”Vasile Alecsandri” University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 25(2015), No. 1, 77-86

FAST ALGORITHMS OF DUALLY CHORDAL GRAPHS

MIHAI TALMACIU

Abstract. We give a characterization of hereditary dually chordal graphs using weak decomposition. We also give a recognition algorithm for hereditary dually chordal graphs and we determine the combinatorial optimization numbers in efficient time.

Work presented at CAIM 2014, September 19-22, 2014, ”Vasile Alecsandri” University of Bacău

1. Introduction

The triangulated graphs (chordal) class has been noticed because of their properties. Among these properties we mention: perfection, recognition algorithms and ability to solve some combinatorial optimization problems (determining the stability number and minimum number of covering cliques) with linear complexity algorithms.

Interest for strongly chordal (M. Farber [5], see [3], [7], Strongly chordal graphs are defined in terms of a stronger ordering condition. A graph G is strongly chordal if and only if every induced subgraph of G has a simple vertex. A vertex v of the graph G is simple in G if the set $N[u] : u \in N[v]$ is linearly ordered by inclusion.) graphs arises in several ways.

Keywords and phrases: chordal graphs, dually chordal graphs, hereditary dually chordal graphs, weak decomposition, recognition algorithm.
(2010)Mathematics Subject Classification: 05C85, 68R10

The problems of locating minimum weight dominating sets and minimum weight independent dominating sets in strongly chordal graphs with real vertex weights can be solved in polynomial time, whereas each of these problems is *NP*-hard for chordal graphs.

A vertex $u \in N[v]$ is a *maximum neighbor* of v iff for all $w \in N[v]$ the inclusion $N[w] \subseteq N[v]$ holds (note that $u = v$ is not excluded). The ordering $(v_1, \cdots, v_n)$ is a

maximum neighborhood ordering if for all $i \in \{1, \cdots, n\}$ there is a maximum neighbor $u_i \in N_i[v_i]$:

for all $w \in N_i[v_i]$, $N_i[w] \subseteq N_i[u_i]$ holds.

The graph G is *dually chordal* [2] iff G has a maximum neighborhood ordering.

Many problems eficiently solvable for strongly chordal graphs remain polynomial-time solvable for dually chordal graphs.

The *k-th power* of a graph G is a graph on the same vertex set as G, where a pair of vertices is connected by an edge if they have distance at most k in G.

Any power of a dually chordal graph [2] is dually chordal.

The *hereditary dually chordal* graphs [2], i.e., graphs for which each induced subgraph is a dually chordal graph.

In [13] specifies that the dually chordal graphs holds:

clique problem is *NP*-complete, independent set is NP-complete, the recognition problem in linear time.

2. Preliminaries

According to ([1]), *G=(V,E)* is a connected, finite and undirected graph, without loops and multiple edges, having *V=V(G)* as the vertex set and *E=E(G)* as the set of edges. $\overline{G}$ (or *co-G*) is the complement of *G*. If $U \subseteq V$, by *G(U)* (or $[U]_G$, or [U]) we denote the subgraph of *G* induced by *U*. By *G-X* we mean the subgraph *G(V-X)*, whenever $X \subseteq V$, but we simply write *G-v*, when *X*=$\{v\}$. If *e=xy* is an edge of a graph *G*, then *x* and *y* are adjacent, while *x* and *e* are incident, as are *y* and *e*. If $xy \in E$, we also use the property $x \sim y$, and $x \aleph y$ whenever *x,y* are not adjacent in *G*. If $A,B \subset V$ are disjoint and $ab \in E$ for every $a \in A$ and $b \in B$, we say that *A,B* are *totally adjacent* and we denote by $A \sim B$, while by $A \aleph B$ we mean that no edge of *G* joins some vertex of *A* to a vertex from *B* and, in this case, we say *A* and *B* are *totally non-adjacent*.

The *neighborhood* of the vertex $v \in V$ is the set $NG(v)=\{u \in V : uv \in E\}$, while $NG[v]=NG(v) \bigcup \{v\}$; we denote *N(v)* and *N[v],* when *G* appears clearly from the context. The *degree* of v in G is $dG(v)=|NG(v)|$. The neighborhood of the vertex v in the complement of G will be denoted by $\overline{N}(v)$.

The neighborhood of $S \subset V$ is the set $N(S)=\bigcup_{v \in S} N(v) - S$ and $N[S]=S \bigcup N(S)$. A graph is complete if every pair of distinct vertices is adjacent.

By Pn, Cn, Kn we mean a chordless path on $n \geq 3$ vertices, a chordless cycle on $n \geq 3$ vertices, and a complete graph on $n \geq 1$ vertices, respectively.

A *complete bipartite* graph is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. A complete bipartite graph with partitions of size $|V_1| = m$ and $|V_2| = n$, is denoted $K_{m,n}$.

Let F denote a family of graphs. A graph G is called *F-free* if none of its subgraphs are in F.

