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AN EXTENSION OF CHEEGER DIFFERENTIAL OPERATOR FROM LIPSCHITZ FUNCTIONS TO ORLICZ-SOBOLEV FUNCTIONS ON METRIC MEASURE SPACES

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Abstract. We introduce two types of Orlicz-Sobolev spaces on a metric measure space. One space is the completion of locally Lipschitz functions in a norm of Orlicz-Sobolev type involving an abstract differentiation operator and the other space is defined via an Orlicz-Poincaré inequality. We prove that these spaces agree and are reflexive provided that the measure is doubling and the Young function defining the underlying Orlicz space is doubling, together with its complementary function. In the case where the Young function is a power function with exponent greater than one, we recover some results of Franchi, Hajłasz and Koskela (1999).

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1. Introduction

A celebrated theorem of Rademacher shows that Lipschitz real-valued functions on $\mathbb{R}^n$ are a.e. differentiable with respect to the Lebesgue measure. Cheeger [2] proved a deep generalization of Rademacher's theorem for a large class of metric measure spaces, showing that each doubling metric measure space supporting a Poincaré inequality admits a differentiable structure with which Lipschitz functions can be differentiated almost everywhere.

In the following, (X, d, μ) is a metric measure space, i.e. (X, d) is a metric space and μ is a Borel regular outer measure that is finite and positive on balls [11].

A strong measurable differentiable structure on (X, d, μ) is a countable collection $\{(X_\alpha, \varphi_\alpha) : \alpha \in \Lambda\}$ of measurable sets $X_\alpha \subset X$ with positive measure and Lipschitz coordinates $\varphi_\alpha = (\varphi_\alpha^1, ..., \varphi_\alpha^{N(\alpha)}) : X \to \mathbb{R}$, such that:

(i) $\mu(X \setminus \bigcup_{\alpha\in\Lambda} X_\alpha) = 0$;

(ii) There exists a non-negative integer N such that $N(\alpha) \leq N$ for all $\alpha \in \Lambda$;

(iii) If $f : X \to \mathbb{R}$ is Lipschitz, then for each $\alpha \in \Lambda$ there exists a unique (up to a set of zero measure) measurable bounded vector valued function $D^\alpha f : X_\alpha \to \mathbb{R}^{N(\alpha)}$ such that

$$\lim_{y\to x} \frac{|f(y) - f(x) - D^\alpha f(x) \cdot (\varphi_\alpha(y) - \varphi_\alpha(x))|}{d(y, x)} = 0, \quad \mu - \text{a.e. } x \in X_\alpha. \tag{1}$$

Such a structure is called *non-degenerate* if $N(\alpha) \geq 1$ holds for all α. The smallest N for which $N(\alpha) \leq N$ for all $\alpha \in \Lambda$ is called the *dimension* of the strong measurable differentiable structure.

Cheeger proved that every doubling metric measure space (X, d, μ) supporting a $(1, p)$ −Poincaré inequality admits a non-degenerate strong measurable differentiable structure.

A function $f : X \to \mathbb{R}$ is said to be (Cheeger) differentiable at $x \in X_\alpha$ if there exists a unique (up to a set of zero measure) measurable bounded vector valued function $D^\alpha f : X_\alpha \to \mathbb{R}^{N(\alpha)}$ such that (1) holds. In this case $D^\alpha f(x)$ is called the (measurable) differential of f at x. Assuming, as we may, that the sets X_α, $\alpha \in \Lambda$ are mutually disjoint and regarding each $\mathbb{R}^{N(\alpha)}$ as a subset of $\mathbb{R}^N$, we define

$$Df = \sum_{\alpha\in\Lambda} D^\alpha f$$

for each $f \in LIP_{loc}(X)$.

We will denote by $LIP(X)$ and $LIP_{loc}(X)$ the collections of all real-valued Lipschitz functions, respectively locally Lipschitz functions. From (1) we see that for every function $f \in LIP_{loc}(X)$ that is constant in some measurable set $E \subset X$ we have $Df = 0$ $\mu-$a.e. in E.

The infinitesimal behavior of a real function on a metric space $u : X \to \mathbb{R}$ at a point $x \in X$ is described by the upper and lower Lipschitz constants

$$Lip\ u(x) = \limsup_{r\to 0} \frac{L(x,u,r)}{r} \text{ and } lip\ u(x) = \liminf_{r\to 0} \frac{L(x,u,r)}{r},$$

where $L(x,u,r) = \sup\{|u(y) - u(x)| : d(x,y) \le r\}$.

There exists a constant $C = C(N) > 0$ depending only on the dimension of the strong measurable differentiable structure [13, Remark 2.1.4] such that for each Lipschitz function $f : X \to \mathbb{R}$ and for a.e. $x \in X$,

$$\frac{1}{C}|Df(x)| \le Lip\ f(x) \le C|Df(x)|. \tag{2}$$

Cheeger considered a Sobolev-type space $H^{1,p}(X)$, $1 \le p < \infty$, which is the completion of $LIP(X)$ in the norm $\|\cdot\|_{1,p}$ defined by $\|f\|_{1,p} = \|f\|_{L^p(X)} + \||Df|\|_{L^p(X)}$[2, Theorem 4.47], [13, Remark 2.1.4], where D is the Cheeger differential operator and proved that $H^{1,p}(X)$ is reflexive if $p > 1$.

An *abstract differential operator* [4, Theorem 10] on $LIP_{loc}(X)$ is a linear operator D which associates with each $u \in LIP_{loc}(X)$ a measurable function $Du : X \to \mathbb{R}^N$, where N is a fixed positive integer, such that the following conditions are satisfied:

(D1) There exists a constant $C_D > 0$ such that $|Du| \le C_D L$ $\mu-$a.e. whenever u is an $L-$Lipschitz function;

(D2) If $u \in LIP_{loc}(X)$ is constant in some measurable set $E \subset X$, then $Du = 0$ $\mu-$a.e. in E.

The above discussion shows that Cheeger differential operator is a special case of abstract differential operator, where N is the dimension of the strong measurable differentiable structure and $C_D = C$ from (2), see also [6, Theorem 11.6].

Franchi, Hajłasz and Koskela [4] compared two different types of Sobolev spaces in the setting of metric spaces, namely $H^{1,p}(X)$ and $P^{1,p}(X)$, $1 \leq p < \infty$. Here $H^{1,p}(X)$ is the closure of the set of locally Lipschitz functions with finite norm $\|f\|_{1,p} = \|f\|_{L^p(X)} + \||Df|\|_{L^p(X)}$, where D is an abstract differential operator. $P^{1,p}(X)$ is the set of all functions $u \in L^p(X)$ for which there exist a nonnegative function $g \in L^p(X)$ and some constants $C_P > 0$ and $\sigma \geq 1$ such that the pair (u, g) satisfies the $(1,p)-$Poincaré inequality (6). The spaces $P^{1,p}(X)$ have been introduced in [7] and developed in [8]. It is proved in [4, Theorem 9] that $P^{1,p}(X) \subset H^{1,p}(X)$ provided that the measure μ is doubling, see also [6, Theorem 10.1]. Note that in [4, Theorem 9] the condition (D2) is replaced by a weaker one, assuming that $Du = 0$ $\mu-$a.e. in each open set where $u \in LIP_{loc}(X)$ is constant. If in addition all pairs $(u, |Du|)$ satisfy a $(1,p)-$Poincaré inequality with fixed constants, for each $u \in LIP_{loc}(X)$, then $P^{1,p}(X) = H^{1,p}(X)$ and D naturally extends from $LIP_{loc}(X)$ to $H^{1,p}(X)$ [4, Theorem 10]; moreover, if $p > 1$, then $H^{1,p}(X)$ is reflexive [6, Theorem 10.2].

