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DENSE SUBCLASSES IN ABSTRACT SOBOLEV SPACES ON METRIC MEASURE SPACES

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Abstract. Given a metric measure space (X, d, μ) and a Banach function space $\mathbf{B}$ over X that has absolutely continuous norm, we prove two results regarding the density in the Newtonian space $N^{1,\mathbf{B}}(X)$ of the subclasses consisting of bounded functions, respectively of bounded functions supported in closed balls. We do not assume that μ is a doubling measure. If $\mathbf{B}$ is rearrangement invariant, (X, d) is proper and the measure μ is nonatomic, it turns out that the class of bounded compactly supported functions from $N^{1,\mathbf{B}}(X)$ is dense in $N^{1,\mathbf{B}}(X)$.

1. Introduction and preliminaries

It is of great importance for the theory and applications of Sobolev spaces on $\mathbb{R}^n$ that smooth functions are dense in the respective Sobolev spaces. In the framework of metric measure spaces the role of smooth functions is played by Lipschitz continuous functions. The density of Lipschitz functions in Newtonian spaces $N^{1,p}(X)$ has been proved in doubling metric measure spaces (X, d, μ) supporting a weak $(1, p)-$Poincaré inequality [16]. Corresponding density results have been proved for Orlicz-Sobolev spaces [17], [1] and Sobolev-Lorentz spaces [6]. These density results rely on some preparatory lemmas, that reduce the proof in the general case to the proof in the special case where the approximated function is bounded and has the support contained in a closed ball.

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The purpose of this paper is to extend the above mentioned lemmas to the case of more general Newtonian spaces, using Banach function spaces as an unifying framework for Orlicz spaces and Lorentz spaces.

Let (X, Σ, μ) be a complete and $\sigma-$finite measure space and let $\boldsymbol{M}^+(X)$ be the collection of all $\mu-$measurable functions $f : X \to [0, \infty]$.

Definition 1. [2] *A function $N : \boldsymbol{M}^+(X) \to [0, \infty]$ is called a Banach function norm if, for all f, g, f_n $(n \geq 1)$ in $\boldsymbol{M}^+(X)$, for all constants $a \geq 0$ and for all measurable sets $E \subset X$, the following properties hold:*

(P1) i) $N(f) = 0$ if and only if $f = 0$ $\mu-$a.e.; ii) $N(af) = aN(f)$; iii) $N(f + g) \leq N(f) + N(g)$.

(P2) If $0 \leq g \leq f$ $\mu-$a.e., then $N(g) \leq N(f)$.

(P3) If $0 \leq f_n \uparrow f$ $\mu-$a.e., then $N(f_n) \uparrow N(f)$.

(P4) If $\mu(E) < \infty$, then $N(\chi_E) < \infty$.

(P5) If $\mu(E) < \infty$, then $\int_E f \; d\mu \leq C_E N(f)$, for some constant $C_E \in (0, \infty)$ depending only on E and ρ.

Let B be the collection of the $\mu-$measurable functions $f : X \to [-\infty, \infty]$ for which $N(|f|) < \infty$.For $f \in B$ define

$$\|f\|_B = N(|f|).$$

Then $(B, \|\cdot\|_B)$ is a seminormed space. As usual we identify two functions that coincide $\mu-$a.e. and denote by $\approx$ the relation of equality $\mu-$a.e. Let $f, g : X \to \overline{\mathbb{R}}$ such that $f \in B$ and $f = g$ $\mu-$a.e. Then g is $\mu-$measurable and $N(|g|) = N(|f|) < \infty$, hence $g \in B$. Moreover, by Definition 1 (P5) and the $\sigma-$finiteness of μ it follows that every function in B is finite $\mu-$a.e. Then $f - g = 0$ $\mu-$a.e. and therefore $\|f - g\|_B = 0$. Consider the quotient vector space $\mathbf{B} = B/\approx$ and define $\|\cdot\|_{\mathbf{B}}$ by $\left\|\hat{f}\right\|_{\mathbf{B}} = \|f\|_B$. Then $(\mathbf{B}, \; \|\cdot\|_{\mathbf{B}})$ is a normed space, that is complete by [2, Theorem I.1.6]

Definition 2. *A function $f \in \mathbf{B}$ is said to have absolutely continuous (AC) norm in $\mathbf{B}$ if and only if $\|f\chi_{E_k}\|_{\mathbf{B}} \to 0$ for every sequence $(E_k)_{k\geq 1}$ of measurable sets satisfying $E_k \to \varnothing$ $\mu-$a.e. (i.e. $\mu\left(\limsup_{k\to\infty} E_k\right) = 0$). The space $\mathbf{B}$ is said to have absolutely continuous norm if every $f \in \mathbf{B}$ has AC norm .*

Note that an Orlicz space $L^{\Psi}(X)$ has absolutely continuous norm if the Young function Ψ is doubling. The $(p,q)-$norm of a Lorentz space $L^{p,q}(X)$ with $1 < p < \infty$ and $1 \leq q < \infty$ is absolutely continuous (see the discussion following Definition 2.1 from [6]).

