"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 24(2014), No. 2, 25-46

COMMON FIXED POINT THEOREMS FOR CONTRACTIVE MAPPINGS WITH ALTERING DISTANCES IN METRIC AND ORDERED METRIC SPACES

SUMIT CHANDOK, MOHAMMAD SAEED KHAN AND BESSEM SAMET

Abstract. We derive some common fixed point theorems for three single valued mappings satisfying a nonlinear contractive condition involving altering distance functions in the setting of metric and ordered metric spaces. The results presented in this paper generalize and extend several well-known results in the literature. As application, we establish an existence result for a nonlinear first order differential equation. Some examples are also presented to illustrate our obtained results.

1. Introduction and Preliminaries

Fixed point theory is one of the famous and traditional theories in mathematics and has a large number of applications. Fixed point and common fixed point theorems for different types of mappings have been investigated extensively by various researchers (see [1]-[33]). The Banach contraction principle [4] is one of the pivotal results of analysis. It is very popular tool for solving existence problems in many different fields of mathematics. There are a lot of generalizations of the Banach fixed point theorem in the literature.

Keywords and phrases: Common fixed point, compatible mappings, (μ, ψ)-generalized Chatterjea type (f, g)-contraction mappings, ordered metric spaces, ordinary differential equation. **(2010)Mathematics Subject Classification:** 47H10, 54H25

Ran and Reurings [28] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. Nieto and Rodŕiguez-López [27] extended the result of Ran and Reurings and applied their main theorems to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Bhaskar and Lakshmikantham [6] introduced the concept of mixed monotone mappings and obtained some coupled fixed point results. Also, they applied their results on a first order differential equation with periodic boundary conditions. For other results related on fixed point theory on ordered metric spaces, we refer the reader to [1, 3, 5, 14, 16, 17, 18, 20, 25, 27, 31, 32, 33].

Alber and Guerre-Delabriere [2] introduced the concept of weakly contractive mappings and proved the existence of fixed points for single-valued weakly contractive mappings in Hilbert spaces. Thereafter, in 2001, Rhoades [30] proved a fixed point theorem for weakly contractive mappings, generalizing Banach contraction principle and showed that some results of [2] are true for any Banach space. In fact, weakly contractive mappings are closely related to the mappings of Boyd and Wong [7] and of Reich types [29]. Recently, Dorić [19] proved a common fixed point theorem for generalized (ψ, ϕ)-weakly contractive mappings. Fixed point problems involving weak contractions and mappings satisfying weak contractive type inequalities have been studied by many authors (see [2, 8, 9, 15, 19, 20, 24, 25] and references cited therein).

In this paper, we generalize the Chatterjea type contraction mappings [13] to (μ, ψ)-generalized Chatterjea type (f, g)-contraction mappings and derive some common fixed point theorems for three single-valued mappings in the setting of metric and ordered metric spaces. Some examples are presented to illustrate our obtained results. We give also an application to the study of the existence of solution to a nonlinear first order differential equation.

First, we recall some basic definitions and notations.

Let (X, d) be a metric space. A map $T : X \to X$ is said to be:

(a) of *Kannan type* (see [23]) if there exists a $k \in (0, \frac{1}{2}]$ such that

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)], \quad \text{for all} \quad x, y \in X;$$

(b) *Chatterjea type* (see [13]) if there exists a $k \in (0, \frac{1}{2}]$ such that

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)], \quad \text{for all} \quad x, y \in X.$$

Khan et al. [24] initiated the use of a control function that alters distance between two points in a metric space, which they called an altering distance function.

Definition 1.1. A function $\mu : [0, \infty) \to [0, \infty)$ is called an *altering distance function* if the following properties are satisfied:

(i) μ is monotone increasing and continuous;
(ii) $\mu(t) = 0$ if and only if $t = 0$.

Using the control function, we generalize the Chatterjea type contraction mappings as follows.

Definition 1.2. Let (X, d) be a metric space and $T, f, g : X \to X$ are self-mappings of X. A mapping T is called (μ, ψ)*-generalized Chatterjea type* (f, g)*-contraction* if for all $x, y \in X$,

(1.1)

$$\mu(d(Tx, fy)) \leq \mu\left(\frac{1}{2}[d(gx, fy) + d(gy, Tx)]\right) - \psi(d(gx, fy), d(gy, Tx)),$$

where μ is an altering distance function and $\psi : [0, \infty)^2 \to [0, \infty)$ is a lower semi-continuous mapping such that $\psi(t, s) = 0$ if and only if $t = s = 0$.

Definition 1.3. Let M be a nonempty subset of a metric space (X, d) and $T, f : M \to M$. A point $x \in M$ is a *common fixed (respectively, coincidence) point* of f and T if $x = fx = Tx$ (respectively, $fx = Tx$). The set of fixed points (respectively, coincidence points) of f and T is denoted by $F(f, T)$ (respectively, $C(f, T)$).
The pair (T, f) is called

a) *commutative* if $Tfx = fTx$ for all $x \in M$;
b) *compatible*[21] if $\lim d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim Tx_n = \lim fx_n = t$ for some t in M;
c) *weakly compatible* [22] if f and T commute at their coincidence points, i.e., if $fTx = Tfx$ whenever $fx = Tx$.

2. Common fixed point theorems in metric spaces

In this section, we prove some common fixed point theorems for three single-valued mappings in the setting of metric spaces.

Our first result is the following.

Theorem 2.1. *Let M be a subset of a metric space (X, d) and $f, g, T : M \to M$. Suppose that T is (μ, ψ)-generalized Chatterjea type (f, g)-contraction, that is, (1.1) is satisfied for all $x, y \in M$. Suppose also that $T(M) \cup f(M) \subseteq g(M)$ and $(g(M), d)$ is complete. Then*

(i) *T, f and g have a coincidence point in M;*
(ii) *If the pairs (g,T) and (g,f) are weakly compatible, then T, f and g have a unique common fixed point.*

Proof. Let $x_0 \in X$. Since $T(M) \cup f(M) \subseteq g(M)$, we can choose $x_1, x_2 \in M$ so that $gx_1 = Tx_0$ and $gx_2 = fx_1$. By induction, we construct a sequence $\{x_n\}$ in X such that $gx_{2n+1} = Tx_{2n}$ and $gx_{2n+2} = fx_{2n+1}$, for every $n \geq 0$.