The aim of this note is to generalize the above mentioned results of Franchi, Hajłasz and Koskela for $1 < p < \infty$ to the case of some new Orlicz-Sobolev spaces $H^{1,\Phi}(X)$ and $P^{1,\Phi}(X)$, where an Young function Φ satisfying the Δ_2-condition together with its complementary function replaces the power function t^p. In our results, (X, d, μ) is a doubling metric measure space and D is an abstract differential operator on X. If in addition X supports a $(1,\Phi)-$Poincaré inequality, then D can be the Cheeger differential operator, whose action extends from locally Lipschitz functions to Orlicz-Sobolev functions.

2. Preliminaries

We will denote by $B(x,r)$ the open ball centered at $x \in X$ of radius $r > 0$, $B(x,r) = \{y \in X : d(y,x) < r\}$. If $B = B(x,r)$ and $\sigma > 0$, we denote the ball $B(x, \sigma r)$ by σB.

The metric measure space (X, d, μ) is said to be *doubling* if there is a constant $C_\mu \geq 1$ so that

$$\mu(B(x,2r)) \leq C_\mu \mu(B(x,r)) \tag{3}$$

for every ball $B(x,r)$ in X.

Lebesgue's differentiation theorem holds on every doubling metric measure space [11, Theorem 1.8 and 2.7], i.e. for every $f \in L^1_{loc}(X)$

and $\mu-$a.e. $x \in X$,

$$\lim_{r\to\infty} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, d\mu(y) = 0. \tag{4}$$

Remark 1. *Assume that (X, d, μ) is a doubling metric measure space, $f \in L^1_{loc}(X)$ and $x \in X$ is a Lebesgue point of f, i.e. (4) holds. Let $B_i = B(x_i, \rho_i)$, $i \geq 1$ be a sequence of balls in X such that $x \in B_i$ for all $i \geq 1$ and $\rho_i \to 0$ as $i \to \infty$. Then $\lim\limits_{i\to\infty} \frac{1}{\mu(B_i)} \int\limits_{B_i} |f(y) - f(x)|\, d\mu(y) = 0$. Indeed, for every $i \geq 1$ we have $B_i \subset B(x, 2\rho_i) \subset 3B_i$, hence $\frac{1}{\mu(B_i)} \int\limits_{B_i} |f(y) - f(x)|\, d\mu(y) \leq (C_\mu)^2 \frac{1}{\mu(B(x,\rho_i))} \int\limits_{B(x,\rho_i)} |f(y) - f(x)|\, d\mu(y)$.*

Given a locally integrable function u on X and $E \subset X$ a measurable set of positive finite measure, we denote the integral mean of u on E by u_E, i. e. $u_E = \frac{1}{\mu(E)} \int\limits_E u\, d\mu$.

The Hardy-Littlewood maximal function $\mathcal{M}(f)$ of a locally integrable real-valued function f in X is defined by

$$\mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f|\, d\mu$$

for all $x \in X$. If the measure μ is doubling, the maximal function theorem [11, Theorem 2.2] shows that $\mathcal{M}$ maps $L^1(X)$ to weak-$L^1(X)$ and $L^p(X)$ to $L^p(X)$ for $p > 1$.

In analysis on metric measure spaces the notion of upper gradient is a substitute for the length of the gradient of a smooth function and was introduced by Heinonen and Koskela in [12]. A Borel measurable function $g : X \to [0, \infty]$ is an *upper gradient* of a real-valued function u on X if γ

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g\, ds$$

for every rectifiable curve $\gamma : [a, b] \to X$.

If $u : X \to \mathbb{R}$ is Lipschitz continuous, then *lip* u is an upper gradient of u [6], hence *Lip* u is also an upper gradient of u.

A function $\Phi : [0, \infty) \to [0, \infty]$ is called a *Young function* if $\Phi(t) = \int\limits_0^t \varphi(s)\, ds$ for an increasing, left-continuous function $\varphi : [0, \infty) \to [0, \infty]$ vanishing at the origin, which is neither identically zero nor identically infinite on $(0, \infty)$. Every Young function is convex, increasing, left continuous and satisfies $\Phi(0) = 0$, $\lim_{t\to\infty} \Phi(t) = \infty$.

In applications of Orlicz spaces, some growth conditions for the corresponding Young functions are very useful. A Young function $\Phi : [0, \infty) \to [0, \infty]$ is said to be *doubling* or to satisfy a Δ_2-condition if there is a constant $C_\Phi \geq 1$ such that $\Phi(2t) \leq C_\Phi \Phi(t)$ for all $t \geq 0$. Every Young function satisfying the Δ_2-condition is finite, strictly increasing and continuous [16, Remark 2.2]. A growth condition in the opposite direction is the ∇_2-condition. A Young function Φ : $[0, \infty) \to [0, \infty]$ is said to satisfy a ∇_2-condition if there is a constant $C > 1$ such that $\Phi(Ct) \geq 2C\Phi(t)$ for all $t \geq 0$. A $N-$function is a continuous Young function $\Phi : [0, \infty) \to [0, \infty)$ satisfying $\lim_{t\to a} \frac{\Phi(t)}{t} = a$ for $a \in \{0, \infty\}$. For a complementary pair of $N-$ functions, one of the functions satisfies the ∇_2-condition if and only if the other satisfies the Δ_2-condition [15]. If $p, q > 1$ are Hölder conjugates to each other, $\Phi(t) = t^p/p$ and $\widetilde{\Phi}(t) = t^q/q$ are complementary $N-$functions satisfying the Δ_2-condition.

Let $(X, \mathcal{A}, \mu)$ be a measure space with μ a complete, $\sigma-$finite measure and let $\Phi : [0, \infty) \to [0, \infty]$ be a Young function. The Orlicz space $L^\Phi(X)$ is the set of all real-valued measurable functions u in X such that $\int_X \Phi(\lambda |u|) d\mu < \infty$ for some $\lambda > 0$. We identify any two functions that agree $\mu-$a.e. $L^\Phi(X)$ is a vector space and $\|u\|_{L^\Phi(X)} = \inf \left\{ k > 0 : \int_X \Phi(\frac{|u|}{k}) d\mu \leq 1 \right\}$ defines a norm on $L^\Phi(X)$, called the Luxemburg norm.

If two complementary Young functions Φ, $\widetilde{\Phi}$ satisfy the Δ_2-condition, then the space $L^\Phi(X)$ is reflexive [5].

We will consider the Orlicz space $L^\Phi(X, \mathbb{R}^N)$ as the set of all Bochner measurable functions $U : X \to \mathbb{R}^N$ such that $|U| \in L^\Phi(X)$. A function $U : X \to \mathbb{R}^N$, $U = (U_1, ..., U_N)$ is Bochner measurable if and only if all the functions U_i, $1 \leq i \leq N$ are measurable. The definition of $L^\Phi(X, \mathbb{R}^N)$ does not depend on the choice of a norm on $\mathbb{R}^N$. In the following we will choose the Euclidean norm on $\mathbb{R}^N$. A Luxemburg norm on $L^\Phi(X, \mathbb{R}^N)$ is defined by analogy to the Luxemburg norm on $L^\Phi(X)$, replacing the modulus on $\mathbb{R}$ by a norm on $\mathbb{R}^N$. It is easy to see that $U = (U_1, ..., U_N)$ belongs to $L^\Phi(X, \mathbb{R}^N)$ if and only if $U_i \in L^\Phi(X)$ for $1 \leq i \leq N$. If $L^\Phi(X)$ is reflexive, then $L^\Phi(X, \mathbb{R}^N)$ is reflexive as a finite product of reflexive spaces.