Let $f : X \to \overline{\mathbb{R}}$ be a $\mu-$measurable function. The distribution function of f is defined by $d_f(t) = \mu(\{x \in X : |f(x)| > t\})$ for $t \geq 0$. The nonincreasing rearrangement of f is defined by

$$f^*(t) = \inf\{s \geq 0 : d_f(s) \leq t\}, \, t \geq 0.$$

Definition 3. *A Banach function space* $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ *is said to be rearrangement invariant if* $f^* = g^*$ *implies* $\|f\|_{\mathbf{B}} = \|g\|_{\mathbf{B}}$.

Lebesgue spaces and some of their generalizations, namely Orlicz spaces and Lorentz spaces are well-known examples of rearrangement invariant Banach function spaces.

Definition 4. *The fundamental function of a* $\mathbf{B}$ *rearrangement invariant space over* (X, μ) *is* $\Phi_{\mathbf{B}} : [0, \infty) \to [0, \infty)$ *defined by* $\Phi_{\mathbf{B}}(t) = \|\chi_E\|_{\mathbf{B}}$, *where* $E \subset X$ *is a* $\mu-$*measurable set with* $\mu(E) = t$.

The above definition is unambiguous, since the characteristic functions of two sets with equal measures have the same distribution function.

Lemma 1. [2, Corollary II. 5.3]*Let* $\mathbf{B}$ *be a rearrangement invariant Banach function space over a resonant measure space* (X, μ). *Then the fundamental function* $\Phi_{\mathbf{B}}$ *satisfies:* $\Phi_{\mathbf{B}}$ *is increasing, vanishes only at the origin, is continuous (except perhaps at the origin) and* $t \mapsto \frac{\Phi_{\mathbf{B}}(t)}{t}$ *is decreasing.*

It is known that if μ is $\sigma-$finite and nonatomic, then (X, μ) is resonant. If μ is doubling and X has no isolated points, then μ is nonatomic.

In the following, the triple (X, d, μ) denotes a metric measure space, which is a metric space (X, d) equipped with a Borel regular measure μ, that is finite and positive on balls. It is known that under the above assumptions μ is regular, i.e. inner regular and outer regular [10, p. 3]. Obviously, μ is $\sigma-$finite.

Denote by $B(x,r) = \{y \in X : d(y,x) < r\}$ and $\overline{B}(x,r) = \{y \in X : d(y,x) \leq r\}$ the open, respectively the closed balls in X.

The measure μ on the metric space (X, d) is said to be *doubling* if there is a constant $C_d \geq 1$ such that for every ball $B(x,r) \subset X$,

$$\mu(B(x,2r)) \leq C_d \mu(B(x,r)). \tag{1.1}$$

A metric space is called *proper* if every closed ball of the space is compact. A metric space equipped with a doubling measure is proper if and only if it is complete.

Remark 1. *Let* (X, τ) *be a normal topological space and* (X, Σ, μ) *is a measure space, such that* $\tau \subset \Sigma$ *and the measure* μ *is regular. Then every* $\mu-$*measurable function* $f : X \to \overline{\mathbb{R}}$ *that is finite* $\mu-$*a.e. coincides* $\mu-$*a.e. with a Borel measurable function* $g : X \to \overline{\mathbb{R}}$*, that belongs to one of the Baire classes* B_0*,* B_1*,* B_2*. Moreover, if* $f \geq 0$*, then we may choose* $g \geq 0$ [15],[4], [14].

If (X, d, μ) *is a metric measure space, then the above assumptions hold, where* τ *is the topology induced by* d *and* Σ *is the family of all* $\mu-$*measurable subsets of* X*. Let* $\mathbf{B}$ *be a Banach function space over* (X, μ)*. Then every function* $f \in \mathbf{B}$*, being* $\mu-$*measurable and finite* $\mu-$*a.e., coincides* $\mu-$*a.e. with a Borel function.*

We will denote by Γ_{rec} the family of all non-constant rectifiable compact curves in X. We will consider in the following, as we may, that every compact rectifiable curve γ is parameterized by arc-length, i.e. $\gamma : [0, l(\gamma)] \to X$ and $l\left(\gamma|_{[0,t]}\right) = t$ for all $t \in [0, l(\gamma)]$. For $g : X \to [0, \infty]$ we may define $\int_\gamma g ds = \int_0^{l(\gamma)} (g \circ \gamma)(t) \, dt$ for all curves $\gamma \in \Gamma_{rec}$ for which $g \circ \gamma : [0, l(\gamma)] \to [0, \infty]$ is Lebesgue measurable. If $g : X \to [0, \infty]$ is Borel measurable, then $g \circ \gamma : I \to [0, \infty]$ is Borel measurable for every continuous function $\gamma : I \to X$, where $I \subset \mathbb{R}$ is an interval.

Definition 5. *A Borel measurable function* $g : X \to [0, \infty]$ *is said to be an upper gradient of a function* $u : X \to \mathbb{R}$ *if for every rectifiable curve* $\gamma : [0, l(\gamma)] \to X$ *the following inequality holds*

$$|u(\gamma(0)) - u(\gamma(l(\gamma)))| \leq \int_\gamma g ds. \tag{1.2}$$

The $\mathbf{B}-$*modulus of a family* Γ of curves in X is defined by $Mod_{\mathbf{B}}(\Gamma) = \inf \|\rho\|_{\mathbf{B}}$,where the infimum is taken over all Borel functions $\rho : X \to [0, \infty]$ satisfying $\int_\gamma \rho \, ds \geq 1$ for all locally rectifiable curves γ in X. The $\mathbf{B}-$modulus of the family of curves that are not rectifiable is zero.

