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λ-CENTRAL BMO ESTIMATES FOR MULTILINEAR COMMUTATORS OF SINGULAR INTEGRAL OPERATOR ON SPACES OF HOMOGENEOUS TYPE

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Abstract. In this paper, we establish λ-central BMO estimates for the multilinear commutator related to the singular integral operator in central Morrey spaces on homogeneous spaces.

1. Introduction

In recent years, research for singular integral operator is becoming more and more popular, and their commutators and multilinear operators have also been well studied(see [6-8][12-14]). Let $b \in BMO(\mathbb{R}^n)$ and T be the Calderón-Zygmund operator, the commutator $[b, T]$ generated by b and T is defined by

$$[b, T](f) = bT(f) - T(bf).$$

In [6][14-15], the authors proved that the commutators and multilinear operators generated by the singular integral operators and BMO functions are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Since $BMO \subset \bigcap_{q>1} CBMO^q$ (see [7]), if we only assume $b \in CBMO^q$, or more generally $b \in CBMO^{q,\lambda}$ with $q > 1$, then $[b, T]$ may not be a bounded operator on $L^p(\mathbb{R}^n)$. However, it has some boundedness properties on other spaces.

As a matter of fact, Grafakos, Li and Yang (see [8]) considered the commutator with $b \in CBMO^q$ on Herz spaces for the first time. Later,Alvarez, Guzmán-Partida and Lakey (see [1]) and Komori (see

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[10][16]) have obtained the λ-central BMO estimates for the commutators of a class of singular integral operators on central Morrey spaces. Inspired by these results, in this paper, we will establish λ-central BMO estimates for the multilinear commutator associated to the singular integral operator in central Morrey spaces on homogeneous spaces.

2. Notations and Results

Given a set X, a function $d : X\times X \to \mathbb{R}^+$ is called a quasi$-$distance on X if the following conditions are satisfied:
(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$, for every x and y in X,
(2) $d(x, y) = d(y, x)$, for every x and y in X,
(3) there exists a constant $K \geq 1$ such that

$$d(x, y) \leq K(d(x, z) + d(z, y))$$

for every x, y and z in X.

Let μ be a positive measure on the σ-algebra of subsets of X which contains the d-balls $B(x, r) = \{y : d(x, y) < r\}$. We assume that μ satisfies a doubling condition, that is, for all $x \in X$ and $r > 0$,there exists a constant A such that

$$0 < \mu(2B(x, r)) \leq A\mu(B(x, r)) < \infty,$$

where, and in what follows, $tB(x, r) = B(x, tr)$ for $t > 0$.

A structure (X, d, μ), with d and μ as above, is called a space of homogeneous type. The constants K and A will be called the constants of the space.

In this paper, $B(x, r)$ will denote a ball centered at x with radius $r > 0$, and $\mu(B(x, r))$ will be the measure of the $B(x, r)$, which satisfies

$$f_{B(x,r)} = \frac{1}{\mu(B(x, r))} \int_{B(x,r)} f(y)d\mu(y).$$

To state our result, we first give some definitions.

Definition 1. Fixed $x_0 \in X$. Let $0 < \lambda < 1$ and $1 < q < \infty$. A function $f \in L^q_{loc}(X)$ is said to belong to the λ-central bounded mean oscillation space $CBMO^{q,\lambda}(X)$ if

$$||f||_{CBMO^{q,\lambda}} = \sup_{r>0} \left(\frac{1}{\mu(B(x_0, r))^{1+\lambda q}} \int_{B(x_0,r)} |f(x) - f_{B(x_0,r)}|^q d\mu(x)\right)^{1/q} < \infty.$$

Remark 1. If two functions which differ by a constant are regarded as a function in the space $CBMO^{q,\lambda}$ becomes a Banach space. Apparently, the formula which mentioned above is equivalent to the following condition (see [7]):

$$||f||_{CBMO^{q,\lambda}} = \sup_{r>0} \inf_{c \in \mathbf{C}} \left(\frac{1}{\mu(B(x_0,r))^{1+\lambda q}} \int_{B(x_0,r)} |f(x) - c|^q d\mu(x) \right)^{1/q} < \infty.$$

And when $\lambda = 0$,the space $CBMO^{q,\lambda}(X)$ is just the space $CBMO^q(X)$ defined as follows:

$$||f||_{CBMO_q} = \sup_{r>0} \left(\frac{1}{\mu(B(x_0,r))} \int_{B(x_0,r)} |f(x) - f_B(x_0,r)|^q d\mu(x) \right)^{1/q} < \infty.$$

Definition 2. Fixed $x_0 \in X$. Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. The central Morrey space $\dot{B}^{q,\lambda}(X)$ is defined by

$$||f||_{\dot{B}^{q,\lambda}} = \sup_{r>0} \left(\frac{1}{\mu(B(x_0,r))^{1+\lambda q}} \int_{B(x_0,r)} |f(x)|^q d\mu(x) \right)^{1/q} < \infty.$$