Tuominen [16] introduced the Orlicz-Sobolev space $N^{1,\Phi}(X)$, which is the set of equivalence classes of functions $u \in L^{\Phi}(X)$ possesing an upper gradient in $L^{\Phi}(X)$, with respect to the equivalence relation defined by $u \sim v$ iff $\|u-v\|_{1,\Phi} = 0$, where

$$\|f\|_{1,\Phi} = \|f\|_{L^{\Phi}(X)} + \inf\left\{\|g\|_{L^{\Phi}(X)} : g \text{ an upper gradient of } f\right\}.$$

We define another two types of Orlicz-Sobolev spaces on a metric measure space, corresponding to a Young function Φ. We follow [4] and [6], where the case $\Phi(t) = t^p, 1 \leq p < \infty$ has been considered. The definition of the Orlicz-Sobolev space $P^{1,\Phi}(X)$, a generalization of $P^{1,p}(X)$, involves a Poincaré inequality and does not require any additional assumption on the metric measure space.

Definition 1. [16] *Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing Young function and $\Omega \subset X$ an open set. We say that a pair (u,g) formed of a function $u \in L^1_{loc}(\Omega)$ and a measurable non-negative function g on Ω satisfies a $(1,\Phi)-$Poincaré inequality in Ω if there exist some constants $C_P > 0$ and $\sigma \geq 1$ such that for each ball $B = B(x,r)$ satisfying $\tau B \subset \Omega$,*

$$\frac{1}{\mu(B)}\int_B |u-u_B|\,d\mu \leq C_P r \Phi^{-1}\left(\frac{1}{\mu(\sigma B)}\int_{\sigma B}\Phi(g)d\mu\right). \tag{5}$$

If the inequality (5) holds for each $u \in L^1_{loc}(\Omega)$ and every upper gradient g of u, with fixed constants, then Ω is said to support a $(1,\Phi)-$Poincaré inequality.

For $\Phi(t) = t^p$, $1 \leq p < \infty$, the $(1,\Phi)-$Poincaré inequality is known as the $(1,p)-$Poincaré inequality

$$\frac{1}{\mu(B)}\int_B |u-u_B|\,d\mu \leq C_P r\left(\frac{1}{\mu(\sigma B)}\int_{\sigma B} g^p d\mu\right)^{\frac{1}{p}}. \tag{6}$$

Definition 2. *We say that a function $u : X \to \mathbb{R}$ in $L^{\Phi}(X)$ belongs to $P^{1,\Phi}(X)$ if there exists a non-negative function $g \in L^{\Phi}(X)$ such that the pair (u,g) satisfies the $(1,\Phi)-$Poincaré inequality (5) for some constants $C_P > 0$ and $\sigma \geq 1$.*

The definition of a generalization of $H^{1,p}(X)$, the Orlicz-Sobolev space $H^{1,\Phi}(X)$, as the closure of a subclass of locally Lipschitz functions under some norm involving an abstract derivative requires a more specialized approach. Let D be an abstract differential operator defined on $LIP_{loc}(X)$.

The set $V_\Phi(X) = \left\{u \in LIP_{loc}(X) \cap L^\Phi(X) : |Du| \in L^\Phi(X)\right\}$ is a vector space and the functional defined by

$$\|u\| = \|u\|_{L^\Phi(X)} + \||Du|\|_{L^\Phi(X)}$$

for $u \in V_\Phi(X)$ is a norm on this space. Then $H^{1,\Phi}(X)$ is defined as the closure of $V_\Phi(X)$ under the above norm. Since $L^\Phi(X)$ is a Banach space, we see that each element of $H^{1,\Phi}(X)$ is represented by a pair (u, G), where $u \in L^\Phi(X)$ and $G : X \to \mathbb{R}^N$ is measurable with $|G| \in L^\Phi(X)$, for which there exists a sequence $(u_n)_{n\geq 1}$ in $V_\Phi(X)$ such that $u_n \to u$ in $L^\Phi(X)$ and $|Du_n - G| \to 0$ in $L^\Phi(X)$ as $n \to \infty$.

As discussed in [4] and [6, Section 10] for $\Phi(t) = t^p$, there may be a problem with the extension of D from $V_\Phi(X)$ to $H^{1,\Phi}(X)$. Assume that two sequences $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ in $V_\Phi(X)$ have the same limit u in $L^\Phi(X)$ and that there exist two measurable functions $G, H : X \to \mathbb{R}^N$ with $|G|, |H| \in L^\Phi(X)$ such that $|Du_n - G| \to 0$ and $|Dv_n - H| \to 0$ in $L^\Phi(X)$ as $n \to \infty$. If G and H are distinct, which may happen, the pairs (u, G) and (u, H) belong to distinct equivalence classes in $H^{1,\Phi}(X)$ and we cannot define Du unambiguously. We say that the property of *uniqueness of the gradient* holds in $H^{1,\Phi}(X)$ if for every sequence $(w_n)_{n\geq 1}$ in $V_\Phi(X)$, such that $w_n \to 0$ in $L^\Phi(X)$ and $|Dw_n - J| \to 0$ in $L^\Phi(X)$ as $n \to \infty$, where $J : X \to \mathbb{R}^N$ is measurable with $|J| \in L^\Phi(X)$, we necessarily have $J = 0$ a.e. Assuming that the uniquess of the gradient holds and taking $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ as above, it follows that $(u_n - v_n) \to 0$ in $L^\Phi(X)$ and $|D(u_n - v_n) - (G - H)| \leq |Du_n - G| + |Dv_n - H| \to 0$ in $L^\Phi(X)$, hence $G = H$ a.e. and we can define $Du = G$.

3. Preliminary results

We prove an elementary lemma for further reference.

Lemma 1. *Assume that $f_n \to f$ in $L^\Phi(X)$ and $f_n \to f$ $\mu-$a.e. on X as $n \to \infty$. If the Young function Φ satisfies the Δ_2-condition, then there exists a subsequence $(f_{n_k})_{k\geq 1}$ such that $\int_X \Phi(|f_{n_k}|)\, d\mu \to \int_X \Phi(|f|)\, d\mu$ as $k \to \infty$.*

Proof. Since $\int\limits_X \Phi(|g|)\,d\mu \le \|g\|_{L^\Phi(X)}$ if $g \in L^\Phi(X)$ and $\|g\|_{L^\Phi(X)} \le 1$ [16, (2.18)], the norm convergence in $L^\Phi(X)$ implies the $\Phi-$mean convergence, namely $f_n \to f$ in $L^\Phi(X)$ implies $\int\limits_X \Phi(|f_n - f|)\,d\mu \to 0$.

By a partial converse of Lebesgue dominated convergence theorem, for each sequence $(h_n)_{n\ge1}$ convergent to zero in $L^1(X)$ there exist a subsequence $(h_{n_k})_{k\ge1}$ and a nonnegative function $h \in L^1(X)$ such that $|h_{n_k}| \le h$ for all $k \ge 1$. Since $\Phi(|f_n - f|) \to 0$ in $L^1(X)$ as $n \to \infty$, there exists a strictly increasing sequence of positive integers $(n_k)_{k\ge1}$ and a function F such that $\Phi(|f_{n_k} - f|) \le F$ for all $k \ge 1$.