A $\mathbf{B}-$*weak upper gradient* of a function $u : X \to \mathbb{R}$ is a Borel measurable function $g : X \to [0, \infty]$ such that (1.2) holds for all rectifiable curves $\gamma : [0, l(\gamma)] \to X$ except for a curve family with zero $\mathbf{B}-$modulus.

We will weaken the assumption that g is Borel measurable in the definition of a $\mathbf{B}-$weak upper gradient, saying that $g : X \to [0, \infty]$ is a *generalized* $\mathbf{B}-$*weak upper gradient* of a function $u : X \to \mathbb{R}$ if there exists a curve family $\Gamma_0 \subset \Gamma_{rec}$ with $Mod_{\mathbf{B}}(\Gamma_0) = 0$ such that for every $\gamma \in \Gamma_{rec} \setminus \Gamma_0$ the function $g \circ \gamma : [0, l(\gamma)] \to [0, \infty]$ is Lebesgue measurable and (1.2) holds. Clearly, a generalized $\mathbf{B}-$weak upper gradient of a function $u : X \to \mathbb{R}$ is a $\mathbf{B}-$weak upper gradient of this function if and only if it is Borel measurable. We will discuss the connections between these two notions in Section 3.

For every function $u : X \to \mathbb{R}$ we will denote by $G_{u,B}$ the family of all $\mathbf{B}-$weak upper gradients $g \in \mathbf{B}$ of u in X. Consider the set $\widetilde{N}^{1,\mathbf{B}}(X)$ formed from the real-valued functions $u \in \mathbf{B}$ for which $G_{u,\mathbf{B}}$ is non-empty. The functional $\|u\|_{1,\mathbf{B}} := \|u\|_{\mathbf{B}} + \inf\{\|g\|_{\mathbf{B}} : g \in G_{u,\mathbf{B}}\}$ is a seminorm on $\widetilde{N}^{1,\mathbf{B}}(X)$. The Sobolev space of Newtonian type $N^{1,\mathbf{B}}(X)$ is defined as the quotient normed space of $\widetilde{N}^{1,\mathbf{B}}(X)$ with respect to the equivalence relation defined by: $u \sim v$ if $\|u - v\|_{1,\mathbf{B}} = 0$. The norm on $N^{1,\mathbf{B}}(X)$ corresponding to the seminorm $\|\cdot\|_{1,\mathbf{B}}$ is denoted by $\|\cdot\|_{N^{1,\mathbf{B}}(X)}$ [12].

For $\mathbf{B} = L^p(X)$, $1 \le p < \infty$, the space $N^{1,\mathbf{B}}(X) = N^{1,p}(X)$ was the first extension, based on upper gradients, of Sobolev spaces to metric measure spaces, introduced and studied by Shanmugalingam [16]. The case $\mathbf{B} = L^\infty(X)$ has been studied by Durand-Cartagena and Jaramillo [7]. The theory of Newtonian spaces $N^{1,p}(X)$ was further generalized by Tuominen [17] and Aïssaoui [1], who studied the case where $\mathbf{B} = L^\Psi(X)$ is an Orlicz space, and very recently by Costea and Miranda [6], who developed the theory for the case where $\mathbf{B} = L^{p,q}(X)$ is a Lorentz space. Note that for $\mathbf{B} = L^{p,q}(X)$ the functions in $N^{1,\mathbf{B}}(X)$ are assumed to extended real-valued, unlike for $\mathbf{B} = L^p(X)$ and $\mathbf{B} = L^\Psi(X)$, the definition of an upper gradient being more general than Definition 5, that is taken from [16], following [11].

Remark 2. *It was proved in* [12, Proposition 2] *that for every* $\mathbf{B}-$*weak upper gradient* $g \in \mathbf{B}$ *of a function* $u : X \to \mathbb{R}$ *there is a decreasing sequence* $(g_i)_{i \ge 1}$ *of upper gradients of* u *such that* $\lim_{i \to \infty} \|g_i - g\|_{\mathbf{B}} = 0$. *Then* $G_{u,\mathbf{B}}$ *is non-empty if and only if* u *has an upper gradient in* $\mathbf{B}$. *For all* $u \in N^{1,\mathbf{B}}(X)$ *we have*

$$\|u\|_{N^{1,\mathbf{B}}(X)} := \|u\|_{\mathbf{B}} + \inf\{\|g\|_{\mathbf{B}} : g \in \mathbf{B} \text{ is an upper gradient of } u\}.$$

2. Approximation by bounded functions in Newtonian spaces

The following lattice property of $N^{1,\mathbf{B}}(X)$ is well-known in the cases where $\mathbf{B}$ is an Orlicz space [17, Lemma 6.14] or a Lorentz space [6, Lemma 3.15, Lemma 3.16].

Lemma 2. *If $g_i \in \mathbf{B}$ is a $\mathbf{B}$-weak upper gradient of $u_i : X \to \mathbb{R}$, for $i = 1, 2$, then $u := \max\{u_1, u_2\}$ and $v := \min\{u_1, u_2\}$ have the $\mathbf{B}-$weak upper gradient $g = \max\{g_1, g_2\}$ and $g \in \mathbf{B}$. Moreover, with the above notations, if $g_i \in \mathbf{B}$ is a generalized $\mathbf{B}-$weak upper gradient of $u_i : X \to \mathbb{R}$, for $i = 1, 2$, then g is a generalized $\mathbf{B}-$weak upper gradient of u and v.*

Proof. If g_i, $i = 1, 2$ are Borel measurable, then g is Borel measurable. More general, if $g_i \circ \gamma : [0, l(\gamma)] \to [0, \infty]$, $i = 1, 2$ are Lebesgue measurable for some $\gamma \in \Gamma_{rec}$, then $g \circ \gamma : [0, l(\gamma)] \to [0, \infty]$ is Lebesgue measurable.