By (1.1), we obtain

$$\begin{aligned}
\mu(d(gx_{2n+1}, gx_{2n+2})) &= \mu(d(Tx_{2n}, fx_{2n+1})) \\
&\leq \mu\left(\frac{1}{2}[d(gx_{2n}, fx_{2n+1}) + d(gx_{2n+1}, Tx_{2n})]\right) \\
&\quad -\psi(d(gx_{2n}, fx_{2n+1}), d(gx_{2n+1}, Tx_{2n})) \\
&= \mu\left(\frac{1}{2}[d(gx_{2n}, gx_{2n+2}) + d(gx_{2n+1}, gx_{2n+1})]\right) \\
&\quad -\psi(d(gx_{2n}, gx_{2n+2}), d(gx_{2n+1}, gx_{2n+1})) \\
&= \mu\left(\frac{1}{2}d(gx_{2n}, gx_{2n+2})\right) - \psi(d(gx_{2n}, gx_{2n+2}), 0) \qquad (2.1)\\
&\leq \mu\left(\frac{1}{2}d(gx_{2n}, gx_{2n+2})\right).
\end{aligned}$$

Since μ is a monotone increasing function, for all $n = 1, 2, \ldots$, we have

$$d(gx_{2n+1}, gx_{2n+2}) \leq \frac{1}{2}d(gx_{2n}, gx_{2n+2}) \leq \frac{1}{2}[d(gx_{2n}, gx_{2n+1}) + d(gx_{2n+1}, gx_{2n+2})]. \qquad (2.2)$$

This implies that

$$d(gx_{2n+1}, gx_{2n+2}) \leq d(gx_{2n}, gx_{2n+1}).$$

Following the similar arguments, we obtain

$$d(gx_{2n+2}, gx_{2n+3}) \leq d(gx_{2n+1}, gx_{2n+2}).$$

Thus we proved that $\{d(gx_n, gx_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists $r \geq 0$ such that $d(gx_n, gx_{n+1}) \to r$ as $n \to \infty$. Letting $n \to \infty$ in (2.2), we obtain that

$$d(gx_{2n}, gx_{2n+2}) \to 2r \quad \text{as} \quad n \to \infty.$$

Letting $n \to \infty$ in (2.1), using the continuity of μ and lower semicontinuity of ψ, we obtain that

$$\mu(r) \leq \mu(r) - \psi(2r, 0).$$

This implies that $\psi(2r,0)=0$ and hence $r=0$. Thus we proved that

$$d(gx_{n+1},gx_n)\to 0 \quad \text{as} \quad n\to\infty. \tag{2.3}$$

Now, we show that $\{gx_n\}$ is a Cauchy sequence. From (2.3), it is sufficient to show that $\{gx_{2n}\}$ is a Cauchy sequence.

On the contrary, suppose that $\{gx_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon>0$ for which we can find subsequences $\{gx_{2m(k)}\}$ and $\{gx_{2n(k)}\}$ of $\{gx_n\}$ with $n(k)>m(k)>k$ such that for all k, we have

$$d(gx_{2m(k)},gx_{2n(k)})\geq\varepsilon \quad \text{and} \quad d(gx_{2m(k)},gx_{2n(k)-2})<\varepsilon.$$

So, we have

$$\begin{aligned}\varepsilon &\leq d(gx_{2m(k)},gx_{2n(k)})\\ &\leq d(gx_{2m(k)},gx_{2n(k)-2})+d(gx_{2n(k)-2},gx_{2n(k)-1})+d(gx_{2n(k)-1},gx_{2n(k)})\\ &< \varepsilon+d(gx_{2n(k)-2},gx_{2n(k)-1})+d(gx_{2n(k)-1},gx_{2n(k)}).\end{aligned}$$

On letting $k\to\infty$ and using (2.3), we have

$$\lim_{k\to\infty} d(gx_{2m(k)},gx_{2n(k)})=\varepsilon. \tag{2.4}$$

Also,

$$|d(gx_{2m(k)-1},gx_{2n(k)})-d(gx_{2m(k)},gx_{2n(k)})|\leq d(gx_{2m(k)-1},gx_{2m(k)}).$$

On letting $k\to\infty$ and using (2.3) and (2.4), we have

$$\lim_{k\to\infty} d(gx_{2m(k)-1},gx_{2n(k)})=\varepsilon=\lim_{k\to\infty} d(gx_{2m(k)},gx_{2n(k)}). \tag{2.5}$$

Again, we have

$$|d(gx_{2n(k)-1},gx_{2m(k)})-d(gx_{2m(k)},gx_{2n(k)})|\leq d(gx_{2n(k)-1},gx_{2n(k)}).$$

On letting $k\to\infty$ and using (2.3) and (2.4), we have

$$\lim_{k\to\infty} d(gx_{2n(k)-1},gx_{2m(k)})=\varepsilon. \tag{2.6}$$

Now, we have

$$\begin{aligned}
\mu(\epsilon) &\leq \mu(d(gx_{2m(k)}, gx_{2n(k)})) \\
&= \mu(d(Tx_{2m(k)-1}, fx_{2n(k)-1})) \\
&\leq \mu\left(\frac{1}{2}[d(gx_{2m(k)-1}, fx_{2n(k)-1}) + d(gx_{2n(k)-1}, Tx_{2m(k)-1})]\right) \\
&\quad -\psi(d(gx_{2m(k)-1}, fx_{2n(k)-1}), d(gx_{2n(k)-1}, Tx_{2m(k)-1})) \\
&= \mu\left(\frac{1}{2}[d(gx_{2m(k)-1}, gx_{2n(k)}) + d(gx_{2n(k)-1}, gx_{2m(k)})]\right) \\
&\quad -\psi(d(gx_{2m(k)-1}, gx_{2n(k)}), d(gx_{2n(k)-1}, gx_{2m(k)})).
\end{aligned}$$