Remark 2. We can observe that $\dot{B}^{q,\lambda}(X)$ is a Banach space continuously included in $CBMO^{q,\lambda}(X)$ from such formulas . We denote by $CMO^{q,\lambda}(X)$ and $B^{q,\lambda}(X)$ the inhomogeneous versions of the λ-central bounded mean oscillation space and the central Morrey space by taking the supremum over $r \geq 1$ in Definition 1 and Definition 2 instead of $r > 0$ there. Obviously, $CBMO^{q,\lambda}(X) \subset CMO^{q,\lambda}(X)$ for $\lambda < 1$ and $1 < q < \infty$, and $\dot{B}^{q,\lambda}(X) \subset B^{q,\lambda}(X)$ for $\lambda \in \mathbb{R}$ and $1 < q < \infty$.
Remark 3. When $\lambda_1 < \lambda_2$, it follows from the property of monotone functions that $B^{q,\lambda_1}(X) \subset B^{q,\lambda_2}(X)$ and $CMO^{q,\lambda_1}(X) \subset CMO^{q,\lambda_2}(X)$ for $1 < q < \infty$. If $1 < q_1 < q_2 < \infty$, then by Hölder's inequality, we know that $\dot{B}^{q_2,\lambda}(X) \subset \dot{B}^{q_1,\lambda}(X)$ for $\lambda \in \mathbb{R}$ and $CBMO^{q_2,\lambda} \subset CBMO^{q_1,\lambda}$, $CMO^{q_2,\lambda}(X) \subset CMO^{q_1,\lambda}(X)$ for $0 < \lambda < 1$.

Definition 3. Suppose b_j $(j = 1, \cdots, m)$ are the fixed locally integrable functions on X. Let T be the singular integral operator as

$$T(f)(x) = \int_X K(x,y) f(y) d\mu(y),$$

where $K(x,y)$ is a standard Calderón-Zygmund kernel(see [3][10]).

The multilinear commutator of the singular integral operator is defined by

$$T_{\vec{b}}(f)(x) = \int_X \prod_{j=1}^{m} (b_j(x) - b_j(y)) K(x,y) f(y) d\mu(y).$$

Note that when $m=1$, $T_{\vec{b}}$ is just the commutator of T and b which is widely studied(see [2-3][6]).

For $b_j \in CBMO^{p_{j+1},\lambda_{j+1}}(X)(j=1,\cdots,m)$, set

$$||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}} = \prod_{j=1}^{m} ||b_j||_{CBMO^{p_{j+1},\lambda_{j+1}}}.$$

Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1),\cdots,\sigma(j)\}$ of $\{1,\cdots,m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1,\cdots,m\}\backslash\sigma$. For $\vec{b} = (b_1,\cdots,b_m)$ and $\sigma = \{\sigma(1),\cdots,\sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)},\cdots,b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)}\cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{CBMO^{\vec{p},\vec{\lambda}}} = ||b_{\sigma(1)}||_{CBMO^{p_2,\lambda_2}}\cdots||b_{\sigma(j)}||_{CBMO^{p_{j+1},\lambda_{j+1}}}$.

Now we state our theorems as following.

Theorem 1. Let $\lambda < 0$ and $1 < q < \infty$, then T is bounded from $\dot{B}^{q,\lambda}(X)$ to $\dot{B}^{q,\lambda}(X)$.

Theorem 2. Let $1 < q < \infty$, $1 < p_k < \infty (1 \le k \le m+1)$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_{m+1}} \le 1$. Suppose $\lambda, \lambda_1 \in \mathbb{R}$, $0 < \lambda_i < 1(i = 2,3,\cdots,m+1)$, $\lambda = \lambda_1+\lambda_2+\cdots\lambda_{m+1} \le 0$. If $b_j \in CBMO^{p_{j+1},\lambda_{j+1}}(X)$ for $j=1,\cdots,m$, then $T_{\vec{b}}$ is bounded from $\dot{B}^{p_1,\lambda_1}(X)$ to $\dot{B}^{q,\lambda}(X)$, and the following inequality holds:

$$||T_{\vec{b}}(f)||_{\dot{B}^{q,\lambda}} \le C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}.$$

3. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1. Let T be the singular integral operator and $1 < p < \infty$. Then T is bounded from $L^p(X)$ to $L^p(X)$.

Lemma 2. Let $1 < p < \infty$, $\lambda > 0$. Suppose $b \in CBMO^{p,\lambda}(X)$, then for any $k \ge 1$,we have

$$|b_{2^{k+1}B} - b_B| \le C||b||_{CBMO^{p,\lambda}} k\mu(2^{k+1}B)^\lambda.$$

Proof.

$$
\begin{aligned}
|b_{2^{k+1}B} - b_B| &\le \sum_{j=0}^{k} |b_{2^{j+1}B} - b_{2^j B}| \\
&\le \sum_{j=0}^{k} \frac{1}{\mu(2^j B)} \int_{2^j B} |b(y) - b_{2^{j+1}B}| d\mu(y) \\
&\le C \sum_{j=0}^{k} \left(\frac{1}{\mu(2^{j+1}B)} \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|^p d\mu(y) \right)^{1/p} \\
&\le C||b||_{CBMO^{p,\lambda}} \sum_{j=0}^{k} \mu(2^{j+1}B)^{\lambda} \\
&\le C||b||_{CBMO^{p,\lambda}} (k+1) \mu(2^{k+1}B)^{\lambda} \\
&\le C||b||_{CBMO^{p,\lambda}} k \mu(2^{k+1}B)^{\lambda}.
\end{aligned}
$$