Since Φ is increasing, convex and satisfies the Δ_2-condition, we have by the above inequality

$$\begin{aligned}\Phi(|f_{n_k}|) &\le \Phi\left(2\frac{|f_{n_k} - f| + |f|}{2}\right) \le \frac{C_\Phi}{2}\left(\Phi(|f_{n_k} - f|) + \Phi(|f|)\right) \\ &\le \frac{C_\Phi}{2}\left(F + \Phi(|f|)\right)\end{aligned}$$

for all $k \ge 1$. Note that the Δ_2-condition shows that $\Phi(|f|) \in L^1(X)$, hence $\frac{C_\Phi}{2}(F + \Phi(|f|)) \in L^1(X)$. Since $\Phi(|f_{n_k}|) \to \Phi(|f|)$ as $k \to \infty$, the claim follows by Lebesgue dominated convergence theorem. ∎

We will denote the mean oscillation of a locally integrable function u on X over a ball $B \subset X$ by

$$MO(u, B) = \frac{1}{\mu(B)} \int\limits_B |u - u_B|\,d\mu.$$

In order to define a discrete convolution operator for locally integrable functions on a doubling metric measure space, we need the notions of $(\varepsilon, \lambda)-$ cover of an open set and of Lipschitz partition of unity subordinated to $\{2B_i : i \ge 1\}$, where $\{B_i : i \ge 1\}$ is an $(\infty, 2)-$ cover of the open set. These notions and their properties have been discussed in [9]. In the following, X is a doubling metric measure space with a doubling constant C_μ and $\Omega \subset X$ is open.

Given the real numbers $\varepsilon > 0$, $\lambda \ge 1$, *an* $(\varepsilon, \lambda)-$ *cover of* Ω [9] is a countable cover $\mathcal{B} = \{B_i = B(x_i, r_i) : i \ge 1\}$ of Ω with the following properties:

(1) $r_i \le \varepsilon$ for all i;
(2) $\lambda B_i \subset \Omega$ for all i;
(3) If λB_i meets λB_j, then $r_i \le 2r_j$;
(4) Each ball λB_i meets at most $C = C(C_\mu, \lambda)$ balls λB_j.

Clearly, if $0 < \varepsilon \leq \varepsilon' \leq \infty$ and $1 \leq \lambda' \leq \lambda < \infty$, then every $(\varepsilon, \lambda) -$ cover of Ω is also an $(\varepsilon', \lambda') -$ cover of Ω.

In [9, Lemma 3.1] it is shown that every open set $\Omega \subset X$ admits an $(\varepsilon, \lambda) -$ cover for arbitrary $\varepsilon > 0$, $\lambda \geq 1$; note that, by the construction of a such a cover from [3, Theorem III.1.3], we can assume that the balls $\frac{1}{5}B_i$, $i \geq 1$ are mutually disjoint. If the balls B_i have the same radius, then each family $\{\tau B_i : i \geq 1\}$ with $\tau > 0$ has bounded overlap: $\sum\limits_{i\geq 1} \chi_{\tau B_i} \leq (C_\mu)^{2\log_2(10\tau+2)}$.

Let $\mathcal{B} = \{B_i = B(x_i, r_i) : i \geq 1\}$ be an $(\infty, 2) -$ cover of Ω. It is shown in [9, Lemma 3.2] that there exists a collection of real functions $\{\varphi_i : i \geq 1\}$ defined on Ω such that

(1) each φ_i is L_i-Lipschitz, where $L_i := C(C_\mu)/r_i$;
(2) $0 \leq \varphi_i \leq 1$ for all i;
(3) $\varphi_i = 0$ on $X \setminus 2B_i$ for all i;
(4) $\sum\limits_{i\geq 1} \varphi_i = 1$ on X.

A collection $\varphi = \{\varphi_i : i \geq 1\}$ as above is called a *(Lipschitz) partition of unity* with respect to $\mathcal{B}$.

Given an $(\infty, 2) -$ cover $\mathcal{B}$ of Ω and a partition of unity $\{\varphi_i : i \geq 1\}$ with respect to $\mathcal{B}$, the corresponding discrete convolution of $u \in L^1_{loc}(\Omega)$ is defined by

$$u_{\mathcal{B}}(x) = \sum_{i\geq 1} u_{B_i}\varphi_i(x),\ x \in \Omega.$$

Note that, for each $x \in \Omega$, there are at most $C(C_\mu, 2)$ non-zero terms in the series defining $u_{\mathcal{B}}(x)$.

Heikkinen [9] proved the following properties of the discrete convolution.

Lemma 2. [9, Lemma 3.3] *Let $u \in L^1_{loc}(\Omega)$ and Φ be an Young function.*

(1) The function $u_{\mathcal{B}}$ is locally Lipschitz. Moreover, for each $x \in B_i = B(x_i, r_i)$

$$Lip\ u_{\mathcal{B}}(x) \leq L_i \frac{1}{\mu(5B_i)} \int_{5B_i} |u - u_{5B_i}|\, d\mu.$$

(2) $\|w_{\mathcal{B}}\|_{L^\Phi(\Omega)} \leq C(C_\mu)\|w\|_{L^\Phi(\Omega)}$ for each $w \in L^\Phi(\Omega)$.

(3) Assume that Φ satisfies the Δ_2-condition and $u \in L^\Phi(\Omega)$. If $\mathcal{B}_k$ is an $(\varepsilon_k, 2) -$ cover of Ω for $k \geq 1$ and if $\varepsilon_k \to 0$ as $k \to \infty$, then $u_{\mathcal{B}_k} \to u$ in $L^\Phi(\Omega)$.

Lemma 3. *Let (X, d, μ) be a doubling metric measure space and D be an abstract differential operator on $LIP_{loc}(X)$. Let $\mathcal{B} = \{B_i(x_i, \varepsilon) : i \geq 1\}$ be an $(\varepsilon, 2)-$ cover of X with balls of fixed radius $\varepsilon > 0$ and let $u_{\mathcal{B}}$ be the discrete convolution of $u \in L^1_{loc}(X)$ with respect to $\mathcal{B}$. Then*

$$|Du_{\mathcal{B}}(x)| \leq \frac{C}{\varepsilon} \sum_{i \geq 1} MO(u, 5B_i)\, \chi_{B_i}(x) \tag{7}$$

for almost every $x \in X$, where $C = 2(C_\mu)^5 C(C_\mu) C(C_\mu, \lambda) C_D$.

Proof. Fix a ball $B_j \in \mathcal{B}$. For all $x \in B_j$,

$$|Du_{\mathcal{B}}(x)| = \left|D\left(u_{\mathcal{B}} - u_{B_j}\right)(x)\right| \leq \sum_{i \geq 1} \left|u_{B_i} - u_{B_j}\right| |D\varphi_i(x)|. \tag{8}$$

If $2B_i \cap 2B_j$ is empty, then $\varphi_i = 0$ on $2B_j$ and, since φ_i is Lipschitz, $D\varphi_i = 0$ $\mu-$a.e. on $2B_j$. In this case, let $E_{ij} \subset 2B_j$ be a set of measure zero such that $D\varphi_i = 0$ on $2B_j \setminus E_{ij}$. Moreover, due to condition (1) for a partition of unity with respect to $\mathcal{B}$ and to condition (D1) satisfied by D, for each $i \geq 1$ there is a set of measure zero F_i such that $|D\varphi_i| \leq C_D C(C_\mu) \varepsilon^{-1}$ on $X \setminus F_i$. By (8) we obtain

$$\begin{aligned} |Du_{\mathcal{B}}(x)| &\leq \sum_{i \geq 1 : 2B_i \cap 2B_j \neq \emptyset} \left|u_{B_i} - u_{B_j}\right| |D\varphi_i(x)| \\ &\leq C_D C(C_\mu) \varepsilon^{-1} \sum_{i \geq 1 : 2B_i \cap 2B_j \neq \emptyset} |u_{B_i} - u_{B^*}|. \end{aligned} \tag{9}$$

for all $x \in B_j \setminus \bigcup_{i \geq 1 : 2B_i \cap 2B_j = \emptyset} (E_{ij} \cup F_i)$, hence for $\mu-$a.e. $x \in B_j$.