Taking $k \to \infty$, using the continuity of μ, the lower semi-continuity of ψ, (2.5) and (2.6), we get that

$$\mu(\varepsilon) \leq \mu\left(\frac{1}{2}[\varepsilon + \varepsilon]\right) - \psi(\varepsilon, \varepsilon).$$

Consequently, $\psi(\varepsilon, \varepsilon) \leq 0$, which is contradiction with $\varepsilon > 0$. Thus $\{gx_{2n}\}$ is a Cauchy sequence and hence $\{gx_n\}$.

As $(g(M), d)$ is complete, there is $t \in M$ such that $gx_n \to gt$ as $n \to \infty$. We shall prove that t is a coincidence point of T, f and g

We have

$$\begin{aligned}
&\mu(d(gx_{2n+1}, ft)) = \mu(d(Tx_{2n}, ft)) \\
&\leq \mu\left(\frac{1}{2}[d(gx_{2n}, ft) + d(gt, Tx_{2n})]\right) - \psi(d(gx_{2n}, ft), d(gt, Tx_{2n})) \\
&= \mu\left(\frac{1}{2}[d(gx_{2n}, ft) + d(gt, gx_{2n+1})]\right) - \psi(d(gx_{2n}, ft), d(gt, gx_{2n+1}))
\end{aligned}$$

On letting $n \to \infty$, we have

$$\mu(d(gt, ft)) \leq \mu\left(\frac{1}{2}d(gt, ft)\right) - \psi(d(gt, ft), 0)) \leq \mu\left(\frac{1}{2}d(gt, ft)\right).$$

This implies that $d(gt, ft) = 0$ and hence $gt = ft$.

Also, we have

$$\begin{aligned}
\mu(d(Tt,gt)) &= \mu(d(Tt,ft)) \\
&\leq \mu\left(\frac{1}{2}[d(gt,ft)+d(gt,Tt)]\right) - \psi(d(gt,ft),d(gt,Tt)) \\
&= \mu\left(\frac{1}{2}d(gt,Tt)\right) - \psi(0,d(gt,Tt)) \\
&\leq \mu\left(\frac{1}{2}d(gt,Tt)\right).
\end{aligned}$$

This implies that $d(Tt,gt)=0$, that is, $Tt=gt$.

Thus we have , $gt=Tt=ft$, that is, t is a coincidence point of T, f and g. Then (i) holds.

Now, suppose that the pairs (g,T) and (g,f) are weakly compatible. Let $z=ft=gt=Tt$. Then we have $gTt=Tgt$ and $gft=fgt$, which implies that $Tz=fz=gz$. On the other hand, we have

$$\begin{aligned}
\mu(d(gz,z)) &= \mu(d(Tz,ft)) \\
&\leq \mu\left(\frac{1}{2}[d(gz,ft)+d(gt,Tz)]\right) - \psi(d(gz,ft),d(gt,Tz)) \\
&= \mu\left(\frac{1}{2}[d(gz,gt)+d(gt,gz)]\right) - \psi(d(gz,gt),d(gt,gz)) \\
&= \mu(d(gz,gt)) - \psi(d(gz,gt),d(gt,gz)) \\
&= \mu(d(gz,z)) - \psi(d(gz,z),d(z,gz)).
\end{aligned}$$

This implies that $d(gz,z)=0$, that is, $gz=z$. Hence, we get that

$$z=gz=Tz=fz,$$

that is, z is a common fixed point of g,T and f. This makes end to the proof.

Suppose now that $z'\in M$ is another common fixed point of g,T and f, that is,

$$z'=gz'=Tz'=fz'.$$

We have

$$\begin{aligned}
\mu(d(z,z')) &= \mu(d(Tz,fz')) \\
&\leq \mu\left(\frac{1}{2}[d(gz,fz')+d(gz',Tz)]\right) - \psi(d(gz,fz'),d(gz',Tz)) \\
&= \mu\left(\frac{1}{2}[d(z,z')+d(z,z')]\right) - \psi(d(z,z'),d(z,z')) \\
&= \mu(d(z,z')) - \psi(d(z,z'),d(z,z')).
\end{aligned}$$

This implies that $d(z,z')=0$, that is, $z=z'$. Thus we proved the uniqueness of the common fixed point. Hence (ii) holds. ∎

Now, we give an example to illustrate our Theorem 2.1.

Example 2.2. *We endow $X=\mathbb{R}$ with the usual metric $d(x,y)=|x-y|$ for all $x,y\in X$. Let $M=[0,1]$ and consider the mappings $T,f,g:M\to M$ defined by*

$$Tx=0, \quad fx=\frac{x^2}{8} \quad \textit{and} \quad gx=x, \quad \textit{for all} \quad x\in M.$$

We have $T(M)\cup f(M)=[0,1/8]\subset[0,1]=g(M)$ and $(g(M),d)=([0,1],d)$ is complete. Obviously the pairs (g,T) and (g,f) are weakly compatible.
Define $\mu:[0,\infty)\to[0,\infty)$ and $\psi:[0,\infty)\times[0,\infty)\to[0,\infty)$ by

$$\mu(t)=\frac{t}{2} \quad \textit{and} \quad \psi(t,s)=\frac{t+s}{16}, \quad \textit{for all} \quad t,s\geq 0.$$