Proof of Theorem 1. Let f be a function in $\dot{B}^{q,\lambda}(X)$. For fixed $r > 0$, set $B = B(x_0, d(x_0, r))$, we write

$$
\begin{aligned}
&\left(\frac{1}{\mu(B)^{1+\lambda q}} \int_B |T(f)(x)|^q d\mu(x) \right)^{\frac{1}{q}} \\
\le\ & \left(\frac{1}{\mu(B)^{1+\lambda q}} \int_B |T(f\chi_{2B})(x)|^q d\mu(x) \right)^{\frac{1}{q}} \\
&+ \left(\frac{1}{\mu(B)^{1+\lambda q}} \int_B |T(f\chi_{(2B)^c})(x)|^q d\mu(x) \right)^{\frac{1}{q}} \\
=\ & I_1 + I_2.
\end{aligned}
$$

For I_1, by the boundedness of T , we have

$$
\begin{aligned}
I_1 &\le C\mu(B)^{-\frac{1}{q}-\lambda} \left(\int_{2B} |f(x)|^q d\mu(x) \right)^{\frac{1}{q}} \\
&\le C\mu(B)^{-\frac{1}{q}-\lambda} \mu(B)^{\frac{1}{q}+\lambda} ||f||_{\dot{B}^{q,\lambda}} \\
&\le C||f||_{\dot{B}^{q,\lambda}}.
\end{aligned}
$$

For I_2, given $x \in B$, by Hölder's inequality, we get

$$\begin{aligned}
|T(f\chi_{(2B)^c})(x)| &\leq \sum_{k=1}^{\infty} \int_{2^{k+1}B\setminus 2^k B} |K(x,y)||f(y)|d\mu(y) \\
&\leq C\sum_{k=1}^{\infty} \frac{1}{\mu(2^k B)} \int_{2^{k+1}B} |f(y)|d\mu(y) \\
&\leq C\sum_{k=1}^{\infty} \frac{1}{\mu(2^k B)} \left(\int_{2^{k+1}B} |f(y)|^q d\mu(y)\right)^{\frac{1}{q}} \mu(2^{k+1}B)^{1-\frac{1}{q}} \\
&\leq C\sum_{k=1}^{\infty} \frac{1}{\mu(2^k B)} \mu(2^{k+1}B)^{\frac{1}{q}+\lambda} ||f||_{\dot{B}^{q,\lambda}} \mu(2^{k+1}B)^{1-\frac{1}{q}} \\
&\leq C||f||_{\dot{B}^{q,\lambda}} \sum_{k=1}^{\infty} \mu(2^k B)^{\lambda} \\
&\leq C||f||_{\dot{B}^{q,\lambda}} \mu(B)^{\lambda}.
\end{aligned}$$

therefore

$$I_2 \leq C\mu(B)^{-\frac{1}{q}-\lambda} ||f||_{\dot{B}^{q,\lambda}} \mu(B)^{\lambda} \mu(B)^{\frac{1}{q}} \leq C||f||_{\dot{B}^{p,\lambda}}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Let f be a function in $\dot{B}^{p_1,\lambda_1}(X)$, we will consider the cases $m = 1$ and $m > 1$,respectively.

We first consider the **Case** $m = 1$**.** Set $(b_1)_B = \frac{1}{\mu(B)} \int_B b_1(y)d\mu(y)$, we have

$$T_{b_1}(f)(x) = (b_1(x) - (b_1)_B)T(f)(x) - T((b_1(y) - (b_1)_B)f)(x).$$

So

$$\begin{aligned}
&\left(\frac{1}{\mu(B)^{1+\lambda q}} \int_B |T_{b_1}(f)(x)|^q d\mu(x)\right)^{\frac{1}{q}} \\
\leq &\left(\frac{1}{\mu(B)^{1+\lambda q}} \int_B |(b_1(x) - (b_1)_B)(T(f\chi_{2B}))(x)|^q d\mu(x)\right)^{\frac{1}{q}} \\
&+ \left(\frac{1}{\mu(B)^{1+\lambda q}} \int_B |(b_1(x) - (b_1)_B)(T(f\chi_{(2B)^c}))(x)|^q d\mu(x)\right)^{\frac{1}{q}} \\
&+ \left(\frac{1}{\mu(B)^{1+\lambda q}} \int_B |T((b_1 - (b_1)_B)f\chi_{2B})(x)|^q d\mu(x)\right)^{\frac{1}{q}} \\
&+ \left(\frac{1}{\mu(B)^{1+\lambda q}} \int_B |T((b_1 - (b_1)_B)f\chi_{(2B)^c})(x)|^q d\mu(x)\right)^{\frac{1}{q}} \\
= &\ J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