For $i \in \{1, 2\}$, let $\Gamma_i \subset \Gamma_{rec}$ be the family of curves $\gamma : [0, l(\gamma)] \to X$ for which the inequality the function $g_i \circ \gamma : [0, l(\gamma)] \to [0, \infty]$ is not Lebesgue measurable or $|u_i(\gamma(0)) - u_i(\gamma(l(\gamma)))| \leq \int_\gamma g_i ds$ does not hold. Then $Mod_{\mathbf{B}}(\Gamma_i) = 0$, $i = 1, 2$.

If g_i, $i = 1, 2$ are Borel measurable, then $\int_\gamma g_i ds \leq \int_\gamma g ds$ for every $\gamma \in \Gamma_{rec}$, therefore $|u_i(\gamma(0)) - u_i(\gamma(l(\gamma)))| \leq \int_\gamma g ds$ for all $\gamma \in \Gamma_{rec} \setminus \Gamma_i$, $i = 1, 2$. In the general case, $\int_\gamma g_i ds \leq \int_\gamma g ds$ for every $\gamma \in \Gamma_{rec} \setminus (\Gamma_1 \cup \Gamma_2)$, hence $|u_i(\gamma(0)) - u_i(\gamma(l(\gamma)))| \leq \int_\gamma g ds$ for all $\gamma \in \Gamma_{rec} \setminus (\Gamma_1 \cup \Gamma_2)$, $i = 1, 2$.

But $|u(x) - u(y)| \leq \max\{|u_1(x) - u_1(y)|, |u_2(x) - u_2(y)|\}$ for all $x, y \in X$. Then $|u(\gamma(0)) - u(\gamma(l(\gamma)))| \leq \int_\gamma g ds$ for all $\gamma \in \Gamma_{rec} \setminus (\Gamma_1 \cup \Gamma_2)$ and $Mod_{\mathbf{B}}(\Gamma_1 \cup \Gamma_2) = 0$, hence g is a (generalized) $\mathbf{B}$-weak upper gradient of u.

Since $v := \min\{u_1, u_2\} = -\max\{-u_1, -u_2\}$ and $(-u_i)$ has a (generalized) $\mathbf{B}$-weak upper gradient g_i, $i = 1, 2$, it follows by the preceding proof that g is also a (generalized) $\mathbf{B}$-weak upper gradient of v.

Obviously, $0 \leq g \leq g_1 + g_2$, hence $g \in \mathbf{B}$ by Definition 1 (P1) iii) and (P2). $\square$

We need the following counterpart of Lemma 3.16 from [6].

Lemma 3. *If $g \in \mathbf{B}$ is a (generalized) $\mathbf{B}$-weak upper gradient of $u : X \to [0, \infty)$, then for every $k \in [0, \infty)$ the function g is a (generalized) $\mathbf{B}$-weak upper gradient of $u_k := \min\{u, k\}$. Moreover, if $u \in \mathbf{B}$, then $u_k \in \mathbf{B}$, with $\|u_k\|_{\mathbf{B}} \leq \|u\|_{\mathbf{B}}$, for every $k \in [0, \infty)$.*

Proof. Let $k \in [0, \infty)$. Since $|u_k(x) - u_k(y)| \leq |u(x) - u(y)|$ for all $x, y \in X$, it is obvious that each (generalized) $\mathbf{B}$-weak upper gradient of u is also a (generalized) $\mathbf{B}$-weak upper gradient of u_k.

Since $0 \leq u_k \leq u$ on X, we have $N(u_k) \leq N(u)$ by Definition 1 (P2), hence $u \in \mathbf{B}$ implies $u_k \in \mathbf{B}$. □

A function $u : X \to \mathbb{R}$ is said to be absolutely continuous (AC) on a compact rectifiable curve γ parameterized by arc-length if $u \circ \gamma : [0, l(\gamma)] \to \mathbb{R}$ is absolutely continuous. The function u is said to be AC on $\mathbf{B}-$almost every curve if there exists a family $\Gamma_0 \subset \Gamma_{rec}$ with $M_{\mathbf{B}}(\Gamma_0) = 0$, such that u is absolutely continuous on each curve $\gamma \in \Gamma_{rec} \setminus \Gamma_0$. We will denote by $ACC_{\mathbf{B}}(X)$ the family of all functions $u : X \to \mathbb{R}$ that are AC on $\mathbf{B}-$almost every curve. It is known that every function $u : X \to R$ that has a $\mathbf{B}-$weak upper gradient $g \in \mathbf{B}$ in X belongs to $ACC_{\mathbf{B}}(X)$, in particular $N^{1,\mathbf{B}}(X) \subset ACC_{\mathbf{B}}(X)$ [13], [12].

