (4.2) for each $x, y \in X$, where $F \in \mathfrak{F}_6$ and $T(X) \subset S(X)$.*

If $S(X)$ or $T(X)$ is a complete subspace of X, then S and T have a unique common fixed point.

Remark 4.7. *If $h(t) = 1$, then by Corollary 4.6 we obtain Corollary 2.11.*

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