For J_1, by Hölder's inequality and boundedness of T from $L^{p_1}(X)$ to $L^{p_1}(X)$, we have

$$\begin{aligned} J_1 &\le \mu(B)^{-\frac{1}{q}-\lambda}\left(\int_B |b_1(x)-(b_1)_B|^{p_2}d\mu(x)\right)^{\frac{1}{p_2}}\left(\int_B |T(f\chi_{2B})(x)|^{p_1}\right)^{\frac{1}{p_1}} \\ &\le C\mu(B)^{-\frac{1}{q}-\lambda}\mu(B)^{\frac{1}{p_2}+\lambda_2}||b_1||_{CBMO^{p_2,\lambda_2}}\left(\int_B |f(x)|^{p_1}d\mu(x)\right)^{\frac{1}{p_1}} \\ &\le C\mu(B)^{-\frac{1}{q}-\lambda+\frac{1}{p_2}+\lambda_2}||b_1||_{CBMO^{p_2,\lambda_2}}\mu(B)^{\frac{1}{p_1}+\lambda_1}||f||_{\dot{B}^{p_1,\lambda_1}} \\ &\le C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}. \end{aligned}$$

For J_2, with the same method which we use above, we get

$$\begin{aligned} |T(f\chi_{(2B)^c})(x)| &\le \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}|K(x,y)||f(y)|d\mu(y) \\ &\le C\sum_{k=1}^{\infty}\frac{1}{\mu(2^kB)}\left(\int_{2^{k+1}B}|f(y)|^{p_1}d\mu(y)\right)^{\frac{1}{p_1}}\mu(2^{k+1}B)^{1-\frac{1}{p_1}} \\ &\le C\sum_{k=1}^{\infty}\frac{1}{\mu(2^kB)}\mu(2^{k+1}B)^{\frac{1}{p_1}+\lambda_1}||f||_{\dot{B}^{p_1,\lambda_1}}\mu(2^{k+1}B)^{1-\frac{1}{p_1}} \\ &\le C||f||_{\dot{B}^{p_1,\lambda_1}}\sum_{k=1}^{\infty}\mu(2^kB)^{\lambda_1} \\ &\le C||f||_{\dot{B}^{p_1,\lambda_1}}\mu(B)^{\lambda_1} \end{aligned}$$

then, we can get

$$\begin{aligned} J_2 &\le C\mu(B)^{-\frac{1}{q}-\lambda}\left(\int_B |(b_1(x)-(b_1)_B)(T(f\chi_{(2B)^c}))(x)|^q d\mu(x)\right)^{\frac{1}{q}} \\ &\le C\mu(B)^{-\frac{1}{q}-\lambda}||f||_{\dot{B}^{p_1,\lambda_1}}\mu(B)^{\lambda_1} \\ &\quad \left(\int_B |b_1(x)-(b_1)_B|^{p_2}d\mu(x)\right)^{1/p_2}\mu(B)^{\frac{1}{q}-\frac{1}{p_2}} \\ &\le C\mu(B)^{-\frac{1}{q}-\lambda+\lambda_1+(\frac{1}{q}-\frac{1}{p_2})+(\frac{1}{p_2}+\lambda_2)}||f||_{\dot{B}^{p_1,\lambda_1}}||b_1||_{CBMO^{p_2,\lambda_2}} \\ &\le C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}. \end{aligned}$$

For J_3, using the boundedness of T and Hölder's inequality, we have

$$
\begin{aligned}
J_3 &\le C\mu(B)^{-\frac{1}{q}-\lambda}\left(\int_B |(b_1(x)-(b_1)_B f(x)|^q d\mu(x)\right)^{\frac{1}{q}} \\
&\le C\mu(B)^{-\frac{1}{q}-\lambda}\left(\int_B |b_1(x)-(b_1)_B|^{p_2} d\mu(x)\right)^{\frac{1}{p_2}}\left(\int_B |f(x)|^{p_1} d\mu(x)\right)^{\frac{1}{p_1}} \\
&\le C\mu(B)^{-\frac{1}{q}-\lambda}\mu(B)^{\frac{1}{p_2}+\lambda_2}||b_1||_{CBMO^{p_2,\lambda_2}}|\mu(B)^{\frac{1}{p_1}+\lambda_1}|f||_{\dot{B}^{p_1,\lambda_1}} \\
&\le C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}
$$