Lemma 4. [13] *Assume that $F \subset X$ is a Borel set and that the function $u \in ACC_{\mathbf{B}}(X)$ is constant $\mu-$a.e. on F. If u has a $\mathbf{B}-$weak upper gradient g in X, then $g\chi_{X \setminus F}$ is also a $\mathbf{B}-$weak upper gradient of u in X.*

We will say that $\mathbf{B}$ has property (C) if $\lim\limits_{k\to\infty} \mu(E_k) = 0$ for every sequence $E_k \subset X$, $k \geq 1$ of measurable sets such that $\lim\limits_{k\to\infty} \|\chi_{E_k}\|_{\mathbf{B}} = 0$.

The following lemma provides an example of class of Banach function spaces that have property (C).

Lemma 5. *Every rearrangement invariant Banach function space over a resonant measure space has property (C).*

Proof. Let $\mathbf{B}$ a rearrangement invariant Banach function space over a resonant measure space. Let $E_k \subset X$, $k \geq 1$ be a sequence of measurable sets such that $\lim\limits_{k\to\infty} \|\chi_{E_k}\|_{\mathbf{B}} = 0$. Denoting $t_k := \mu(E_k)$, $k \geq 1$, this means that $\lim\limits_{k\to\infty} \Phi_{\mathbf{B}}(t_k) = 0$. Let $t := \limsup\limits_{k\to\infty} t_k$. Then $t \geq 0$ and there exists a subsequence $\left(t_{k_j}\right)_{j\geq 1}$ such that $t = \lim\limits_{j\to\infty} t_{k_j}$. If $t > 0$, we see for $t < \infty$ that $\lim\limits_{j\to\infty} \Phi_{\mathbf{B}}\left(t_{k_j}\right) = \Phi_{\mathbf{B}}(t) > 0$ for $t < \infty$

and for $t = \infty$ that $\lim_{j\to\infty} \Phi_{\mathbf{B}}\left(t_{k_j}\right) = \lim_{\tau\to\infty} \Phi_{\mathbf{B}}(\tau)$ (see Lemma 1). We get a contradiction, that proves that $t = 0$, hence there exists $\lim_{k\to\infty} t_k = 0$. □

Proposition 1. *Let* $\mathbf{B}$ *be a Banach function space over* X *that has absolutely continuous norm and has property (C). Let* $u \in N^{1,\mathbf{B}}(X)$ *be nonnegative. For each integer* $k \geq 0$ *we define* $u_k := \min\{u, k\}$*. Then* $u_k \in \mathbf{B}$ *for each* $k \geq 0$ *and the sequence* $(u_k)_{k\geq 0}$ *converges to* u *in the norm of* $N^{1,\mathbf{B}}(X)$.

Proof. Let $g \in \mathbf{B}$ be a $\mathbf{B}-$weak upper gradient of u. By Lemma 3, for each $k \geq 0$, $u_k \in \mathbf{B}$ and g is a $\mathbf{B}-$weak upper gradient of u_k. Then $2g$ is a $\mathbf{B}-$weak upper gradient of $u_k - u$.

For each integer $k \geq 0$, let $E_k := \{x \in X : u(x) > k\}$. Since the set E_k is measurable and the measure μ is Borel regular and outer regular, there exists an open set $O_k \subset X$ such that $E_k \subset O_k$ and $\mu(O_k) \leq \mu(E_k) + 2^{-k}$.

Having $E_{k+1} \subset E_k$ for each $k \geq 0$, we may choose the sequence $(O_k)_{k\geq 0}$ such that $O_{k+1} \subset O_k$ for each $k \geq 0$. Since $u \geq k\chi_{E_k}$ on X, $\|u\|_{\mathbf{B}} \geq \|k\chi_{E_k}\|_{\mathbf{B}}$, for each $k \geq 0$. Then $\lim_{k\to\infty} \|\chi_{E_k}\|_{\mathbf{B}} = 0$. Since $\mathbf{B}$ has property (C), it follows that $\lim_{k\to\infty} \mu(E_k) = 0$. Then $\lim_{k\to\infty} \mu(O_k) = 0$ and, since $O_{k+1} \subset O_k$ for each $k \geq 0$ and there exists k_0 such that $\mu(O_{k_0}) < \infty$, we have $\mu\left(\limsup_{k\to\infty} O_k\right) = \mu\left(\bigcap_{k=1}^{\infty} O_k\right) = \lim_{k\to\infty} \mu(O_k) = 0$.

Let $k \geq 0$ be an integer. Since $u_k - u = 0$ on the closed set $X \setminus O_k$ and $2g$ is a $\mathbf{B}-$weak upper gradient of $u_k - u$, by Lemma 4 $2g\chi_{O_k}$ is also a $\mathbf{B}-$weak upper gradient of $u_k - u$ in X. For $x \in E_k$ we have $|u_k(x) - u(x)| = u(x) - k \leq u(x)$, while for $x \in O_k \setminus E_k$ we have $|u_k(x) - u(x)| = 0 \leq u(x)$, therefore $|u_k - u| \leq u\chi_{O_k}$. Since $\|u_k - u\|_{N^{1,\mathbf{B}}(X)} \leq \|u_k - u\|_{\mathbf{B}} + \|2g\chi_{O_k}\|_{\mathbf{B}}$, we have

$$\|u_k - u\|_{N^{1,\mathbf{B}}(X)} \leq \|u\chi_{O_k}\|_{\mathbf{B}} + 2\|g\chi_{O_k}\|_{\mathbf{B}}.$$