For all $x,y\in M$, we have

$$\mu(d(Tx,fy))=\mu\left(d\left(0,\frac{y^2}{8}\right)\right)=\mu\left(\frac{y^2}{8}\right)=\frac{y^2}{16}$$

and

$$\begin{aligned}
&\mu\left(\frac{1}{2}[d(gx,fy)+d(gy,Tx)]\right)-\psi(d(gx,fy),d(gy,Tx))\\
&=\mu\left(\frac{1}{2}\left[d\left(x,\frac{y^2}{8}\right)+d(y,0)\right]\right)-\psi\left(d\left(x,\frac{y^2}{8}\right),d(y,0)\right)\\
&=\mu\left(\frac{1}{2}\left[\left|x-\frac{y^2}{8}\right|+y\right]\right)-\psi\left(|x-\frac{y^2}{8}|,y\right)\\
&=\frac{1}{4}\left[\left|x-\frac{y^2}{8}\right|+y\right]-\frac{\left|x-\frac{y^2}{8}\right|+y}{16}\\
&=\frac{3}{16}\left[\left|x-\frac{y^2}{8}\right|+y\right]\geq\frac{3y}{16}\geq\frac{y^2}{16}.
\end{aligned}$$

Thus we have

$$\mu(d(Tx,fy))\leq\mu\left(\frac{1}{2}[d(gx,fy)+d(gy,Tx)]\right)-\psi(d(gx,fy),d(gy,Tx)),$$

for all $x,y\in M$. By Theorem 2.1, T, f and g have a unique common fixed point $z=0$.

From our Theorem 2.1, we can deduce many other results.

Theorem 2.3. *Let M be a subset of a metric space (X,d) and $f,g,T:M\to M$. Suppose that for all $x,y\in M$, we have*

(2.7)

$$\mu(d(Tx,fy))\leq\mu\left(\frac{1}{2}[d(gx,fy)+d(gy,Tx)]\right)-\varphi\left(\frac{1}{2}[d(gx,fy)+d(gy,Tx)]\right),$$

where μ is an altering distance function and $\varphi:[0,\infty)\to[0,\infty)$ is a lower semi-continuous mapping such that $\varphi(t)=0$ if and only if $t=0$. Suppose also that $T(M)\cup f(M)\subseteq g(M)$ and $(g(M),d)$ is complete. Then

(i) *T, f and g have a coincidence point in M;*
(ii) *If the pairs (g,T) and (g,f) are weakly compatible, then T, f and g have a unique common fixed point.*

Proof. It is sufficient to show that condition (2.8) can be reduced to (1.1), with

$$\psi(t,s)=\varphi\left(\frac{t+s}{2}\right)\quad\text{for all}\quad t,s\geq 0.$$

Clearly ψ a lower semi-continuous mapping and $\psi(t,s)=0$ if and only if $t=s=0$. Then the desired result follows from Theorem 2.1. ∎

Taking in Theorem 2.3 $\mu(t) = t$ and $\varphi(t) = 2\left(\frac{1}{2} - k\right) t$ with $k \in (0, \frac{1}{2}]$ is a constant, we get immediately the following common fixed point result of Chatterjea type.

Corollary 2.4. *Let M be a subset of a metric space (X, d) and $f, g, T : M \to M$. Suppose that there exists $k \in (0, \frac{1}{2}]$ such that for all $x, y \in M$, we have*

$$d(Tx, fy) \leq k[d(gx, fy) + d(gy, Tx)].$$

Suppose also that $T(M) \cup f(M) \subseteq g(M)$ and $(g(M), d)$ is complete. Then

(i) *T, f and g have a coincidence point in M;*
(ii) *If the pairs (g, T) and (g, f) are weakly compatible, then T, f and g have a unique common fixed point.*

We can also obtain a common fixed point result for a contractive condition of integral type. At first, we denote by Λ the set of mappings $\alpha : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

(a) α is Lebesgue integrable on each compact subset of $[0, \infty)$;
(b) for all $\varepsilon > 0$, we have

$$\int_0 \varepsilon\alpha(s)\ ds > 0.$$

Corollary 2.5. *Let M be a subset of a metric space (X, d) and $f, g, T : M \to M$. Suppose that for all $x, y \in M$, we have*

(2.8)
$$\int_0^{d(Tx,fy)} \alpha(s)\, ds \leq \int_0^{\frac{1}{2}[d(gx,fy)+d(gy,Tx)]} \alpha(s)\, ds - \int_0^{\frac{1}{2}[d(gx,fy)+d(gy,Tx)]} \beta(s)\, ds,$$

where $\alpha, \beta \in \Lambda$. Suppose also that $T(M) \cup f(M) \subseteq g(M)$ and $(g(M), d)$ is complete. Then

(i) *T, f and g have a coincidence point in M;*
(ii) *If the pairs (g, T) and (g, f) are weakly compatible, then T, f and g have a unique common fixed point.*

Proof. We take $\mu(t) = \int_0^t \alpha(s)\ ds$ and $\varphi(t) = \int_0^t \beta(s)\ ds$ in Theorem 2.3. ∎

3. Common fixed point theorems in ordered metric spaces

In this section, we extend Theorem 2.1 to the setting of ordered metric spaces.

At first, we recall the following concept introduced recently by Nashine and Samet in [25].

Definition 3.1. Let $(X, \preceq)$ be a partially ordered set and $T, f, g : X \to X$ are mappings such that $T(X) \subseteq g(X)$ and $f(X) \subseteq g(X)$. Then T and f are *weakly increasing with respect to* g if and only if for all $x \in X$, we have

(a) $Tx \preceq fy$ for all $y \in g^{-1}(Tx)$;
(b) $fx \preceq Ty$ for all $y \in g^{-1}(fx)$.

Various examples of such mappings are given in [25, 26].

Remark 3.1. If $gx = x$ for all $x \in X$, then T and f are weakly increasing with respect to g implies that T and f are weakly increasing mappings. Note that the concept of weakly increasing mappings was introduced by Altun and Simsek in [3].

We have the following result.