For J_4, given $x \in B$, by Hölder's inequality and lemma 2, we have

$$
\begin{aligned}
& |T((b_1-(b_1)_B)f\chi_{(2B)^c})(x)| \\
\le\ & \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^k B} |b_1(y)-(b_1)_B||K(x,y)||f(y)|d\mu(y) \\
\le\ & C\sum_{k=1}^{\infty}\frac{1}{\mu(2^kB)}\left(\int_{2^{k+1}B}|b_1(y)-(b_1)_B|^{p_2}d\mu(y)\right)^{\frac{1}{p_2}}\left(\int_{2^{k+1}B}|f(y)|^{p_1}d\mu(y)\right)^{\frac{1}{p_1}} \\
\times\ & \mu(2^{k+1}B)^{1-\frac{1}{p_1}-\frac{1}{p_2}} \\
\le\ & C\sum_{k=1}^{\infty}\times\mu(2^{k+1}B)^{\frac{1}{p_1}+\lambda_1}||f||_{\dot{B}^{p_1,\lambda_1}}\mu(2^{k+1}B)^{1-\frac{1}{p_1}-\frac{1}{p_2}} \\
\times\ & \frac{1}{\mu(2^kB)}\left[\left(\int_{2^{k+1}B}|b_1(y)-(b_1)_{2^{k+1}B}|^{p_2}d\mu(y)\right)^{\frac{1}{p_2}}+|(b_1)_{2^{k+1}B}-(b_1)_B|\mu(2^{k+1}B)^{\frac{1}{p_2}}\right]
\end{aligned}
$$

$$
\begin{aligned}
\le\ & C\sum_{k=1}^{\infty}\mu(2^{k+1}B)^{\frac{1}{p_1}+\lambda_1}||f||_{\dot{B}^{p_1,\lambda_1}}\mu(2^{k+1}B)^{1-\frac{1}{p_1}-\frac{1}{p_2}} \\
\times\ & \frac{1}{\mu(2^kB)}\left[\mu(2^{k+1}B)^{\frac{1}{p_2}+\lambda_2}||b_1||_{CBMO^{p_2,\lambda_2}}+k\mu(2^{k+1}B)^{\lambda_2}||b_1||_{CBMO^{p_2,\lambda_2}}\mu(2^{k+1}B)^{\frac{1}{p_2}}\right] \\
\le\ & C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}\sum_{k=1}^{\infty}k\mu(2^kB)^{\lambda_1+\lambda_2} \\
\le\ & C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}\mu(B)^{\lambda_1+\lambda_2},
\end{aligned}
$$

therefore,

$$
\begin{aligned}
J_4 &\le C\mu(B)^{-\frac{1}{q}-\lambda}||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}\mu(B)^{\lambda_1+\lambda_2}\mu(B)^{\frac{1}{q}} \\
&\le C||b_1||_{CBMO^{p_2,\lambda_2}}||f||_{\dot{B}^{p_1,\lambda_1}}
\end{aligned}
$$

This completes the proof of the case $m=1$.

Now, we consider the **Case** $m > 1$. Set $\vec{b}_B = ((b_1)_B, \cdots, (b_m)_B)$, where $(b_j)_B = \frac{1}{\mu(B)}\int_B |b_j(y)|d\mu(y)$ for $1 \le j \le m$, we have

$$\begin{aligned} T_{\vec{b}}(f)(x) &= \int_X \prod_{j=1}^m (b_j(x) - b_j(y))K(x,y)f(y)d\mu(y) \\ &= \sum_{j=0}^m \sum_{\sigma \in C_j^m} (-1)^{m-j}(b_j(x) - (b_j)_B)_\sigma \int_X (b_j(y) - (b_j)_B)_{\sigma^c} K(x,y)f(y)d\mu(y) \\ &= \prod_{j=1}^m (b_j(x) - (b_j)_B)\int_X K(x,y)f(y)d\mu(y) \\ &\quad +(-1)^m \int_X \prod_{j=1}^m (b_j(y) - (b_j)_B)K(x,y)f(y)d\mu(y) \\ &\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j}(b_j(x) - (b_j)_B)_\sigma \int_X (b_j(y) - (b_j)_B)_{\sigma^c} K(x,y)f(y)d\mu(y) \\ &= \prod_{j=1}^m (b_j(x) - (b_j)_B)T(f)(x) + (-1)^m T(\prod_{j=1}^m (b_j - (b_j)_B)f)(x) \\ &\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j}((b_j(x) - (b_j)_B)_\sigma T(b_j - (b_j)_B)_{\sigma^c}(f)(x), \end{aligned}$$

thus

$$\begin{aligned} &\left(\frac{1}{\mu(B)^{1+\lambda q}}\int_B |T_{\vec{b}}(f)(x)|^q d\mu(x)\right)^{\frac{1}{q}} \\ \le &\left(\frac{1}{\mu(B)^{1+\lambda q}}\int_B |(b_1(x) - (b_1)_B)\cdots(b_m(x) - (b_m)_B)(T(f\chi_{2B}))(x)|^q d\mu(x)\right)^{\frac{1}{q}} \\ &+ \left(\frac{1}{\mu(B)^{1+\lambda q}}\int_B |(b_1(x) - (b_1)_B)\cdots(b_m(x) - (b_m)_B)(T(f\chi_{(2B)^c}))(x)|^q d\mu(x)\right)^{\frac{1}{q}} \\ &+ \left(\frac{1}{\mu(B)^{1+\lambda q}}\int_B |T((b_1 - (b_1)_B)\cdots(b_m - (b_m)_B)f\chi_{2B})(x)|^q d\mu(x)\right)^{\frac{1}{q}} \\ &+ \left(\frac{1}{\mu(B)^{1+\lambda q}}\int_B |T((b_1 - (b_1)_B)\cdots(b_m - (b_m)_B)f\chi_{(2B)^c})(x)|^q d\mu(x)\right)^{\frac{1}{q}} \\ &+ \left(\frac{1}{\mu(B)^{1+\lambda q}}\int_B \left|\sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (b_j(x) - (b_j)_B)_\sigma T((b_j - (b_j)_B)_{\sigma^c} f\chi_{2B})(x)\right|^q d\mu(x)\right)^{\frac{1}{q}} \\ &+ \left(\frac{1}{\mu(B)^{1+\lambda q}}\int_B \left|\sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (b_j(x) - (b_j)_B)_\sigma T((b_j - (b_j)_B)_{\sigma^c} f\chi_{(2B)^c})(x)\right|^q d\mu(x)\right)^{\frac{1}{q}} \\ = &\; B_1 + B_2 + B_3 + B_4 + B_5 + B_6. \end{aligned}$$