Note that for all i we have $\left|u_{B_i} - u_{B_j}\right| \leq \frac{1}{\mu(B_i)} \int_{B_i} \left|u(y) - u_{B_j}\right| d\mu(y)$ and $\left|u(y) - u_{B_j}\right| \leq \frac{1}{\mu(B_j)} \int_{B_j} |u(y) - u(z)|\, d\mu(z)$, hence

$$\left|u_{B_i} - u_{B_j}\right| \leq \frac{1}{\mu(B_i)} \frac{1}{\mu(B_j)} \int_{B_i} \int_{B_j} |u(y) - u(z)|\, d\mu(z)\, d\mu(y). \tag{10}$$

Assume that $2B_i \cap 2B_j$ is non-empty. By the triangle inequality, $B_i \subset 5B_j$ and $B_j \subset 5B_i$. Using the doubling property of μ we get $\mu(5B_j) \leq (C_\mu)^5 \mu(B_i)$ and $\mu(5B_j) \leq (C_\mu)^3 \mu(B_j)$. Integrating the inequality $|u(y) - u(z)| \leq \left|u(y) - u_{5B_j}\right| + \left|u(z) - u_{5B_j}\right|$ over $B_i \times B_j$ and taking account of (10) we get $\left|u_{B_i} - u_{B_j}\right| \leq \frac{1}{\mu(B_i)} \int_{B_i} \left|u(y) - u_{5B_j}\right| d\mu(y) +$

$\frac{1}{\mu(B_j)} \int\limits_{B_j} \left|u(z) - u_{5B_j}\right| d\mu(z)$. Using the estimates of $\mu(B_i)$ and $\mu(B_j)$ in terms of $\mu(5B_j)$ this implies

$$\left|u_{B_i} - u_{B_j}\right| \le 2\left(C_\mu\right)^5 \frac{1}{\mu(5B_j)} \int\limits_{5B_j} \left|u(y) - u_{5B_j}\right| d\mu(y). \tag{11}$$

From (9) and (11), using the bounded overlap of the family $\{2B_i : i \ge 1\}$, we obtain

$$\begin{aligned} |Du_{\mathcal{B}}(x)| &\le \frac{C}{\varepsilon} MO(u, 5B_j) \\ &\le \frac{C}{\varepsilon} \sum_{i\ge 1} MO(u, 5B_i)\, \chi_{B_i}(x). \end{aligned} \tag{12}$$

for all $x \in B_j \setminus \bigcup\limits_{i\ge 1: 2B_i \cap 2B_j = \emptyset} (E_{ij} \cup F_i)$, where $C = 2\left(C_\mu\right)^5 C\left(C_\mu\right) C\left(C_\mu, \lambda\right) C_D$.

Since $\{B_j : j \ge 1\}$ is a cover of X, from (12) we get $|Du_{\mathcal{B}}(x)| \le \frac{C}{\varepsilon} \sum\limits_{i\ge 1} MO(u, 5B_i)\, \chi_{B_i}(x)$ for all $x \in X$ belonging to the complement of $\cup \{E_{ij} : i, j \ge 1 \text{ and } 2B_i \cap 2B_j = \emptyset\} \cup \bigcup\limits_{i\ge 1} F_i$. Therefore, (7) holds for almost every $x \in X$. ∎

4. Comparison between two Orlicz-Sobolev spaces and extension of a differential operator

Next we prove generalizations of Theorem 9 and Theorem 10 in [4], see also Theorem 10.1 and Theorem 10.2 in [6].

Theorem 4. *Let (X, d, μ) be a doubling metric measure space and D be a abstract differential operator on $LIP_{loc}(X)$. If Φ is a Young function satisfying the Δ_2-condition together with its complementary function $\widetilde{\Phi}$, then $P^{1,\Phi}(X) \subset H^{1,\Phi}(X)$.*

Proof. Let $u \in P^{1,\Phi}(X)$.

Assume that, for each integer $k \ge 1$, $\mathcal{B}_k = \{B_{ki} : i \ge 1\}$ is an $(\varepsilon_k, 2)-$ cover of X with balls of fixed radius $\varepsilon_k > 0$ and that $\varepsilon_k \to 0$ as $k \to \infty$. Denote $u_k = u_{\mathcal{B}_k}$ for $k \ge 1$. By Lemma 2, $u_k \to u$ in $L^\Phi(\Omega)$ as $k \to \infty$.

Let $k \ge 1$. By the proof of Lemma 3,

$$|Du_k(x)| \le \frac{C}{\varepsilon_k} MO(u, 5B_{ki})$$

for almost every $x \in B_{ki}$, for each $i \ge 1$.

Since $u \in P^{1,\Phi}(X)$, there exists a nonnegative function $g \in L^{\Phi}(X)$ and some constants $C_P > 0$, $\sigma \geq 1$ such that (5) holds for every ball $B \subset X$ of radius r. Then

$$MO(u, 5B_{ki}) \leq C_P \varepsilon_k \Phi^{-1}\left(\frac{1}{\mu(5\sigma B_{ki})} \int\limits_{5\sigma B_{ki}} \Phi(g)\, d\mu\right)$$

for all $i \geq 1$. It follows that

$$|Du_k(x)| \leq CC_P \Phi^{-1}\left(\frac{1}{\mu(5\sigma B_{ki})} \int\limits_{5\sigma B_{ki}} \Phi(g)\, d\mu\right), \tag{13}$$

for almost every $x \in B_{ki}$, for each $i \geq 1$.From the above inequality and the Δ_2-condition for Φ,

$$\Phi(|Du_k(x)|) \leq C' \frac{1}{\mu(5\sigma B_{ki})} \int\limits_{5\sigma B_{ki}} \Phi(g)\, d\mu,$$

for almost every $x \in B_{ki}$, for each $i \geq 1$. Here $C' = \max\{1, (C_\Phi)^{\log_2(2CC_P)}\}$. Therefore,

$$\int\limits_{B_{ki}} \Phi(|Du_k|)\, d\mu \leq C' \int\limits_{5\sigma B_{ki}} \Phi(g)\, d\mu \text{ for all } i \geq 1.$$

Since $\{B_{ki} : i \geq 1\}$ is a cover of X,

$$\begin{aligned}\int\limits_X \Phi(|Du_k|)\, d\mu &\leq \sum_{i\geq 1} \int\limits_{B_{ki}} \Phi(|Du_k|)\, d\mu \\ &\leq (C_\Phi)^{\log_2(2CC_P)} \sum_{i\geq 1} \int\limits_X \Phi(g)\, \chi_{5\sigma B_{ki}} d\mu,\end{aligned}$$

The family $\{5\sigma B_{ki} : i \geq 1\}$ has bounded overlap: there exists a constant $C(C_\mu, \sigma)$ such that each ball $5\sigma B_{ki}$ meets at most $C(C_\mu, \sigma)$ balls $5\sigma B_{kj}$. The above inequality implies

$$\int\limits_X \Phi(|Du_k|)\, d\mu \leq C'C(C_\mu, \sigma) \int\limits_X \Phi(g)\, d\mu. \tag{14}$$

Since $g \in L^{\Phi}(X)$ and Φ satisfies the Δ_2-condition, $\int\limits_X \Phi(g)\, d\mu$ is finite. By (14), there exists $M = C'C(C_\mu, \sigma) \geq 1$ so that $\int\limits_X \Phi(|Du_k|)\, d\mu \leq$

M. Using the convexity of Φ and $\Phi(0)=0$ we get

$$\int_X \Phi\left(\frac{|Du_k|}{M}\right) d\mu \leq 1.$$

Therefore, $|Du_k| \in L^\Phi(X)$ and $\||Du_k|\|_{L^\Phi(X)} \leq M$.