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d. Suppose that the mappings $T, f, g : X \to X$ satisfy*

$$\mu(d(Tx, fy)) \leq \mu\left(\frac{1}{2}[d(gx, fy) + d(gy, Tx)]\right) - \psi(d(gx, fy), d(gy, Tx)) \tag{3.1}$$

for all $x, y \in X$ such that $gx \preceq gy$, where μ is an altering distance function and $\psi : [0, \infty)^2 \to [0, \infty)$ is a lower semi-continuous mapping such that $\psi(t, s) = 0$ if and only if $t = s = 0$. Suppose that

(i) *$T(X) \subseteq g(X)$, $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X;*
(ii) *T and f are weakly increasing with respect to g.*

Also suppose that either

(a) *if $\{g(x_n)\} \subset X$ is a nondecreasing sequence with $g(x_n) \to g(z)$ in $g(X)$, then $g(x_n) \preceq g(z)$ for every n; or*
(b) *the pair (T, g) is compatible and T, g are continuous; or*
(c) *the pair (f, g) is compatible and f, g are continuous.*

Then T, f and g have a coincidence point, that is, there exists $t \in X$ such that $gt = ft = Tt$.

Proof. Let $x_0 \in X$. From (i), we can choose $x_1, x_2 \in X$ such that $gx_1 = Tx_0$ and $gx_2 = fx_1$. By induction, we construct a sequence $\{gx_n\}$ in X such that $gx_{2n+1} = Tx_{2n}$ and $gx_{2n+2} = fx_{2n+1}$, for every $n \geq 0$.

We claim that

$$gx_n \preceq gx_{n+1}, \quad \text{for all} \quad n \geq 1. \tag{3.2}$$

Since T and f are weakly increasing mappings with respect to g, we obtain

$$gx_1 = Tx_0 \preceq fy, \forall y \in g^{-1}(Tx_0).$$

Since $gx_1 = Tx_0$, then $x_1 \in g^{-1}(Tx_0)$, and we get

$$gx_1 = Tx_0 \preceq fx_1 = gx_2.$$

Again,

$$gx_2 = fx_1 \preceq Ty, \forall y \in g^{-1}(fx_1).$$

Since $x_2 \in g^{-1}(fx_1)$, we get

$$gx_2 = fx_1 \preceq Tx_2 = gx_3.$$

By induction on n, we conclude that

$$gx_1 \preceq gx_2 \preceq \ldots \preceq gx_{2n+1} \preceq gx_{2n+2} \preceq \ldots$$

Thus our claim (3.2) holds.

Since $gx_{2n} \preceq gx_{2n+1}$, by inequality (3.1), we have

$$\begin{aligned}
\mu(d(gx_{2n+1}, gx_{2n+2})) &= \mu(d(Tx_{2n}, fx_{2n+1})) \\
&\leq \mu\left(\frac{1}{2}[d(gx_{2n}, fx_{2n+1}) + d(gx_{2n+1}, Tx_{2n})]\right) \\
&\quad -\psi(d(gx_{2n}, fx_{2n+1}), d(gx_{2n+1}, Tx_{2n})) \\
&= \mu\left(\frac{1}{2}d(gx_{2n}, gx_{2n+2})\right) - \psi(d(gx_{2n}, gx_{2n+2}), 0) \qquad (3.3) \\
&= \mu\left(\frac{1}{2}d(gx_{2n}, gx_{2n+2})\right).
\end{aligned}$$

Since μ is a monotone increasing function, for all $n = 1, 2, \ldots$, we have

$$d(gx_{2n+1}, gx_{2n+2}) \leq \frac{1}{2}d(gx_{2n}, gx_{2n+2}) \leq \frac{1}{2}[d(gx_{2n}, gx_{2n+1}) + d(gx_{2n+1}, gx_{2n+2})]. \tag{3.4}$$

This implies that

$$d(gx_{2n+1}, gx_{2n+2}) \leq d(gx_{2n}, gx_{2n+1}).$$

Following the similar arguments, we obtain

$$d(gx_{2n+2}, gx_{2n+3}) \leq d(gx_{2n+1}, gx_{2n+2}).$$

Thus $\{d(gx_n, gx_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists $r \geq 0$ such that $d(gx_n, gx_{n+1}) \to r$ as $n \to \infty$. Letting $n \to \infty$ in (3.4), we obatin that

$$d(gx_{2n}, gx_{2n+2}) \to 2r \quad \text{as} \quad n \to \infty.$$

Letting $n \to \infty$ in (3.4), using the continuity of μ and the lower semi-continuity of ψ, we get that

$$\mu(r) \leq \mu(r) - \psi(2r, 0),$$

which implies that $\psi(2r, 0) = 0$ and hence $r = 0$. Thus we proved that

$$d(gx_n, gx_{n+1}) \to 0 \quad \text{as} \quad n \to \infty.$$

Now, proceeding as in Theorem 2.1 we can prove that $\{gx_n\}$ is a Cauchy sequence.

Since $(g(X), d)$ is complete, there exists $t \in X$ such that $gx_n \to gt$ as $n \to \infty$.

Suppose that (a) holds. Then we get

$$g(x_n) \preceq g(t), \quad \text{for all} \quad n.$$

From (3.1), we get

$$\begin{aligned}
\mu(d(gx_{2n+1}, ft)) &= \mu(d(Tx_{2n}, ft)) \\
&\leq \mu\left(\frac{1}{2}[d(gx_{2n}, ft) + d(gt, Tx_{2n})]\right) - \psi(d(gx_{2n}, ft), d(gt, Tx_{2n})) \\
&= \mu\left(\frac{1}{2}[d(gx_{2n}, ft) + d(gt, gx_{2n+1})]\right) - \psi(d(gx_{2n}, ft), d(gt, gx_{2n+1})) \\
&\leq \mu\left(\frac{1}{2}[d(gx_{2n}, ft) + d(gt, gx_{2n+1})]\right).
\end{aligned}$$

Letting $n \to \infty$, we get

$$\mu(d(gt, ft)) \leq \mu\left(\frac{1}{2}d(gt, ft)\right),$$

which implies that $d(gt, ft) = 0$, that is, $gt = ft$.