Since $\mu\left(\limsup_{k\to\infty} O_k\right) = 0$, by the absolute continuity of the norm on $\mathbf{B}$ and the fact that $u, g \in \mathbf{B}$, we get $\lim_{k\to\infty} \|u\chi_{O_k}\|_{\mathbf{B}} = 0$ and $\lim_{k\to\infty} \|g\chi_{O_k}\|_{\mathbf{B}} = 0$. The claim follows using the last inequality. □

Corollary 1. *Let* (X, d, μ) *and* $\mathbf{B}$ *be as in Proposition 1. For each* $u \in N^{1,\mathbf{B}}(X)$ *and every* $\varepsilon > 0$ *there is a bounded function* $v \in N^{1,\mathbf{B}}(X)$ *such that* $\|u - v\|_{N^{1,\mathbf{B}}(X)} < \varepsilon$.

Proof. Let $u \in N^{1,\mathbf{B}}(X)$ and $\varepsilon > 0$. Define $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. Then u^+, $u^- \in N^{1,\mathbf{B}}(X)$ are nonnegative and $u = u^+ - u^-$. By Proposition 1, the sequences $u_k^+ := \min\{u^+, k\}$ and $u_k^- := \min\{u^-, k\}$, $k \geq 1$, converge , respectively, to u^+ and u^- in the norm of $N^{1,\mathbf{B}}(X)$. There exists a positive integer $N = N(\varepsilon)$ so that $\left\|u^\pm - u_k^\pm\right\|_{N^{1,\mathbf{B}}(X)} < \frac{\varepsilon}{2}$ for every $k \geq N$, hence $\left\|u - \left(u_N^+ - u_N^-\right)\right\|_{N^{1,\mathbf{B}}(X)} < \varepsilon$. □

Remark 3. *Proposition 1 and Corollary 1 extend, respectively, Proposition 6.5 from* [6] *and Proposition 6.16 from* [17].

3. Approximation by compactly supported functions in Newtonian spaces

We investigate some natural connections between the notions of $\mathbf{B}-$weak upper gradient and generalized $\mathbf{B}-$weak upper gradient.

Lemma 6. *Let* $u : X \to \mathbb{R}$.

1) If g *is a generalized* $\mathbf{B}-$*weak upper gradient of* u *and* $h = g$ $\mu-$*almost everywhere in* X*, then* h *is a generalized* $\mathbf{B}-$*weak upper gradient of* u.

2) For every generalized $\mathbf{B}-$*weak upper gradient* g_1 *of* u *that is finite* $\mu-$*a.e. there exists a* $\mathbf{B}-$*weak upper gradient* h_1 *of* u *such that* $h_1 = g_1$ $\mu-$*almost everywhere in* X.

Proof. 1) Assume that $g : X \to [0, \infty]$ is a generalized $\mathbf{B}-$weak upper gradient of u. There exists a exists a curve family $\Gamma_0 \subset \Gamma_{rec}$ with $Mod_{\mathbf{B}}(\Gamma_0) = 0$ such that for every $\gamma \in \Gamma_{rec} \setminus \Gamma_0$ the function $g \circ \gamma : [0, l(\gamma)] \to [0, \infty]$ is Lebesgue measurable and (1.2) holds.

Assume that $h = g$ $\mu-$a. e. in X. Let $E_1 := \{x \in X : g(x) \neq h(x)\}$. Then E_1 is measurable and $\mu(E_1) = 0$. Since μ is Borel regular, there exists a Borel set $E \subset X$ such that $E_1 \subset E$ and $\mu(E) = \mu(E_1)$. Let Γ_1 be the family of all curves for which $\mathcal{L}^1(\gamma^{-1}(E)) > 0$. The Borel function $\rho_E := \infty \cdot \chi_E$ is an admissible function for Γ_1 and $\|\rho_E\|_{\mathbf{B}} = 0$, hence $Mod_{\mathbf{B}}(\Gamma_1) = 0$. For every $\gamma \in \Gamma_{rec} \setminus \Gamma_1$ we have $\mathcal{L}^1(\gamma^{-1}(E)) = 0$, but $\gamma^{-1}(E_1) \subset \gamma^{-1}(E)$, hence $\gamma^{-1}(E_1)$ is $\mathcal{L}^1$-measurable and $\mathcal{L}^1(\gamma^{-1}(E_1)) = 0$.

Let $\gamma \in \Gamma_{rec} \setminus (\Gamma_0 \cup \Gamma_1)$, with $\gamma : [0, l(\gamma)] \to X$ parameterized by arc-length. Then $(h \circ \gamma)(t) = (g \circ \gamma)(t)$ for $\mathcal{L}^1-$a.e. $x \in [0, l(\gamma)]$ and

$g\circ\gamma:[0,l(\gamma)]\to[0,\infty]$ is Lebesgue measurable, therefore $h\circ\gamma:[0,l(\gamma)]\to[0,\infty]$ is also Lebesgue measurable and

$$\int_\gamma h\circ\gamma\,ds=\int_\gamma g\circ\gamma\,ds\geq\left|u(\gamma(0))-u(\gamma(l(\gamma)))\right|.$$

Since $Mod_{\mathbf{B}}(\Gamma_0\cup\Gamma_1)=0$, it follows that h is a generalized $\mathbf{B}-$weak upper gradient of u.