We proved that the sequence $(Du_k)_{k\geq 1}$ is bounded in $L^\Phi\left(X,\mathbb{R}^N\right)$. Since Φ and its complementary Young function satisfy the Δ_2-condition, the space $L^\Phi\left(X,\mathbb{R}^N\right)$ is reflexive. Passing to a subsequence, we may assume that $(Du_k)_{k\geq 1}$ is weakly convergent in $L^\Phi\left(X,\mathbb{R}^N\right)$ to some $h\in L^\Phi\left(X,\mathbb{R}^N\right)$. By Mazur's lemma, there exists a sequence of convex combinations $W_k=\sum_{j=k}^{N(k)}\lambda_{kj}Du_j$, $k\geq 1$, that converges strongly in $L^\Phi\left(X,\mathbb{R}^N\right)$ to h. Note that $W_k=Dv_k$, where

$$v_k=\sum_{j=k}^{N(k)}\lambda_{kj}u_j,$$

for $k\geq 1$.

Since the sequence of Lipschitz functions $(v_k)_{k\geq 1}$ converges to u in $L^\Phi(X)$ and $(Dv_k)_{k\geq 1}$ converges in $L^\Phi\left(X,\mathbb{R}^N\right)$, it follows that $u\in H^{1,\Phi}(X)$, q.e.d. ∎

Remark 2. *For $\Phi(t)=t^p$ with $1<p<\infty$, the above theorem gives Theorem 9 in* [4], *see also Theorem 10.1 in* [6].

Theorem 5. *Let (X,d,μ) be a doubling metric measure space, Φ a Young function satisfying the Δ_2-condition and let D be an abstract differential operator on $LIP_{loc}(X)$. Assume that for every locally Lipschitz function u, the pair $(u,|Du|)$ satisfies the $(1,\Phi)-$Poincaré inequality with fixed constants. Then $H^{1,\Phi}(X)\subset P^{1,\Phi}(X)$ and the uniqueness of the gradient holds in $H^{1,\Phi}(X)$. Moreover, if the complementary function $\widetilde{\Phi}$ of Φ also satisfies the Δ_2-condition, then:*

a) $|Du|\leq Cg$ a.e. for some constant C, whenever $u\in H^{1,\Phi}(X)$, $g\in L^\Phi(X)$ and the pair (u,g) satisfies the $(1,\Phi)-$Poincaré inequality;

b) $H^{1,\Phi}(X)=P^{1,\Phi}(X)$;

c) $H^{1,\Phi}(X)$ is reflexive.

Proof. **Step 1.** First we prove that the uniqueness of the gradient holds in $H^{1,\Phi}(X)$. Assume that $(f_n)_{n\geq 1}$ is a sequence in $V_\Phi(X)$, such that $f_n\to 0$ in $L^\Phi(X)$ and $|Df_n-G|\to 0$ in $L^\Phi(X)$, where $G:X\to\mathbb{R}^N$ is measurable with $|G|\in L^\Phi(X)$. We prove that $G=0$ a.e. We choose a sequence $0 < r_n < 1$, $n \geq 1$ such that $\sum\limits_{n\geq 1} r_n < \infty$. We will specify later some additional conditions on this sequence. By passing to a subsequence we may assume that

$$\|f_{n+1} - f_n\|_{L^\Phi(X)} + \||Df_{n+1} - Df_n|\|_{L^\Phi(X)} \leq r_n, \ n \geq 1.$$

Then $(f_n)_{n\geq 1}$ converges pointwise a.e. in $L^\Phi(X)$ to 0 and $(Df_n)_{n\geq 1}$ converge pointwise a.e. to G in $L^\Phi\left(X, \mathbb{R}^N\right)$ to 0, by [1, Theorem 1.4 and Theorem 1.6].

Set $\varphi_n = f_{n+1} - f_n$ for $n \geq 1$. Since d_n is locally Lipschitz, the pair $(\varphi_n, |D\varphi_n|)$ satisfies the $(1,\Phi)$ −Poincaré inequality (5). Set $\psi_n = \mathcal{M}\left(\Phi\left(|D\varphi_n|\right)\right)$, where $\mathcal{M}$ is the Hardy-Littlewood maximal operator. The following uniform Lipschitz-type estimate follows from the $(1,\Phi)$ −Poincaré inequality, by [16, Lemma 5.15]:

$$|\varphi_n(x) - \varphi_n(y)| \leq C_L d(x,y)\left(\Phi^{-1}(\psi_n(x)) + \Phi^{-1}(\psi_n(y))\right), \tag{15}$$

for $\mu-$almost all $x, y \in X$, for all $n \geq 1$. The constant $C_L > 0$ depends only on the doubling constant C_μ of μ and on the constant C_P of the $(1,\Phi)$ −Poincaré inequality. Denote $g_k(x) = \sum\limits_{n=k}^{\infty} \Phi^{-1}(\psi_n(x))$ for $x \in X$, $k \geq 1$.

Fix $k \geq 1$. By (15) it follows that for $\mu-$almost all $x, y \in X$, for all $m \geq l \geq k$ we have

$$|(f_m - f_l)(x) - (f_m - f_l)(y)| \leq C_L d(x,y)\left(g_k(x) + g_k(y)\right).$$

Letting $m \to \infty$ in the above inequality, we obtain that for $\mu-$almost all $x, y \in X$ and for all $l \geq k$

$$|f_l(x) - f_l(y)| \leq C_L d(x,y)\left(g_k(x) + g_k(y)\right). \tag{16}$$

In order to get an uniform estimate for the Lipschitz constant of some restriction of f_l for $l \geq k$ we consider the upper level sets $E_{k,t} = \{g_k > t\}$, where $t > 0$.

Fix some $t > 0$. By (16), for $\mu-$almost all $x, y \in X \setminus E_{k,t}$ and for all $l \geq k$ we have

$$|f_l(x) - f_l(y)| \leq 2C_L t d(x,y).$$

Since f_l is continuous, the above inequality holds for all $x, y \in X \setminus E_{k,t}$, i.e. the restriction of f_l to $X \setminus E_{k,t}$ is $2C_L t-$Lipschitz. Using the MacShane extension to X of this restriction and condition (D1) from the definition of the abstract differential operator D, we conclude that

$$|Df_l(x)| \leq 2C_D C_L t$$

for a.e. $x \in X \setminus E_{k,t}$ and for all $l \geq k$. Letting $l \to \infty$ in the above inequality we get $|G(x)| \leq 2C_D C_L t$ for a.e. $x \in X \setminus E_{k,t}$, hence

$$\mu\{|G| > 2C_D C_L t\} \leq \mu(E_{k,t}) \text{ for } k \geq 1.$$

We will show that $\mu(E_{k,t}) \to 0$ as $k \to \infty$, hence $\mu\{|G| > 2C_D C_L t\} = 0$. Since $t > 0$ is arbitrary, it follows that $G = 0$, as claimed.