The *Zykov sum* of the graphs $G1$, $G2$ is the graph $G=G1+G2$ having:

$V(G) = V(G1) \bigcup V(G2)$,

$E(G) = E(G1) \bigcup E(G2) \bigcup \{uv : u \in V(G1), v \in V(G2)\}$.

Let G be connected and u and v be two nonadjacent vertices of G. A *uv-separator* is a set $S \subseteq V(G)$ such that u and v are in different connected components of G - S. The separator is *minimal* if no proper subset of S has the same property.

Let $G=(V,E)$ be a connected graph. A non-empty set of vertices T is called *star-cutset* ([2]) if G-T is not connected and there exists a vertex v in T that is adjacent to any other vertex in T. The cutset is *minimal* if no proper subset of T has the same property.

We call a graph *Berge* if neither the graph nor its complement contains C_n, as an induced subgraph, for n odd and $n \geq 5$.

The *chromatic number* of a graph G ($\chi(G)$) is the least number of colors it takes to color its vertices so that adjacent vertices have different colors. The *stability number* $\alpha(G)$ of a graph G is the cardinality of the largest stable set. Recall that a stable set of G is a subset of the vertices such that no two of them are connected by an edge. The *clique number* of a graph G is a number of the vertices in a maximum clique of G, denoted by $\omega(G)$.

A graph G is *perfect* if, for each induced subgraph S of G, the chromatic number of S is equal to the clique number of S. A graph is *minimal imperfect* if it is not perfect and yet every proper induced subgraph is perfect.

A class H of graphs is called *hereditary* if every induced subgraph of a graph in H is in H.

3. A new characterization of hereditary dually chordal graphs

We recall a characterization of the weak decomposition of a graph.

Definition 1. ([10]) *A set $A \subset V(G)$ is called a weak set of the graph G if $NG(A) \neq V(G)$-A and G(A) is connected. If A is a weak set, maximal with respect to set inclusion, then G(A) is called a weak component. For simplicity, the weak component G(A) will be denoted with A.*

Definition 2. ([10]) *Let G=(V,E) be a connected and non-complete graph. If A is a weak set, then the partition $\{A, N(A), V\text{-}A \bigcup N(A)\}$ is called a weak decomposition of G with respect to A.*

The name of *weak component* is justified by the following result.

Theorem 1.([11]) *Every connected and non-complete graph G=(V,E) admits a weak component A such that $G(V\text{-}A)=G(N(A))+G(\overline{N}(A))$.*

Theorem 2.([11]) *Let G=(V,E) be a connected and non-complete graph and $A \subset V$. Then A is a weak component of G if and only if G(A) is connected and $N(A) \sim \overline{N}(A)$.*

The next result, that follows from Theorem 2, ensures the existence of a weak decomposition in a connected and non-complete graph.

Corollary 1. *If G=(V,E) is a connected and non-complete graph, then V admits a weak decomposition (A,B,C), such that G(A) is a weak component and G(V-A)=G(B)+G(C).*

Theorem 2 provides an *O(n+m)* algorithm for building a weak decomposition for a non-complete and connected graph.

Algorithm for the weak decomposition of a graph ([8], see [12])

Input: A connected graph with at least two nonadjacent vertices, *G=(V,E)*.

Output: A partition *V=(A,N,R)* such that *G(A)* is connected, *N=N(A)*, A$\aleph$R=$\overline{N}$(A).

Begin

A:= any set of vertices such that A$\bigcup$N(A)$\neq$V N:= N(A)

R := V-A$\bigcup$N(A)

While ($\exists$n$\in$N, $\exists$r$\in$R such that nr$\notin$E) *do*

Begin

A:= A$\bigcup$\{n\}

N:= (N-{n})$\bigcup$(N(n)$\bigcap$R)
R:=R-(N(n) $\bigcap$R)
end
end

Definition 3. ([5]) *A strong elimination ordering of a graph* $G = (V, E)$ *is an ordering* $v_1, v_2, \cdots, v_n$ *of* V *with the property that for each i, j, k and l, if* $i < j, k < l$, $v_k, v_l \in N[v_i]$, *and* $v_k \in N[v_j]$, *then* $v_l \in N[v_j]$.

Definition 4. ([5]) A graph is strongly chordal if it admits a strong elimination ordering.

Remark 1. ([5]) *Every induced subgraph of a strongly chordal graph is strongly chordal.*

A graph G is *hereditary dually chordal* graph if any induced subgraph of G is dually chordal.

If C is a cycle of even length in the graph G, then a *strong chord* of C is an edge of G joining two vertices, u and v, of C such that $d_C(u, v)$is odd and greater than 1.

Theorem 3. ([5]) *A graph G is strongly chordal if and only if it is chordal and every*

even cycle of length at least 6 in G has a strong chord.

Theorem 4. [4, 6] *Let G be a graph. Then, G is chordal if and only if every minimal vertex separator of G is complete.*

Theorem 6. [8] *G is triangulated if and only if for any weak decomposition (A;N;R) with G(A) weak component:*

1) N is a clique

2) [*R*] *and G- R are triangulated.*

Corollary 2. [2] *For a graph G the following conditions are equivalent:*

(i) G is a strongly chordal graph;

(ii) G is a hereditary dually chordal graph.