For B_1, by Hölder's inequality and the boundedness of T , we have

$$\begin{aligned} B_1 &\le \mu(B)^{-\frac{1}{q}-\lambda}\prod_{j=1}^{m}\left(\int_B |b_j(x)-(b_j)_B|^{p_{j+1}}d\mu(x)\right)^{\frac{1}{p_{j+1}}}\left(\int_B |(T(f\chi_{2B}))(x)|^{p_1}d\mu(x)\right)^{\frac{1}{p_1}}\\ &\le C\mu(B)^{-\frac{1}{q}-\lambda}\prod_{j=1}^{m}\left(\mu(B)^{\frac{1}{p_{j+1}}+\lambda_{j+1}}||b_j||_{CBMO^{p_{j+1},\lambda_{j+1}}}\right)\left(\int_B |f(x)|^{p_1}d\mu(x)\right)^{\frac{1}{p_1}}\\ &\le C\mu(B)^{-\frac{1}{q}-\lambda}\mu(B)^{\frac{1}{p_2}+\cdots+\frac{1}{p_{m+1}}+\lambda_2+\cdots+\lambda_{m+1}}||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}\mu(B)^{\frac{1}{p_1}+\lambda_1}||f||_{\dot{B}^{p_1,\lambda_1}}\\ &\le C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}. \end{aligned}$$

For B_2, by the inequality $|T(f\chi_{(2B)^c})(x)|\le C||f||_{\dot{B}^{p_1,\lambda_1}}\mu(B)^{\lambda_1}$ from the proof of Theorem 1, we can get

$$\begin{aligned} B_2 &\le C\mu(B)^{-\frac{1}{q}-\lambda}||f||_{\dot{B}^{p_1,\lambda_1}}\mu(B)^{\lambda_1}\left(\int_B |\prod_{j=1}^{m}(b_j(x)-(b_j)_B)|^q d\mu(x)\right)^{\frac{1}{q}}\\ &\le C\mu(B)^{-\frac{1}{q}-\lambda}||f||_{\dot{B}^{p_1,\lambda_1}}\mu(B)^{\lambda_1}\\ &\quad\times\prod_{j=1}^{m}\left(\int_B |(b_j(x)-(b_j)_B)|^{p_{j+1}}d\mu(x)\right)^{\frac{1}{p_{j+1}}}\mu(B)^{\frac{1}{q}-\frac{1}{p_2}-\cdots-\frac{1}{p_{m+1}}}\\ &\le C\mu(B)^{-\frac{1}{q}-\lambda}||f||_{\dot{B}^{p_1,\lambda_1}}\mu(B)^{\lambda_1}\prod_{j=1}^{m}\mu(B)^{\frac{1}{p_{j+1}}+\lambda_{j+1}}\\ &\quad\times||b_j||_{CBMO^{p_{j+1},\lambda_{j+1}}}\mu(B)^{\frac{1}{q}-\frac{1}{p_2}-\cdots-\frac{1}{p_{m+1}}}\\ &\le C\prod_{j=1}^{m}||b_j||_{CBMO^{p_{j+1},\lambda_{j+1}}}||f||_{\dot{B}^{p_1,\lambda_1}}\\ &\le C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}. \end{aligned}$$

For B_3, using the boundedness of T and Hölder's inequality, we have

$$\begin{aligned} B_3 &\le C\mu(B)^{-\frac{1}{q}-\lambda}\left(\int_{2B}|(b_1(x)-(b_1)_B)\cdots(b_m(x)-(b_m)_B)f\chi_{2B}(x)|^q d\mu(x)\right)^{\frac{1}{q}}\\ &\le C\mu(B)^{-\frac{1}{q}-\lambda}\prod_{j=1}^{m}\left(\int_{2B}|(b_j(x)-(b_j)_B)|^{p_{i+1}}d\mu(x)\right)^{\frac{1}{p_{i+1}}}\left(\int_2 B|f(x)|^{p_1}d\mu(x)\right)^{\frac{1}{p_1}}\\ &\le C\mu(B)^{-\frac{1}{q}-\lambda}\prod_{j=1}^{m}\mu(2B)^{\frac{1}{p_{i+1}}+\lambda_{i+1}}||b_i||_{CBMO^{p_{i+1},\lambda_{i+1}}}\mu(2B)^{\frac{1}{p_1}+\lambda_1}||f||_{\dot{B}^{p_1,\lambda_1}}\mu(B)^{\lambda_1}\\ &\le C\prod_{i=1}^{m}||b_i||_{CBMO^{p_{i+1},\lambda_{i+1}}}||f||_{\dot{B}^{p_1,\lambda_1}}\\ &\le C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}. \end{aligned}$$