In order to prove that $\mu(E_{k,t}) \to 0$ as $k \to \infty$, note that $\Phi^{-1}(\psi_n(x)) \leq \frac{t}{2^{n-k+1}}$ for all $n \geq k$ implies $g_k(x) \leq t$, therefore

$$E_{k,t} \subset \bigcup_{n=k}^{\infty} \left\{\Phi^{-1}(\psi_n(x)) > \frac{t}{2^{n-k+1}}\right\}. \tag{17}$$

Since Φ is strictly increasing,

$$\left\{\Phi^{-1}(\psi_n(x)) > \frac{t}{2^{n-k+1}}\right\} \subset \left\{\mathcal{M}(\Phi(|D\varphi_n|)) > \Phi\left(\frac{t}{2^{n-k+1}}\right)\right\}. \tag{18}$$

Since Φ satisfies the Δ_2-condition, $|D\varphi_n| \in L^{\Phi}(X)$ implies $\Phi(|D\varphi_n|) \in L^1(X)$. The weak L^1- estimate for the maximal function in doubling metric measure spaces [11, Theorem 2.2] yields

$$\mu\left(\left\{\mathcal{M}(\Phi(|D\varphi_n|)) > \Phi\left(\frac{t}{2^{n-k+1}}\right)\right\}\right) \leq C_1\left(\Phi\left(\frac{t}{2^{n-k+1}}\right)\right)^{-1} \int_X \Phi(|D\varphi_n|)\, d\mu. \tag{19}$$

Here C_1 depends only on the doubling constant C_μ.

From (17), (18) and (19) we obtain

$$\mu(E_{k,t}) \leq C_1 \sum_{n=k}^{\infty} \left(\Phi\left(\frac{t}{2^{n-k+1}}\right)\right)^{-1} \int_X \Phi(|D\varphi_n|)\, d\mu. \tag{20}$$

Recall that $\int_X \Phi(|f|)\, d\mu \leq \|f\|_{L^{\Phi}(X)}$ if $f \in L^{\Phi}(X)$ and $\|f\|_{L^{\Phi}(X)} \leq 1$ [16, (2.18)]. Since $\||D\varphi_n|\|_{L^{\Phi}(X)} \leq r_n < 1$, we have

$$\int_X \Phi(|D\varphi_n|)\, d\mu \leq r_n \tag{21}$$

for all $n \geq 1$.

Since $\Phi(t) \leq (C_\Phi)^{n-k+1} \Phi\left(\frac{t}{2^{n-k+1}}\right)$ by the Δ_2-condition, we get from (20) and (21)

$$\mu(E_{k,t}) \leq \frac{C_1}{\Phi(t)} (C_\Phi)^{1-k} \sum_{n=k}^{\infty} r_n (C_\Phi)^n. \tag{22}$$

Now we choose some $0 < a < 1$ such that the condition $S_k := (C_\Phi)^{1-k} \sum\limits_{n=k}^{\infty} r_n (C_\Phi)^n \to 0$ as $k \to \infty$ is satisfied with $r_n = a^n$, $n \geq 1$. In this case $S_k = C_\Phi a^k \sum\limits_{n=0}^{\infty} (aC_\Phi)^n$ and it suffices to assume that $a < \frac{1}{C_\Phi}$. Then $0 < r_n < 1$ for $n \geq 1$, $\sum\limits_{n\geq 1} r_n < \infty$ and $S_k \to 0$ as $k \to \infty$.

By (22) we obtain $\mu(E_{k,t}) \to 0$ as $k \to \infty$, which completes the proof of the first step.

Step 2. We prove that $H^{1,\Phi}(X) \subset P^{1,\Phi}(X)$ showing that the validity of the $(1,\Phi)$ −Poincaré inequality extends from locally Lipschitz functions to functions in $H^{1,\Phi}(X)$ via approximation.

Let $u \in H^{1,\Phi}(X)$. As in the proof of Step 1, we find a sequence $(\omega_n)_{n\geq 1}$ in $V_\Phi(X)$ such that $\omega_n \to u$ in $L^\Phi(X)$, $|D\omega_n - Du| \to 0$ in $L^\Phi(X)$, while $\omega_n \to u$ and $D\omega_n \to Du$ pointwise a.e. on X. For every ball $B = B(x,r)$ and all $n \geq 1$

$$\frac{1}{\mu(B)} \int_B |\omega_n - (\omega_n)_B| \, d\mu \leq C_P r \Phi^{-1} \left(\frac{1}{\mu(\sigma B)} \int_{\sigma B} \Phi(|D\omega_n|) d\mu \right). \tag{23}$$

Since $|D\omega_n| \to |Du|$ in $L^\Phi(X)$, passing to a subsequence we may assume that $\int_{\sigma B} \Phi(|D\omega_n|)d\mu \to \int_{\sigma B} \Phi(|Du|)d\mu$, by Lemma 1. Since Φ^{-1} is continuous, the right hand side of (23) converges to $\Phi^{-1}\left(\frac{1}{\mu(\sigma B)} \int_{\sigma B} \Phi(|Du|)d\mu\right)$ as $n \to \infty$.

On the other hand,

$$\begin{aligned} \frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu &\leq \frac{1}{\mu(B)} \int_B |\omega_n - (\omega_n)_B| \, d\mu + \\ &\quad + \frac{1}{\mu(B)} \int_B |u - \omega_n| \, d\mu + |u_B - (\omega_n)_B| \\ &\leq \frac{1}{\mu(B)} \int_B |\omega_n - (\omega_n)_B| \, d\mu + \frac{2}{\mu(B)} \int_B |u - \omega_n| \, d\mu. \end{aligned} \tag{24}$$

Since $\mu(B) < \infty$, convergence in $L^\Phi(B)$ implies convergence in $L^1(B)$ [16, Lemma 4.18 and Remark 4.19], therefore $\int_B |u - \omega_n| \, d\mu \to 0$

as $n \to \infty$. Using (24) and (23), then letting $n \to \infty$ we obtain

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu d\mu \leq C_P r \Phi^{-1} \left(\frac{1}{\mu(\sigma B)} \int_{\sigma B} \Phi(|Du|) d\mu \right).$$

From now on, we assume that the complementary function $\widetilde{\Phi}$ of Φ also satisfies the Δ_2-condition. By Theorem 4 and Step 2, $H^{1,\Phi}(X) = P^{1,\Phi}(X)$.

Step 3. Let $u \in H^{1,\Phi}(X)$ and $g \in L^{\Phi}(X)$ such that the pair (u, g) satisfies the $(1, \Phi)-$Poincaré inequality. We will use the approximating sequence $(u_k)_{k \geq 1}$ for u from the proof of Theorem 4, associated to the sequence $\mathcal{B}_k = \{B_{ki} : i \geq 1\}$ of $(\varepsilon_k, 2)-$ covers of X with balls of fixed radius $\varepsilon_k > 0$ for $k \geq 1$, such that $\varepsilon_k \to 0$ as $k \to \infty$.

By (13) for all k, $i \geq 1$ the following inequality holds

$$|Du_k(x)| \leq CC_P \Phi^{-1} \left(\frac{1}{\mu(5\sigma B_{ki})} \int_{5\sigma B_{ki}} \Phi(g) \, d\mu \right),$$

for $x \in B_{ki} \setminus A_{ki}$, where $A_{ki} \subset B_{ki}$ is some set of zero measure. Since $\Phi(g) \in L^1(X)$, the complement of the set of Lebesgue points of $\Phi(g)$ is a set $A_g \subset X$ of zero measure, see Remark 1. Let $x \in X \setminus \left(A_g \cup \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} A_{ki} \right)$. For each k we select from the cover $\mathcal{B}_k$ of X a ball B_{ki_k} containing x, and $x \notin A_{ki_k}$, hence (13) holds with $i = i_k$. Since $x \notin A_g$, we get $\frac{1}{\mu(5\sigma B_{ki_k})} \int_{5\sigma B_{ki_k}} \Phi(g) \, d\mu \to 0$ for $k \to \infty$, hence

$$\limsup_{k \to \infty} |Du_k(x)| \leq CC_P g(x). \tag{25}$$

We took into account that Φ^{-1} is continuous. Note that (25) holds for $\mu-$a.e. $x \in X$.