Again, we have

$$\begin{aligned}
\mu(d(Tt,gt)) &= \mu(d(Tt,ft))\\
&\leq \mu\left(\frac{1}{2}[d(gt,ft)+d(gt,Tt)]\right)-\psi(d(gt,ft),d(gt,Tt))\\
&= \mu\left(\frac{1}{2}d(gt,Tt)\right)-\psi(0,d(gt,Tt))\\
&\leq \mu\left(\frac{1}{2}d(gt,Tt)\right),
\end{aligned}$$

which implies that $d(gt,Tt)=0$, that is, $gt=Tt$. Thus we proved that if (a) holds then t is a coincidence point of T,g and f.

Suppose that condition (b) holds. Let $z=gt$. Then we have

$$\lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} gx_{2n} = z.$$

Since the pair (T,g) is compatible, then

$$\lim_{n\to\infty} d(g(Tx_{2n}),T(gx_{2n}))=0. \tag{3.5}$$

Also, from the continuity of T and g, we have

$$\lim_{n\to\infty} d(g(Tx_{2n}),T(gx_{2n}))=d(gz,Tz). \tag{3.6}$$

Now, using (3.5) and (3.6), by the uniqueness of the limit, we have $d(gz,Tz)=0$, that is, $gz=Tz$. Using (3.1), we have

$$\begin{aligned}
\mu(d(gz,fz)) &= \mu(d(Tz,fz))\\
&\leq \mu\left(\frac{1}{2}[d(gz,fz)+d(gz,Tz)]\right)-\psi(d(gz,fz),d(gz,Tz))\\
&= \mu\left(\frac{1}{2}d(gz,fz)\right)-\psi(d(gz,fz),0)\\
&\leq \mu\left(\frac{1}{2}d(gz,fz)\right),
\end{aligned}$$

which implies that $gz=fz$. Thus, we have $gz=fz=Tz$, and z is a coincidence point of T,g and f.

If condition (c) holds, then by following the same arguments, we get the result. ∎

Following the same arguments as in the proof of Theorem 2.3, we get the following result.

Theorem 3.2. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d. Suppose that the mappings $T, f, g : X \to X$ satisfy*

$$\mu(d(Tx, fy)) \leq \mu\left(\frac{1}{2}[d(gx, fy) + d(gy, Tx)]\right) - \varphi\left(\frac{1}{2}[d(gx, fy) + d(gy, Tx)]\right)$$

for all $x, y \in X$ such that $gx \preceq gy$, where μ is an altering distance function and $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous mapping such that $\varphi(t) = 0$ if and only if $t = 0$. Suppose that

(i) *$T(X) \subseteq g(X)$, $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X;*
(ii) *T and f are weakly increasing with respect to g.*

Also suppose that either

(a) *if $\{g(x_n)\} \subset X$ is a nondecreasing sequence with $g(x_n) \to g(z)$ in $g(X)$, then $g(x_n) \preceq g(z)$ for every n; or*
(b) *the pair (T, g) is compatible and T, g are continuous; or*
(c) *the pair (f, g) is compatible and f, g are continuous.*

Then T, f and g have a coincidence point, that is, there exists $t \in X$ such that $gt = ft = Tt$.

Corollary 3.3. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d. Suppose that the mappings $T, f, g : X \to X$ satisfy*

$$d(Tx, fy) \leq k[d(gx, fy) + d(gy, Tx)]$$

for all $x, y \in X$ such that $gx \preceq gy$, where $k \in [0, 1/2)$. Suppose that

(i) *$T(X) \subseteq g(X)$, $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X;*
(ii) *T and f are weakly increasing with respect to g.*

Also suppose that either

(a) *if $\{g(x_n)\} \subset X$ is a nondecreasing sequence with $g(x_n) \to g(z)$ in $g(X)$, then $g(x_n) \preceq g(z)$ for every n; or*
(b) *the pair (T, g) is compatible and T, g are continuous; or*
(c) *the pair (f, g) is compatible and f, g are continuous.*

Then T, f and g have a coincidence point, that is, there exists $t \in X$ such that $gt = ft = Tt$.

Corollary 3.4. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d such that (X, d) is complete. Suppose that the mapping $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$ *such that* $x \preceq y$*, where* $k \in [0, 1/2)$*. Suppose that*

$$Tx \preceq T(Tx), \quad \text{for all} \quad x \in X.$$

Also suppose that either

(a) *if* $\{x_n\} \subset X$ *is a nondecreasing sequence with* $x_n \to z$ *then* $x_n \preceq z$ *for every* n*; or*

(b) T *is continuous.*

Then T *has a fixed point.*

Proof. Taking in Corollary 3.3 $T = f$ and $gx = x$ for all $x \in X$, we get the desired result. ∎

Corollary 3.5. *Let* $(X, \preceq)$ *be a partially ordered set endowed with a metric* d*. Suppose that the mappings* $T, f, g : X \to X$ *satisfy*

$$\int_0^{d(Tx,fy)} \alpha(s)\, ds \leq \int_0^{\frac{1}{2}[d(gx,fy)+d(gy,Tx)]} \alpha(s)\, ds - \int_0^{\frac{1}{2}[d(gx,fy)+d(gy,Tx)]} \beta(s)\, ds$$

for all $x, y \in X$ *such that* $gx \preceq gy$*, where* $\alpha, \beta \in \Lambda$*. Suppose that*

(i) $T(X) \subseteq g(X)$*,* $f(X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of* X*;*

(ii) T *and* f *are weakly increasing with respect to* g*.*

Also suppose that either

(a) *if* $\{g(x_n)\} \subset X$ *is a nondecreasing sequence with* $g(x_n) \to g(z)$ *in* $g(X)$*, then* $g(x_n) \preceq g(z)$ *for every* n*; or*

(b) *the pair* (T, g) *is compatible and* T, g *are continuous; or*

(c) *the pair* (f, g) *is compatible and* f, g *are continuous.*

Then T*,* f *and* g *have a coincidence point, that is, there exists* $t \in X$ *such that* $gt = ft = Tt$*.*

4. Application to nonlinear first order ordinary differential equation

Consider the nonlinear differential equation

$$\begin{cases} x'(t) &= f(t, x(t)), \quad t \in I, \\ x(t_0) &= x_0, \end{cases} \tag{4.1}$$

where $t_0 \in \mathbb{R}$, $I = [t_0, t_0 + a]$, $a > 0$, and $f : I \times \mathbb{R} \to \mathbb{R}$.

