"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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ON FUZZY CLOSED MAPS IN FUZZY BICLOSURE SPACES

U. D. TAPI AND R. NAVALAKHE

Abstract. The purpose of this paper is to introduce the notions of fuzzy closed map and fuzzy open map in fuzzy biclosure spaces and investigate some of their properties.

1. Introduction

With the introduction of fuzzy sets by Zadeh [7] and fuzzy topology by Chang [3], the theory of fuzzy topological spaces was subsequently developed by several authors by considering the basic concepts of general topology.

Closure spaces were introduced by E. Čech [2]. The notions of closure system and closure operators are very useful tools in several areas of mathematics playing an important role in the study of topological spaces , Boolean algebra , convex sets etc.

Fuzzy closure spaces have been introduced and studied as a generalization of closure spaces, by A.S Mashhour and M.H Ghanim [4].

Recently, Boonpok [1] introduced the notion of biclosure space, as a space equipped with two arbitrary closure operators. He extended some of the standard results of separation axioms from closure spaces to biclosure spaces. Thereafter a large number of papers have been written to generalize the concept of closure space to biclosure space. The authors [6] have introduced the notion of fuzzy biclosure spaces and generalized the concept of fuzzy closure space to fuzzy biclosure space.

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In this paper we introduce the concepts of fuzzy closed map and fuzzy open map in fuzzy biclosure spaces and investigate some properties of these maps.

2. Preliminaries

Let X be an arbitrary set, $I = [1, 0]$ and I^X be a family of all fuzzy subsets of X. For a fuzzy set A of $X, cl(A), int(A)$ and $1 - A$ will denote the closure of A, the interior of A and the complement of A respectively whereas the constant fuzzy sets taking on the values 0 and 1 on X are denoted by 0_X and 1_X respectively.

Definition 2.1. [4] Let X be a nonempty set. A function $u : I^X \to I^X$ is said to be a *fuzzy closure operator* for X, if it satisfies the following three conditions :

(i) $u\phi = \phi$
(ii) $A \leq u(A)$ for all $A \in I^X$.
(iii) $u(A \vee B) = u(A) \vee u(B)$ for all A , $B \in I^X$.

The pair (X, u) , where u is a fuzzy closure operator for X, is a fuzzy closure space (or a fcs, for short). For a fuzzy subset A of a fcs (X, u) the fuzzy subset uA is called the closure of A.

Remark 2.2. Note that $A \leq B \leq X$ implies $u(A) \leq u(B)$, since $u(B) = u(A) \vee u(B \wedge (1 - A))$.

Definition 2.3. [4] A fuzzy subset A of a fuzzy closure space (fcs) (X, u) is said to be *fuzzy closed*, if $uA = A$ and it is fuzzy open if its complement $1 - A$ is fuzzy closed.

Definition 2.4. [4] A fuzzy closure space (Y, v) is said to be a *subspace* of (X, u) if $Y \leq X$ and $vA = uA \wedge Y$ for each fuzzy subset $A \leq Y$. If Y is fuzzy closed in (X, u), then the subspace (Y, v) of (X, u) is also said to be fuzzy closed.

Definition 2.5. [5] Let (X, u) and (Y, v) fuzzy closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *fuzzy continuous* if $f(uA) \leq vf(A)$ for every fuzzy subset $A \leq X$.

In other words, a map $f : (X, u) \to (Y, v)$ is fuzzy continuous if and only if $uf^{-1}(B) \leq f^{-1}v(B)$ for every fuzzy subset $B \leq Y$.

Clearly, if a map $f : (X, u) \to (Y, v)$ is fuzzy continuous, then $f^{-1}(F)$ is a fuzzy closed subset of (X, u) for every fuzzy closed subset F of(Y, v).

Definition 2.6. [5] Let (X, u) and (Y, v) be fuzzy closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *fuzzy closed (* resp. *fuzzy open)* if $f(F)$ is a fuzzy closed (resp. fuzzy open) subset of (Y, v) whenever F is a fuzzy closed (resp. fuzzy open) subset of (X, u).