Since $(Du_k)_{k \geq 1}$ is bounded in the reflexive space $L^{\Phi}(X, \mathbb{R}^N)$, passing to a subsequence, we may assume that $(Du_k)_{k \geq 1}$ is weakly convergent in $L^{\Phi}(X, \mathbb{R}^N)$ to some $h \in L^{\Phi}(X, \mathbb{R}^N)$. By Mazur's lemma, there exists a sequence of convex combinations $W_k = \sum_{j=k}^{N(k)} \lambda_{kj} Du_j$, $k \geq 1$, that converges strongly in $L^{\Phi}(X, \mathbb{R}^N)$ to h. By the uniqueness of the gradient, $h = Du$ a.e. Hence, $|W_k| \to |Du|$ as $k \to \infty$ in $L^{\Phi}(X)$. Passing again to a subsequence, we may assume that $|W_k| \to |Du|$ as $k \to \infty$ a.e. on X.

We have $W_k = Dv_k$, where

$$v_k = \sum_{j=k}^{N(k)} \lambda_{kj} u_j,$$

for $k \geq 1$.

Then

$$|Dv_k| \leq \sum_{j=k}^{N(k)} \lambda_{kj} |Du_j| \text{ on } X.$$

Now (25) and the above inequality show that for $\mu-$a.e. $x \in X$ we have

$$|Du(x)| = \lim_{k\to\infty} |Dv_k(x)| \leq CC_P g(x).$$

Step 4. Since $L^\Phi(X)$ is reflexive and $F = (F_0, F_1, ..., F_N) \in L^\Phi(X, \mathbb{R}^{N+1})$ if and only if $F_i \in L^\Phi(X)$ for all $0 \leq i \leq N$, the space $L^\Phi(X, \mathbb{R}^{N+1})$ is reflexive. By the uniqueness of the gradient, $\|u\|_{H^{1,\Phi}(X)} = \|u\|_{L^\Phi(X)} + \||Du|\|_{L^\Phi(X)}$ defines a norm on $H^{1,\Phi}(X)$. The application $u \longmapsto (u, Du)$ from $H^{1,\Phi}(X)$ into $L^\Phi(X, \mathbb{R}^{N+1})$ is linear and injective. Denote by $(Du)_i$ the $i-$th component of Du for $1 \leq i \leq N$. We have

$$\|(u, Du)\|_{L^\Phi(X,\mathbb{R}^{N+1})} \leq \|u\|_{L^\Phi(X)} + \sum_{i=1}^{N} \|(Du)_i\|_{L^\Phi(X)} \leq N \|u\|_{H^{1,\Phi}(X)}$$

and

$$\|(u, Du)\|_{L^\Phi(X,\mathbb{R}^{N+1})} \geq \|u\|_{L^\Phi(X)} + \max_{1\leq i\leq N} \|(Du)_i\|_{L^\Phi(X)} \geq \frac{1}{\sqrt{N}} \|u\|_{H^{1,\Phi}(X)}.$$

Then the Banach space $H^{1,\Phi}(X)$ is isomorphic as a normed space with a closed subspace of the reflexive space $L^\Phi(X, \mathbb{R}^{N+1})$, therefore $H^{1,\Phi}(X)$ is reflexive. ∎

Remark 3. *For $\Phi(t) = t^p$ with $1 < p < \infty$, the above theorem gives Theorem 10 in* [4]*, see also Theorem 10.2 in* [6]*.*

Assume that (X, d, μ) is a doubling metric measure space supporting a $(1, \Phi)$ −Poincaré inequality. By [16, Theorem 5.7], (X, d, μ) supports a $(1, p)$ −Poincaré inequality for all $\log_2 C_\mu \leq p < \infty$. By Cheeger's theorem, X admits a non-degenerate strong measurable differentiable structure. So, the Cheeger differential operator D is well defined on $LIP_{loc}(X)$ and is an example of abstract differential operator. Let u be a locally Lipschitz function. By (2) and the fact that *Lip* u is an upper gradient of u, it follows that $|Du|$ is also an upper gradient of u, hence the pair $(u, |Du|)$ satisfies the $(1, \Phi)-$Poincaré inequality with fixed constants. The above theorem implies the following

Corollary 6. *Let Φ be a Young function satisfying the Δ_2-condition and let (X, d, μ) be a doubling metric measure space supporting a $(1, \Phi)-$Poincaré inequality. Denote by D the Cheeger differential operator on $LIP_{loc}(X)$. Then $H^{1,\Phi}(X) \subset P^{1,\Phi}(X)$ and the uniqueness of the gradient holds in $H^{1,\Phi}(X)$. Moreover, if the complementary Young function $\widetilde{\Phi}$ also satisfies the Δ_2-condition, then $H^{1,\Phi}(X) = P^{1,\Phi}(X)$ and this space is reflexive.*

References

[1] C. Bennett and R. Sharpley, **Interpolation of Operators**, Pure and Applied Mathematics, vol. 129, Academic Press, London, 1988.

[2] J. Cheeger, **Differentiability of Lipschitz functions on metric measure spaces**, Geom. Funct. Anal. 9 (1999), 428-517.

[3] R. R. Coifman and G. Weiss, **Analyse Harmonique Non-Commutative Sur Certains Espaces Homogènes**, vol. 242 of Lecture Notes in Mathematics, Springer, Berlin, Germany, 1971.

[4] B. Franchi, P. Hajlasz and P. Koskela, **Definitions of Sobolev classes on metric spaces**, Ann. Inst. Fourier (Grenoble) 49 (1999), 1903–1924.

[5] S. Fučik, O. John and A. Kufner, **Function Spaces**, Noordhoff International Publishing, Leyden, Academia, Prague, 1977.

[6] P. Hajłasz, **Sobolev spaces on metric-measure spaces**, Contemporary Math., 338 (2003), 173-218.

[7] P. Hajłasz and P. Koskela, **Sobolev meets Poincaré**, C. R. Acad. Sci. Paris Ser. I Math. 320 (1995), 1211–1215.

[8] P. Hajłasz and P. Koskela, **Sobolev met Poincaré,** Memoirs Amer. Math. Soc. 145 (2000), no. 688, x+101 pp.

[9] T. Heikkinen, **Characterizations of Orlicz-Sobolev spaces by means of generalized Orlicz-Poincaré inequalities**, Journal of Function Spaces and Applications, Volume 2012, Article ID 426067, 15 pages, doi:10.1155/2012/426067

[10] T. Heikkinen, P. Koskela and H. Tuominen, **Sobolev-type spaces from generalized Poincaré inequalities,** Studia Math 181 (1), 2007, 1-16

[11] J. Heinonen, **Lectures on Analysis on Metric Measure Spaces**, Springer- Verlag, New York, 2001.

[12] J. Heinonen and P. Koskela, **Quasiconformal maps on metric spaces with controlled geometry**, Acta Math. 181 (1998), 1–61.

[13] S. Keith, **A differentiable structure for metric measure spaces**, Advances in Math., 183 (2) (2004), 271-315

[14] M. Mocanu, **Differentiability properties of Orlicz-Sobolev functions on metric measure spaces**, ROMAI J., 10 (2) (2014), 153-168

[15] M. M. Rao and Z. D. Ren, **Theory of Orlicz Spaces**, Monographs and Textbooks in Pure and Applied Mathematics 146, Marcel Dekker, New York, 1991.
[16] H. Tuominen, **Orlicz-Sobolev spaces on metric measure spaces**, Ann. Acad. Sci. Fenn., Diss.135 (2004) 86 pp

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