Let $X = C(I, \mathbb{R})$ denotes the space of all continuous $\mathbb{R}$-valued functions on I. We endow this space with the metric d given by

$$d(u, v) = \max_{t \in I} |u(t) - v(t)|, \quad \text{for all} \quad u, v \in X.$$

It is well known that (X,d) is a complete metric space. We define an order relation $\preceq$ on X by

$$u,v\in X,\quad u\preceq v\Longleftrightarrow u(t)\le v(t),\ \text{for all } t\in I.$$

We consider the following assumptions:

(H1) $f: I\times\mathbb{R}\to\mathbb{R}$ is continuous;

(H2) for all $t\in I$, $f(t,\cdot):\mathbb{R}\to\mathbb{R}$ is nondecreasing;

(H3) we have

$$f(t,z)\le f(t,f(t,z)),\quad \text{for all}\quad t\in I,\ z\in\mathbb{R};$$

(H4) for all $t\in I$, for all $u\in C(I,\mathbb{R})$,

$$f(t,u(t))\le x_0+\int_{t_0}^{t} f(\tau,x(\tau))\,d\tau;$$

(H5) there exists $k\in(0,1/2)$ such that for all $u,v\in C(I,\mathbb{R})$ with $u\preceq v$, we have

$$\begin{aligned}\int_{t_0}^{t}[f(s,v(s))-f(s,u(s))]\,ds \ &\le\ k\bigg(\left|u(t)-x_0-\int_{t_0}^{t} f(s,v(s))\,ds\right|+\\ &\qquad\left|v(t)-x_0-\int_{t_0}^{t} f(s,u(s))\,ds\right|\bigg)\end{aligned}$$

for all $t\in I$.

We have the following result.

Theorem 4.1. *Suppose that (H1)-(H5) hold. Then (4.1) has at least one solution $x^*\in C(I,\mathbb{R})$.*

Proof. Consider the mapping $T: C(I,\mathbb{R})\to C(I,\mathbb{R})$ defined by

$$Tu(t)=x_0+\int_{t_0}^{t} f(s,u(s))\,ds,\quad t\in I,$$

for all $u\in C(I,\mathbb{R})$. Clearly, $x^*\in C(I,\mathbb{R})$ is a solution of (4.1) if and only if x^* is a fixed point of T.

Let $x,y\in C(I,\mathbb{R})$ such that $x\preceq y$. From (H5), we have

$$\begin{aligned}\int_{t_0}^{t}[f(s,y(s))-f(s,x(s))]\,ds \ &\le k\ \bigg(\left|x(t)-x_0-\int_{t_0}^{t} f(s,y(s))\,ds\right|+\\ &\qquad\left|y(t)-x_0-\int_{t_0}^{t} f(s,x(s))\,ds\right|\bigg)\\ &\le\ k(|x(t)-Ty(t)|+|y(t)-Tx(t)|)\\ &\le\ k(d(x,Ty)+d(y,Tx)).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|Tx(t)-Ty(t)| &= \left|\int_{t_0}^{t}[f(s,x(s))-f(s,y(s))]\,ds\right| \\
&\leq \int_{t_0}^{t}|f(s,x(s))-f(s,y(s))|\,ds \\
\text{(from (H2))} &= \int_{t_0}^{t}[f(s,y(s))-f(s,x(s))]\,ds.
\end{aligned}$$

Then we have

$$|Tx(t)-Ty(t)|\leq k(d(x,Ty)+d(y,Tx)), \quad \text{for all} \quad t\in I.$$

This implies that

$$d(Tx,Ty)\leq k(d(x,Ty)+d(y,Tx)).$$

Let $x\in C(I,\mathbb{R})$. For all $t\in I$, we have

$$\begin{aligned}
Tx(t) &= x_0+\int_{t_0}^{t}f(s,x(s))\,ds \\
\text{(from (H3))} &\leq x_0+\int_{t_0}^{t}f(s,f(s,x(s)))\,ds \\
\text{(from (H2) and (H4))} &\leq x_0+\int_{t_0}^{t}f\left(s,x_0+\int_{t_0}^{s}f(\tau,x(\tau))\,d\tau\right)\,ds \\
&= x_0+\int_{t_0}^{t}f(s,Tx(s))\,ds \\
&= T(Tx(t)).
\end{aligned}$$

Thus we have

$$Tx\preceq T(Tx), \quad \text{for all} \quad x\in C(I,\mathbb{R}).$$

Also, it is proved in [27] that if $\{x_n\}\subset C(I,\mathbb{R})$ is a nondecreasing sequence with $x_n\to z$ then $x_n\preceq z$ for every n.

Now, applying Corollary 3.4, we obtain that there exists $x^*\in C(I,\mathbb{R})$, a fixed point of T. This makes end to the proof. ∎

Acknowledgements. The authors are thankful to the learned referee for valuable suggestions.

References

[1] R.P. Agarwal, M.A. El-Gebeily and D. O'Regan, **Generalized contractions in partially ordered metric spaces**, Appl. Anal. 87 (2008) 109-116.

[2] Ya.I. Alber and S. Guerre-Delabriere, **Principles of weakly contractive maps in Hilbert spaces**, New results in Operator Theory, Advances Appl., Vol. 8 (I. Gohberg and Yu. Lyubich, eds.), Birkhauser, Basel, 1997, 7-22.