Definition 2.7. [5] The *product of a family* $\{(X_\alpha, u_\alpha)\alpha \in J\}$ *of fuzzy closure spaces* denoted by $\prod_{\alpha\in J}(X_\alpha, u_\alpha)$, is the fuzzy closure space $(\prod_{\alpha\in J} X_\alpha, u)$ where $\prod_{\alpha\in J} X_\alpha$ denotes the cartesian product of fuzzy sets $X_\alpha, \alpha \in J$ and u is the fuzzy closure operator defined by $uA = \prod_{\alpha\in J} u_\alpha \pi_\alpha(A)$ for each $A \leq \prod_{\alpha\in J} X_\alpha$.

The following statement is evident :

Proposition 2.8. [6] Let $\{(X_\alpha,\ u_\alpha) : \alpha\ \in\ J\}$ be a family of fuzzy closure spaces. Then the projection map $p_\beta : \prod_{\alpha\in J}(X_\alpha, u_\alpha) \to\ (X_\beta, u_\beta)$ is fuzzy closed and fuzzy continuous for every $\beta \in J$.

Definition 2.9. [6] A triple (X, u_1, u_2) is said to a be a *fuzzy biclosure space* if X is a nonempty set and $u_1\ ,\ u_2 : I^X \to I^X$ are fuzzy closure operators.

Definition 2.10. [6] A subset A of a fuzzy biclosure space (X, u_1, u_2) is called *fuzzy closed* if $u_1u_2A\ = A$. The complement of a fuzzy closed set is called fuzzy open.

Lemma 2.11. *Let (X, u_1, u_2) be a fuzzy biclosure space and let $A \leq X$ be a fuzzy subset. Then following conditions are equivalent :*

(i) *A is a fuzzy closed subset of (X, u_1, u_2);*
(ii) *$u_1A\ = A$ and $u_2A\ = A$;*
(iii) *$u_2u_1A\ = A$.*

Proof. (i) $\Rightarrow$ (ii) Assume $u_1u_2A\ =\ A$. By Definition 2.1(ii) and Remark (2.2) $A \leq u_1A \leq u_1u_2A$ and $A \leq u_2A \leq u_1u_2A$. Then $u_1A\ = A$ and $u_2A\ = A$.

(ii) $\Rightarrow$ (i) Assume $u_1A\ \ = A$ and $u_2A\ \ = A$. Then $u_1u_2A\ \ = u_1(u_2A)\ =\ u_1A\ =\ A$, hence A is fuzzy closed in (X, u_1, u_2). The proofs for the implications

(iii) $\Rightarrow$ (ii) and (ii)$\Rightarrow$ (iii) are obtained from the proofs of (i)$\Rightarrow$ (ii) and (ii)$\Rightarrow$ (i), respectively by interchanging u_1 and u_2.

Corollary 2.12. Let (X, u_1, u_2) be fuzzy biclosure space and let $A \leq X$ be a fuzzy subset. Then A is a fuzzy closed subset of (X, u_1, u_2) if and only if A is both a fuzzy closed subset of (X, u_1) and (X, u_2).

Definition 2.13. The product of a family $\{(X_\alpha,\ u^1_\alpha, u^2_\alpha) : \alpha \in J\}$ of fuzzy biclosure spaces denoted by $\prod_{\alpha\in J}(X_\alpha,\ u^1_\alpha, u^2_\alpha)$ is the fuzzy biclosure space
$\left(\prod_{\alpha\in J} X_\alpha,\ u^1, u^2\right)$ where $\left(\prod_{\alpha\in J} X_\alpha, u^i\right)$ for $i \in \{1,2\}$ is the product of the family of fuzzy closure spaces $\{X_\alpha, u^i_\alpha : \alpha \in J\}$.