For B_4, given $x \in B$, for $\lambda, \lambda_1 \in \mathbb{R}$ and $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_{m+1} \leq 0$, by Hölder's inequality and lemma 2, we have

$$\begin{aligned}
& |T((b_1 - (b_1)_B) \cdots (b_m - (b_m)_B) f \chi_{(2B)^c})(x)| \\
\leq & \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |b_1(y) - (b_1)_B| \cdots |b_m(y) - (b_m)_B| |K(x,y)| |f(y)| d\mu(y) \\
\leq & C \sum_{k=1}^{\infty} \frac{1}{\mu(2^k B)} \prod_{j=1}^{m} \left(\int_{2^{k+1}B} |(b_j(y) - (b_j)_B)|^{p_{j+1}} d\mu(y) \right)^{\frac{1}{p_{j+1}}} \\
& \times \left(\int_{2^{k+1}B} |f(y)|^{p_1} d\mu(y) \right)^{\frac{1}{p_1}} \mu(2^{k+1}B)^{1 - \frac{1}{p_1} - \frac{1}{p_2} \cdots - \frac{1}{p_{m+1}}} \\
\leq & C \sum_{k=1}^{\infty} \frac{1}{\mu(2^k B)} \prod_{j=1}^{m} \left[\left(\int_{2^{k+1}B} |b_j(y) - (b_j)_{2^{k+1}B}|^{p_{j+1}} d\mu(y) \right)^{\frac{1}{p_{j+1}}} \right. \\
& \left. + |(b_j)_{2^{k+1}B} - (b_j)_B| \mu(2^{k+1}B)^{\frac{1}{p_{j+1}}} \right] \\
& \times \mu(2^{k+1}B)^{\frac{1}{p_1} + \lambda_1} ||f||_{\dot{B}^{p_1,\lambda_1}} \mu(2^{k+1}B)^{1 - \frac{1}{p_1} - \frac{1}{p_2} \cdots - \frac{1}{p_{m+1}}} \\
\leq & C ||f||_{\dot{B}^{p_1,\lambda_1}} \prod_{j=1}^{m} ||b_j||_{CBMO^{p_{j+1},\lambda_{j+1}}} \sum_{k=1}^{\infty} k^m \mu(2^{k+1}B)^{\lambda_1 + \lambda_2 + \cdots + \lambda_{m+1}} \\
\leq & C ||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}} ||f||_{\dot{B}^{p_1,\lambda_1}} \mu(B)^{\lambda_1 + \lambda_2 + \cdots + \lambda_{m+1}} \\
= & C ||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}} ||f||_{\dot{B}^{p_1,\lambda_1}} \mu(B)^{\lambda},
\end{aligned}$$

so, we obtain

$$\begin{aligned}
J_4 & \leq \mu(B)^{-\frac{1}{q} - \lambda} ||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}} ||f||_{\dot{B}^{p_1,\lambda_1}} \mu(B)^{\lambda} \mu(B)^{\frac{1}{q}} \\
& \leq C ||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}} ||f||_{\dot{B}^{p_1,\lambda_1}}
\end{aligned}$$

For B_5, let$1 < q_1, q_2, q_3 < \infty$, set $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\frac{1}{q_1} = \frac{1}{q_3} + \frac{1}{p_1}$, we denote $\frac{1}{q_2} = \sum \frac{1}{p_{j+1}}$, $\lambda' = \sum \lambda_{j+1}$ where j satisfies $\sigma(j) \in \sigma$, $\frac{1}{q_3} = \sum \frac{1}{p_{j+1}}$, $\lambda'' = \sum \lambda_{j+1}$ where j satisfies $\sigma(j) \in \sigma^c$ and $\lambda_1 + \lambda'' < 0$, by the boundedness of T and Hölder's inequality, we have

$$B_5 \leq C \mu(B)^{-\frac{1}{q} - \lambda} \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left(\int_B |(b_j(x) - b_{j\,B})_\sigma|^{q_2} d\mu(x) \right)^{\frac{1}{q_2}}$$