2) Let g_1 be a generalized $\mathbf{B}-$weak upper gradient of u that is finite $\mu-$a.e. By Remark 1, there exists a Borel function $h_1:X\to[0,\infty]$ such that $h_1=g_1$ $\mu-$a.e. The preceding argument shows that h_1 is a generalized $\mathbf{B}-$weak upper gradient of u, therefore, being Borel measurable, h_1 is a $\mathbf{B}-$weak upper gradient of u. □

Corollary 2. *A function $u\in\mathbf{B}$ belongs to $N^{1,\mathbf{B}}(X)$ if and only if there exists in $\mathbf{B}$ a generalized $\mathbf{B}-$weak upper gradient of u. For every $u\in N^{1,\mathbf{B}}(X)$ we have $\|u\|_{N^{1,\mathbf{B}}(X)}=\|u\|_{\mathbf{B}}+\inf\{\|h\|_{\mathbf{B}}:h\in\mathbf{B}$ is a generalized $\mathbf{B}-$weak upper gradient of $u\}$.*

The following counterpart of the product rule extends Lemma 6.7 from [6] in the case of real-valued functions and slightly generalizes Theorem 2 from [13], where u_1 and u_2 were assumed to be bounded Borel measurable functions.

Lemma 7. *Assume that $u_k:X\to\mathbb{R}$ is a $\mu-$measurable function which has a $\mathbf{B}-$weak upper gradient $g_k\in\mathbf{B}$ in X, for $k\in\{1,2\}$. Then the function $g:=|u_1|\,g_2+|u_2|\,g_1$ is a generalized $\mathbf{B}-$weak upper gradient of $u:=u_1u_2$ in X . Moreover, if u_1 and u_2 are bounded, then $g\in\mathbf{B}$.*

Proof. Let Γ_0 be the family of all $\gamma\in\Gamma_{rec}$ for which $\int_\gamma(g_1+g_2)\,ds=\infty$. Since $g_1+g_2\in\mathbf{B}$, we have $Mod_{\mathbf{B}}(\Gamma_0)=0$ by [12, Proposition 1 (b)]. Let Γ_k, $k\in\{1,2\}$ be the family of all $\gamma\in\Gamma_{rec}$ for which $|u_k(\gamma(0))-u_k(\gamma(l(\gamma)))|\leq\int_\gamma g_k\,ds$ does not hold. Then $Mod_{\mathbf{B}}(\Gamma_k)=0$, since g_k is a $\mathbf{B}-$weak upper gradient of u_k. Let $\Gamma_3\subset\Gamma_{rec}$ be the family of curvesthat have a subcurve in $\Gamma_1\cup\Gamma_2$.

Assume that $\gamma\in\Gamma_{rec}\setminus\Gamma_3$. Then $u_k\circ\gamma$ is absolutely continuous on $[0,l(\gamma)]$ for $k=1,2$, hence $g\circ\gamma:[0,l(\gamma)]\to[0,\infty]$ is Borel measurable. As in the proof of [13, Theorem 2], using a method from [5, Lemma 1.7] it follows that $|u(\gamma(0))-u(\gamma(l))|\leq\int_\gamma[|u_1|\,g_2+|u_2|\,g_1]\,ds$.

Since $Mod_{\mathbf{B}}(\Gamma_0 \cup \Gamma_3) = 0$, it follows that $g := |u_1| g_2 + |u_2| g_1$ is a generalized $\mathbf{B}-$weak upper gradient of $u := u_1 u_2$.

If u_1 and u_2 are bounded, let $M_k := \sup_{x \in X} |u_k(x)|$, $k = 1, 2$. Since g is $\mu-$measurable, $0 \leq g \leq M_1 g_2 + M_2 g_1$ and $M_1 g_2 + M_2 g_1 \in \mathbf{B}$, we get $g \in \mathbf{B}$. $\square$

Fix a point $x_0 \in X$. As in [17, Lemma 6.15] and [6, Lemma 6.8] we define a sequence of cut-off functions, as follows:

$$\varphi_k(x) = \begin{cases} 1, \text{ if } d(x_0, x) \leq k-1 \\ k - d(x_0, x), \text{ if } k-1 < d(x_0, x) < k \\ 0, \text{ if } d(x_0, x) \geq k \end{cases}, \ k \geq 1.$$

Note that for each $k \geq 1$ the function φ_k is $1-$Lipschitz.

Proposition 2. *Assume that (X, d, μ) is a metric measure space and $\mathbf{B}$ is Banach function space over (X, μ) that has absolutely continuous norm. If $u \in N^{1,\mathbf{B}}(X)$ is bounded, then the function $u\varphi_k$ is in $N^{1,\mathbf{B}}(X)$, for each $k \geq 1$, and the sequence $(u\varphi_k)_{k \geq 1}$ converges to u in the norm of $N^{1,\mathbf{B}}(X)$.*

Proof. Let $k \geq 1$. Denote $v_k := u\varphi_k$ The function φ_k is Borel measurable, hence v_k is $\mu-$measurable. Since $0 \leq |v_k| \leq |u|$, and $u \in \mathbf{B}$, it follows that $v_k \in \mathbf{B}$.