Remark 2.14. Let $\prod_{\alpha\in J}(X_\alpha, u^1_\alpha, u^2_\alpha) = \left(\prod_{\alpha\in J} X_\alpha, u^1, u^2\right)$. Then for each $A \leq \prod_{\alpha\in J} X_\alpha$,

$$u^1u^2A = \prod_{\alpha\in J} u^1_\alpha u^2_\alpha \pi_\alpha(A).$$

We characterize fuzzy closed subsets in fuzzy biclosure space as follows:

Proposition 2.15. Let $\{(X_\alpha,\ u^1_\alpha, u^2_\alpha) : \alpha \in J\}$ be a family of fuzzy biclosure spaces and let $\beta \in J$. Then F is a fuzzy closed subset of $\prod_{\alpha\in J}(X_\alpha,\ u^1_\alpha, u^2_\alpha)$ if and only if F is a fuzzy closed subset of $\prod_{\alpha\in J}(X_\alpha,\ u^1_\alpha)$ and $\prod_{\alpha\in J}(X_\alpha,\ u^2_\alpha)$.

Proof. Denoting $X = \prod_{\alpha\in J} X_\alpha$ and $\prod_{\alpha\in J}(X_\alpha, u^i_\alpha) = (X, u^i)$ for $i \in \{1,2\}$, we have $\prod_{\alpha\in J}(X_\alpha, u^1_\alpha, u^2_\alpha) = (X, u^1, u^2)$, according to Definition 2.13 . By Corollary 2.12, F is a fuzzy closed subset of $(X_\alpha,\ u^1, u^2)$ if and only if F is a fuzzy closed subset of $(X,\ u^1)$ and $(X,\ u^2)$.

Proposition 2.16. [6] Let $\{(X_\alpha,\ u^1_\alpha, u^2_\alpha) : \alpha \in J\}$ be a family of fuzzy biclosure spaces and let $\beta \in J$. Then $F \subset X_\beta$ is a fuzzy closed subset of $(X_\beta,\ u^1_\beta, u^2_\beta)$ if and only if $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is a fuzzy closed subset of $\prod_{\alpha\in J}(X_\alpha,\ u^1_\alpha, u^2_\alpha)$.

Proposition 2.17. [6] Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in J\}$ be a family of fuzzy biclosure spaces and let $\beta \in J$. Then $G \subset X_\beta$ is a fuzzy open subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if

$G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$

is a fuzzy open subset of $\prod_{\alpha\in J}(X_\alpha,\ u^1_\alpha, u^2_\alpha)$.

Definition 2.18. [6] Let (X, u_1, u_2) be fuzzy biclosure space. A fuzzy biclosure space (Y, v_1, v_2) is called a *subspace* of (X, u_1, u_2) if $Y \leq X$ and $v_iA = u_iA \wedge Y$ for each $i \in \{1,2\}$ and each subset $A \leq Y$.

Proposition 2.19. [6] Let (X, u_1, u_2) be fuzzy biclosure space and let (Y, v_1, v_2) be a fuzzy closed subspace of (X, u_1, u_2). If F is a fuzzy closed subset of (Y, v_1, v_2), then F is a fuzzy closed subset of (X, u_1, u_2).

3. Fuzzy Closed Maps in Fuzzy Biclosure Spaces

In this section we introduce the notions of fuzzy closed map and fuzzy open map in fuzzy biclosure spaces and investigate some of their characterizations.

Definition 3.1. Let (X, u_1, u_2) and (Y, v_1, v_2) be fuzzy biclosure spaces. A map
$f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is said to be *fuzzy closed (* resp. *fuzzy open)* if $f(F)$ is fuzzy closed (resp. fuzzy open) subset of (Y, v_1, v_2) whenever F is a fuzzy closed (resp. fuzzy open) subset of (X, u_1, u_2).

The following statement is evident .

Proposition 3.2. Let (X, u_1, u_2) , (Y, v_1, v_2) and (Z, w_1, w_2) be fuzzy biclosure spaces. If the maps $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ are fuzzy closed (resp. fuzzy open), then $g \circ f : (X, u_1, u_2) \to (Z, w_1, w_2)$ is fuzzy closed (resp. fuzzy open).

Proposition 3.3. Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in J\}$ be a family of fuzzy biclosure spaces. Then for each $\beta \in J$, the projection map $p_\beta : \prod_{\alpha\in J}(X_\alpha, u_\alpha^1, u_\alpha^2) \to (X_\beta, u_\beta^1, u_\beta^2)$ is fuzzy closed.