$$
\begin{aligned}
& \times\left(\int_B |T((b_j - b_{jB})_{\sigma^c} f\chi_{2B})(x)|^{q_1} d\mu(x)\right)^{\frac{1}{q_1}} \\
\leq\ & C\mu(B)^{-\frac{1}{q}-\lambda} \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} \left(\int_B |(b_j(x) - b_{jB})_\sigma|^{q_2} d\mu(x)\right)^{\frac{1}{q_2}} \\
& \times\left(\int_B |(b_j - (b_j)_B)_{\sigma^c} f(x)|^{q_1} d\mu(x)\right)^{\frac{1}{q_1}} \\
\leq\ & C\mu(B)^{-\frac{1}{q}-\lambda} \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} \left(\int_B |(b_j(x) - b_{jB})_\sigma|^{q_2} d\mu(x)\right)^{\frac{1}{q_2}} \\
& \times\left(\int_B |(b_j - (b_j)_B)_{\sigma^c}|^{q_3} d\mu(x)\right)^{\frac{1}{q_3}} \left(\int_B |f(x)|^{p_1} d\mu(x)\right)^{\frac{1}{p_1}} \\
\leq\ & C\mu(B)^{-\frac{1}{q}-\lambda} \sum_{j=1}^{m-1} \sum_{\sigma\in C_j^m} \mu(B)^{\frac{1}{q_2}+\lambda'} ||\vec{b}_\sigma||_{CBMO^{q_2,\lambda'}} \mu(B)^{\frac{1}{q_3}+\lambda''} \\
& \times ||\vec{b}_{\sigma^c}||_{CBMO^{q_3,\lambda''}} \mu(B)^{\frac{1}{p_1}+\lambda_1} ||f||_{\dot{B}^{p_1,\lambda_1}} \\
\leq\ & C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}} ||f||_{\dot{B}^{p_1,\lambda_1}}.
\end{aligned}
$$

For B_6, given $x \in B$, using the same notations in B_5, $\lambda = \lambda_1 + \lambda' + \lambda''$, by Hölder's inequality and lemma 2, we have

$$
\begin{aligned}
& |T((b_j - b_{jB})_{\sigma^c} f\chi_{(2B)^c})(x)| \\
\leq\ & \sum_{k=1}^{\infty} \int_{2^{k+1}B\setminus 2^k B} |(b_j - b_{jB})_{\sigma^c}||K(x,y)||f(y)| d\mu(y) \\
\leq\ & C\sum_{k=1}^{\infty} \frac{1}{\mu(2^k B)} \left(\int_{2^{k+1}B} |(b_j - b_{jB})_{\sigma^c}|^{q_3} d\mu(y)\right)^{\frac{1}{q_3}} \\
& \times\left(\int_{2^{k+1}B} |f(y)|^{p_1} d\mu(y)\right)^{\frac{1}{p_1}} \mu(2^{k+1}B)^{1-\frac{1}{p_1}-\frac{1}{q_3}} \\
\leq\ & C\sum_{k=1}^{\infty} \frac{1}{\mu(2^k B)} \mu(2^{k+1}B)^{\frac{1}{q_3}+\lambda''} ||\vec{b}_{\sigma^c}||_{CBMO^{q_3,\lambda''}} \mu(2^{k+1}B)^{\frac{1}{p_1}+\lambda_1} \\
& \times ||f||_{\dot{B}^{p_1,\lambda_1}} \mu(2^{k+1}B)^{1-\frac{1}{p_1}-\frac{1}{q_3}} \\
\leq\ & C||\vec{b}_{\sigma^c}||_{CBMO^{q_3,\lambda''}}
\end{aligned}
$$

$$\begin{aligned} &\times||f||_{\dot{B}^{p_1,\lambda_1}}\sum_{k=1}^{\infty}\mu(2^kB)^{\lambda_1+\lambda''}\\ \leq\ & C||\vec{b}_{\sigma^c}||_{CBMO^{q_3,\lambda''}}||f||_{\dot{B}^{p_1,\lambda_1}}\mu(B)^{\lambda_1+\lambda''}, \end{aligned}$$

thus

$$\begin{aligned} B_6 \ \leq\ & C\mu(B)^{-\frac{1}{q}-\lambda}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_{\sigma^c}||_{CBMO^{q_3,\lambda''}}||f||_{\dot{B}^{p_1,\lambda_1}}\mu(B)^{\lambda_1+\lambda''}\\ &\times\left(\int_B|(b_j(x)-(b_j)_B)_\sigma|^{q_2}d\mu(x)\right)^{\frac{1}{q_2}}\mu(B)^{\frac{1}{q}-\frac{1}{q_2}}\\ \leq\ & C\mu(B)^{-\frac{1}{q}-\lambda}||\vec{b}_{\sigma^c}||_{CBMO^{q_3,\lambda''}}\mu(B)^{\lambda_1+\lambda''}\\ &\times||f||_{\dot{B}^{p_1,\lambda_1}}||\vec{b}_\sigma||_{CBMO^{q_2,\lambda'}}\mu(B)^{\frac{1}{q_2}+\lambda'}\mu(B)^{\frac{1}{q}-\frac{1}{q_2}}\\ \leq\ & C||\vec{b}||_{CBMO^{\vec{p},\vec{\lambda}}}||f||_{\dot{B}^{p_1,\lambda_1}}. \end{aligned}$$

This completes the total proof of the Theorem 2.

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