Since φ_k is $1-$Lipschitz, the constant function 1 is an upper gradient of φ_k. Moreover, since φ_k is constant on the closed sets $\overline{B}(x_0, k-1)$ and $X \setminus B(x_0, k)$, applying twice Lemma 4 we get that the characteristic function of $B(x_0, k) \setminus \overline{B}(x_0, k-1)$ is a $\mathbf{B}-$ weak upper gradient of φ_k. The same remark applies to $1-\varphi_k$. Denote by h_k the characteristic function of $B(x_0, k) \setminus \overline{B}(x_0, k-1)$.

We have $u - v_k = u(1 - \varphi_k)$. It follows by the product rule from Lemma 7 that $|u| h_k + g(1 - \varphi_k)$ is a generalized $\mathbf{B}-$ weak upper gradient of v_k. But

$$g_k := (|u| + g)\, \chi_{X \setminus \overline{B}(x_0, k-1)} \geq |u| h_k + g(1 - \varphi_k) \tag{3.1}$$

on X.

Since $u \in ACC_{\mathbf{B}}$, we find a curve family Γ such that $u \circ \gamma$ is absolutely continuous on $[0, l(\gamma)]$, in particular Borel measurable, for all $\gamma \in \Gamma_{rec} \setminus \Gamma$. Consequently, $g_k \circ \gamma$ is Borel measurable on $[0, l(\gamma)]$ for all $\gamma \in \Gamma_{rec} \setminus \Gamma$. By inequality (3.1) it follows that g_k is a generalized $\mathbf{B}-$weak upper gradient of v_k. Moreover, $g_k \in \mathbf{B}$ by Definition 1 (P2).

Then $\|u-v_k\|_{N^{1,\mathbf{B}}(X)} \le \|u-v_k\|_{\mathbf{B}} + \|g_k\|_{\mathbf{B}}$, therefore

(3.2)

$$\|u-v_k\|_{N^{1,\mathbf{B}}(X)} \le \left\| u\chi_{X\setminus \overline{B}(x_0,k-1)} \right\|_{\mathbf{B}} + \left\| (|u|+g)\,\chi_{X\setminus \overline{B}(x_0,k-1)} \right\|_{\mathbf{B}}.$$

Since the sequence of sets $\left(X \setminus \overline{B}\,(x_0,k-1)\right)_{k\ge 1}$ converges to the empty set μ–a.e. and u, $|u|+g \in \mathbf{B}$, it follows that $\left\| u\chi_{X\setminus \overline{B}(x_0,k-1)} \right\|_{\mathbf{B}} \to 0$ and $\left\| (|u|+g)\,\chi_{X\setminus \overline{B}(x_0,k-1)} \right\|_{\mathbf{B}} \to 0$ as $k\to\infty$, by the absolute continuity of the norm of $\mathbf{B}$.

By (3.2) we obtain $\|u-v_k\|_{N^{1,\mathbf{B}}(X)} \to 0$ as $k\to\infty$. □

Corollary 3. *Let $\mathbf{B}$ be a Banach function space over X that has absolutely continuous norm and has property (C). For each $u\in N^{1,\mathbf{B}}(X)$ and every $\varepsilon>0$ there is a bounded function $w\in N^{1,\mathbf{B}}(X)$ supported in a closed ball, such that $\|u-w\|_{N^{1,\mathbf{B}}(X)} < \varepsilon$.*

Proof. Let $u\in N^{1,\mathbf{B}}(X)$ and $\varepsilon>0$. By Corollary 1, there is a bounded function $v\in N^{1,\mathbf{B}}(X)$ such that $\|u-v\|_{N^{1,\mathbf{B}}(X)} < \frac{\varepsilon}{2}$.

By Proposition 2 applied to the bounded function $v\in N^{1,\mathbf{B}}(X)$, there is an integer $k\ge 1$ such that $w := v\varphi_k$ satisfies $\|v-w\|_{N^{1,\mathbf{B}}(X)} < \frac{\varepsilon}{2}$. Then $\|u-w\|_{N^{1,\mathbf{B}}(X)}$. The function $w\in N^{1,\mathbf{B}}(X)$ is bounded, since $|w|\le|v|$, and w is supported in the closed ball $\overline{B}\,(x_0,k)$. □

Corollary 4. *Assume that (X,d,μ) is a proper metric measure space, with μ nonatomic. Let $\mathbf{B}$ be a rearrangement invariant Banach function space over (X,μ) that has absolutely continuous norm. For each $u\in N^{1,\mathbf{B}}(X)$ and every $\varepsilon>0$ there is a bounded compactly supported function $w\in N^{1,\mathbf{B}}(X)$ such that $\|u-w\|_{N^{1,\mathbf{B}}(X)} < \varepsilon$.*

Proof. Let $u\in N^{1,\mathbf{B}}(X)$ and $\varepsilon>0$. Since $\mathbf{B}$ is rearrangement invariant, it has property (C), by Lemma 5. By Corollary 3, there is a bounded function $w\in N^{1,\mathbf{B}}(X)$ supported in some closed ball $\overline{B}\,(x_0,k)$, where $k\ge 1$ is an integer, such that $\|u-w\|_{N^{1,\mathbf{B}}(X)} < \varepsilon$. Since X is proper, $\overline{B}\,(x_0,k)$ is compact. □

Remark 4. *Proposition 2 extends Lemma 6.15 from* [17] *and Lemma 6.8 from* [6] *to the setting of Newtonian spaces based on Banach function spaces.*

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