Proof. Let F be a fuzzy closed subset of $\prod_{\alpha\in J}(X_\alpha, u_\alpha^1, u_\alpha^2)$. Then F is a fuzzy closed subset of $\prod_{\alpha\in J}(X_\alpha, u_\alpha^1)$ and $\prod_{\alpha\in J}(X_\alpha, u_\alpha^2)$ respectively. Since
$p_\beta : \prod_{\alpha\in I}(X_\alpha, u_\alpha^1) \to (X_\beta, u_\beta^1)$ is fuzzy closed, $p_\beta\ (F)$ is a fuzzy closed subset of (X_β, u_β^1). Similarly, since $p_\beta : \prod_{\alpha\in I}(X_\alpha, u_\alpha^2) \to (X_\beta, u_\beta^2)$ is fuzzy closed, hence $p_\beta\ (F)$ is a fuzzy closed subset of (X_β, u_β^2). Consequently, $p_\beta\ (F)$ is a fuzzy closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$, hence p_β is fuzzy closed.

Proposition 3.4. Let (X, u_1, u_2) be a fuzzy biclosure space, $\{(Y_\alpha, v_\alpha^1, v_\alpha^2) : \alpha \in J\}$ be a family of fuzzy biclosure spaces and $f : X \to \prod_{\alpha\in J} Y_\alpha$ be a map. Then $f : (X, u_1, u_2) \to \prod_{\alpha\in J}(Y_\alpha, v_\alpha^1, v_\alpha^2)$ is fuzzy closed if and only if
$\pi_\alpha \circ f : (X, u_1, u_2) \to (Y_\alpha, v_\alpha^1, v_\alpha^2)$ is fuzzy closed for each $\alpha \in J$, where $\pi_\beta : \prod_{\alpha\in J} Y_\alpha \to Y_\beta$.

Proof. Let f be fuzzy closed. Since π_α is fuzzy closed for each $\alpha \in J$, it follows that $\pi_\alpha \circ f$ is fuzzy closed for each $\alpha \in J$.

Conversely, let $\pi_\alpha \circ f$ be fuzzy closed for each $\alpha \in J$. Suppose that f is not fuzzy closed. Then there exists a fuzzy closed subset F of

(X, u_1, u_2) such that

$$\prod_{\alpha \in J} v_\alpha^1 v_\alpha^2 \pi_\alpha(f(F)) \not\leq f(F)$$

Therefore, there exists $\beta \in J$ such that $v_\beta^1 v_\beta^2 \pi_\beta(f(F)) \not\leq \pi_\beta(f(F))$. But $\pi_\beta \circ f$ is fuzzy closed, hence $\pi_\beta(f(F))$ is a fuzzy closed subset of $(Y_\beta, v_\beta^1, v_\beta^2)$. This is a contradiction.

Proposition 3.5. Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in J\}$ and $\{(Y_\alpha, v_\alpha^1, v_\alpha^2) : \alpha \in J\}$ be families of fuzzy biclosure spaces. For each $\alpha \in J$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a surjection and let $f : \prod_{\alpha \in J} X_\alpha \to \prod_{\alpha \in J} Y_\alpha$ be defined by $f((x_\alpha)_{\alpha \in J}) = (f_\alpha(x_\alpha))_{\alpha \in J}$.Then $f : \prod_{\alpha \in J}(X_\alpha, u_\alpha^1, u_\alpha^2) \to \prod_{\alpha \in J}(Y_\alpha, v_\alpha^1, v_\alpha^2)$ is fuzzy closed if and only if
$f_\alpha : (X_\alpha, u_\alpha^1, u_\alpha^2) \to (Y_\alpha, v_\alpha^1, v_\alpha^2)$ is fuzzy closed for each $\alpha \in J$.

Proof. **Necessity :** Let $\beta \in J$ and let F be a fuzzy closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$. Then $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is a fuzzy closed subset of $\prod_{\alpha \in J}(X_\alpha, u_\alpha^1, u_\alpha^2)$ by Proposition 2.16. Since f is fuzzy closed,$f(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha$ is a fuzzy closed subset of $\prod_{\alpha \in J}(Y_\alpha, v_\alpha^1, v_\alpha^2)$. But $f(F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} X_\alpha) = f_\beta(F) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} Y_\alpha$, hence $f_\beta(F) \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in J}} Y_\alpha$ is a fuzzy closed subset of $\prod_{\alpha \in J}(Y_\alpha, v_\alpha^1, v_\alpha^2)$. By Proposition 2.16, $f_\beta(F)$ is a fuzzy closed subset of $(Y_\beta, v_\beta^1, v_\beta^2)$, hence$f_\beta$ is fuzzy closed.