[3] I. Altun and H. Simsek, **Some fixed point theorems on ordered metric spaces and application**, Fixed Point Theory Appl. 2010 (2010) Article ID 621492, 17 pages.

[4] S. Banach, **Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales**, Fund. Math. 3 (1922) 133-181.

[5] I. Beg and A.R. Butt, **Fixed point for set-valued mappings satisfying an implicit relation in ordered metric spaces**, Nonlinear Anal. 71 (2009) 3699-3704.

[6] T.G. Bhaskar and V. Lakshmikantham, **Fixed point theorems in partially ordered metric spaces and applications**, Nonlinear Anal. 65 (2006) 1379-1393.

[7] D.W. Boyd and T.S.W. Wong, **On nonlinear contractions**, Proc. Amer. Math. Soc. 20 (1969) 458-464.

[8] S. Chandok, **Some common fixed point theorems for generalized f-weakly contractive mappings**, J. Appl. Math. Informatics. 29 (2011) 257-265.

[9] S. Chandok, **Some common fixed point theorems for generalized nonlinear contractive mappings**, Comput. Math. Appl. 62 (2011) 3692-3699.

[10] S. Chandok, **Common fixed points, invariant approximation and generalized weak contractions**, Internat. J. Math. Math. Sci. Vol. 2012 (2012) Article ID 102980.

[11] S. Chandok, **Some common fixed point results for generalized weak contractive mappings in partially ordered metric spaces**, J. Nonlinear Anal. Opt. 4(1)(2013), 45–52.

[12] S. Chandok, **Common fixed points for generalized nonlinear contractive mappings in metric spaces**, Mat. Vesnik 65(1)(2013), 29–34.

[13] S.K. Chatterjea, **Fixed point theorem**, C.R. Acad. Bulgare Sci. 25 (1972) 727-730.

[14] C. Chifu and G. Petruşel, **Fixed-point results for generalized contractions on ordered Gauge spaces with applications**, Fixed Point Theory Appl. 2011 (2011) Article ID 979586, 10 pages.

[15] B.S. Choudhury, **Unique fixed point theorem for weakly C-contractive mappings**, Kathmandu University J. Sci. Engg. Tech. 5 (2009) 6-13.

[16] Lj. Ćirić, N. Cakić, M. Rajović and J.S. Ume, **Monotone generalized nonlinear contractions in partially ordered metric spaces**, Fixed Point Theory Appl. 2008 (2008) Article ID 131294, 11 pages.

[17] Lj. Ćirić, R.P. Agarwal and B. Samet, **Mixed monotone-generalized contractions in partially ordered probabilistic metric spaces**, Fixed Point Theory Appl. 2011, 2011:56.

[18] Lj. Ćirić and V. Lakshmikantham, **Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces**, Stoch. Anal. Appl. 27 (2009) 1246-1259.

[19] D. Dorić, **Common fixed point for generalized (ψ, ϕ)-weak contractions**, Appl. Math. Letters. 22 (2009) 1896-1900.

[20] J. Harjani, B. López and K. Sadarangani, **Fixed point theorems for weakly C-contractive mapping in ordered metric spaces**, Comput. Math. Appl. 61(4)(2011) 790-796.

[21] G. Jungck, **Compatible mapping and common fixed points**, Int. J. Math. Math Sci. 9 (1986) 771-779.

[22] G. Jungck and B.E. Rhoades, **Fixed point for set valued functions without continuity**, Indian J. Pure Appl. Math. 29 (1998) 227-238.

[23] R. Kannan, **Some results on fixed points-II**, Amer. Math. Monthly 76 (1969) 405-408.

[24] M.S. Khan, M. Swaleh and S. Sessa, **Fixed point theorems by altering distances between the points**, Bull. Aust. Math. Soc. 30 (1984) 1-9.

[25] H.K. Nashine and B. Samet, **Fixed point results for mappings satisfying (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces**, Nonlinear Anal. 74 (2011) 2201-2209.

[26] H.K. Nashine, B. Samet and J.K. Kim, **Fixed point results for contractions involving generalized altering distances in ordered metric spaces**, Fixed Point Theory Appl. 2011, 2011:5.

[27] J.J. Nieto and R.R. López, **Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations**, Order. 22 (2005) 223-239.

[28] A.C.M. Ran and M.C.B. Reurings, **A fixed point theorem in partially ordered sets and some applications to matrix equations**, Proc. Amer. Math. Soc. 132 (5)(2004) 1435-1443.

[29] S. Reich, **Some fixed point problems**, Atti Acad. Naz. Lincei Ren. Cl. Sci. Fis. Mat. Natur. 57 (1975) 194-198.

[30] B.E. Rhoades, **Some theorems on weakly contractive maps**, Nonlinear Anal. 47(2001), 2683-2693.

[31] B. Samet, **Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces**, Nonlinear Anal. 72 (2010) 4508-4517.

[32] B. Samet and C. Vetro, **Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces**, Nonlinear Anal. 74 (2011) 4260-4268.

[33] B. Samet, C. Vetro and P. Vetro, **Fixed point theorems for $\alpha - \psi$-contractive type mappings**, Nonlinear Analysis (2011), doi:10.1016/j.na.2011.10.014.

Sumit Chandok

Department of Mathematics, Khalsa College of Engineering & Technology,

Punjab Technical University, Amritsar-143001, INDIA. email: chansok.s@gmail.com; chandhok.sumit@gmail.com

Mohammad Saeed Khan
Department of Mathematics and Statistics, College of Science, Sultan Qaboos University, Al-Khod, Post Box 36, PCode 123, Muscat, Sultanate of OMAN. email: mohammad@squ.edu.om
Bessem Samet
University of Tunisia, Department of Mathematics, Tunis College of Sciences and Techniques, 5 Avenue Taha Hussein, BP, 56, Bab Manara, TUNIS. email: bessem.samet@gmail.com