Sufficiency : Let f_β be fuzzy closed for each $\beta \in J$. Suppose that f is not fuzzy closed. Then there exist a fuzzy closed subset F of $\prod_{\alpha \in J}(X_\alpha, u_\alpha^1, u_\alpha^2)$ such that $\prod_{\alpha \in J} v_\alpha^1 v_\alpha^2 \pi_\alpha(f(F)) \not\leq f(F)$ Therefore, there exists $\beta \in J$ such that
$v_\beta^1 v_\beta^2 \pi_\beta(f(F)) \not\leq \pi_\beta(f(F))$. Denote the projection
$p_\beta : \prod_{\alpha \in J}(X_\alpha, u_\alpha^1, u_\alpha^2) \to (X_\beta, u_\beta^1, u_\beta^2)$, $\beta \in J$. We have $\pi_\beta(f(A)) = f_\beta(p_\beta(A))$, for each $A \leq \prod_{\alpha \in J} X_\alpha$ and all $\beta \in J$. Since $p_\beta(F)$ is a fuzzy closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$ and f_β is fuzzy closed , it follows that $f_\beta(p_\beta(F))$ is a fuzzy closed subset of $(Y_\beta, v_\beta^1, v_\beta^2)$. This is a contradiction.

Proposition 3.6. Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in J\}$ and $\{(Y_\alpha, v_\alpha^1, v_\alpha^2) : \alpha \in J\}$ be families of fuzzy biclosure spaces. For each $\alpha \in J$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a surjection and let $f : \prod_{\alpha \in J} X_\alpha \to \prod_{\alpha \in J} Y_\alpha$ be

defined by $f((x_\alpha)_{\alpha\in J}) = (f_\alpha(x_\alpha))_{\alpha\in J}$. If $f : \prod_{\alpha\in J}(X_\alpha, u^1_\alpha, u^2_\alpha) \to \prod_{\alpha\in J}(Y_\alpha, v^1_\alpha, v^2_\alpha)$ is fuzzy open, then
$f_\alpha : (X_\alpha, u^1_\alpha, u^2_\alpha) \to (Y_\alpha, v^1_\alpha, v^2_\alpha)$ is fuzzy open for each $\alpha \in J$.

Proof. Let $\beta \in J$ and let G be a fuzzy open subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then
$G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in J}} X_\alpha$ is a fuzzy open subset of $\prod_{\alpha\in J}(X_\alpha, u^1_\alpha, u^2_\alpha)$.
Since f is fuzzy open, $f(G \times \prod_{\substack{\alpha\neq\beta,\\ \alpha\in J}} X_\alpha)$ is a fuzzy open subset of $\prod_{\alpha\in J}(Y_\alpha, v^1_\alpha, v^2_\alpha)$. But
$f(G\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}} X_\alpha) = f_\beta(G)\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}} Y_\alpha$, hence $f_\beta(G)\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in J}} Y_\alpha$
is a fuzzy open subset of $\prod_{\alpha\in J}(Y_\alpha, v^1_\alpha, v^2_\alpha)$. By Proposition 2.17, $f_\beta(G)$ is a fuzzy open subset of $(Y_\beta, v^1_\beta, v^2_\beta)$, hencef_β is fuzzy open.

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U. D. Tapi

Department Of Applied Mathematics and Computational Science, Shri G.S.Institute of Technology and Science, 23, Sir M. Visvesvaraya Marg (Park Road), Indore (M.P.), INDIA, e-mail: utapi@sgsits.ac.in

R. Navalakhe

Department Of Applied Mathematics and Computational Science, Shri G.S.Institute of Technology and Science, 23, Sir M. Visvesvaraya Marg, (Park Road), Indore (M.P.), INDIA, e-mail: rnavalakhe@sgsits.ac.in