Teorema 7. *G is hereditary dually chordal if and only if for any weak decomposition (A,N,R) with G(A)weak component:*

(1) *N is clique*
(2) *G(R) and G(V-R) are hereditary dually chordal.*

Proof. I. We suppose that G is *hereditary dually* chordal. From Corollary 2 it follows that G is strongly chordal. Fom G strongly chordal it follows that, according to Remark 1., G(R) and G(V-R) are chordals. From Corollary 2, (2) holds. Since G is strongly chordal, according to Theorem 3, it follows that G is chordal. From Theorem 6 it follows that N is the clique. So, (1) holds.

II. Suppose (1) and (2) hold. We show that G is *hereditary dually* chordal. From (2) and from Corollary 2 and Theorem 3, it follows that G(R), G(V-R) are chordals. From (1) and from Theorem 6, it follows that G is chordal. If G is not strongly chordal then, according to Theorem 3, there is an even cycle, C, of length at least 6 which has no strong chord. Because N is clique and $R \sim N$ it follows that either $C \subseteq G(R)$ or $C \subseteq G(N \bigcup A)$. So, either *G(R)* or *G(V-R)* is not strongly chordal. So, according to Corollary 3, either *G(R)* or *G(V-R)* is not *hereditary dually chordal*, contradicting (2).

The above results lead to the following recognition algorithm.

Input: A connected graph with at least two nonadjacent vertices, G=(V,E).

Output: An answer to the question: is G a *hereditary dually chordal* graph

begin

L:={*G*}*; //L a list of graphs*

While (L $\neq \phi$*)*

Extract an element F from L;

Find a weak decomposition (A,N,R) for F;

If (N not clique in F) then

Return: G is not hereditary dually chordal

else introduce in L the connected components of G(R), G(V-R) incomplete

Return: G is hereditary dually chordal

end

The fact that N is the clique is proved in the following: If there is a vertex v in N with the degree in G(N) less than |N|-1 (thus we determine the grades of the vertices in the subgraph G(N)) then N is not the clique. So the algorithm is O(n(n+m)) time (just because the weak decomposition is O(n+m)).

Corollary 3 [9, 12]. *If G is a connected graph and (A,N,R) a weak decomposition with A weak component then the following holds:*

$$\alpha(G) = \max\left\{\alpha[A] + \alpha[R], \alpha(A\bigcup N)\right\};$$

$\omega(G)$=*max*$\{\omega([N]) + \omega([R]),\, \omega([A \bigcup N])\}$.

Corollary 4. *Let G=(V,E) be a connected and non-complete graph with G(A) a weak component in G. If G is hereditary dually chordal then holds:*

$$\alpha(G) = \alpha(G(A)) + \alpha(G(R)); \tag{1}$$

$$\omega(G) = \max\left\{|N| + \omega(G(R)), \omega(G(A \bigcup N))\right\}. \tag{2}$$

Proof. From Corollary 3, because N is clique in G results (2) . Let $T \subset A \bigcup N$ such that T is stable set and $|T| = \alpha([A \bigcup N]_G)$. Because N is clique in G results $|T \bigcap N| \leq 1$. If $T \bigcap N = \phi$ then $T \bigcup \{r\}$ is a stable set in $[A \bigcup R]_G$ else if $T \bigcap N = \{v\}$ then $(T - \{v\}) \bigcup \{r\}$ is a stable set in $[A \bigcup R]_G$, $\forall r \in R$. So, (1) holds.

Corollary 4 implies an algorithm for the construction of a stable set of maximum cardinal and a clique of maximum cardinal in a *hereditary dually chordal* graph.

Input: G =(V, E) a connected and noncomplete graph satisfying conditions in Corollary 4

Output: A stable set S with |S|=α(G)

begin

S = ϕ

L = {G} // L is a list of graphs

while ($L \neq \phi$)

begin

extract an element F from L

if (F is complete) *then*

$$S = S \bigcup \{v\}, \forall v \in V(F)$$

else

Determine a weak decomposition (A,N, R) for F

Put [A]F and the connected components of [R]F in L

end

end.

Input: G=(V, E) a connected and noncomplete graph satisfying conditions in Corollary 4

Output: A clique Q with |Q|=ω(G)

Begin

$Q = \phi$

k=0

L = {G} // L is a list of graphs

while ($L \neq \phi$)

begin

extract an element F from L
if (F is complete) *then*
$Q = Q \bigcup V(F)$
$k = k + 1$
$Q_k = Q$
$Q = \phi$
else
Determine a weak decomposition (A,N, R) for L
Put $[A \bigcup N]_F$ and $[N \bigcup R]_F$ in L
end
Let m be so that $|Q_m| = \max\{|Q_1|, \cdots, |Q_k|\}$and $Q = Q_m$
Then $\omega(G) = |Q|$
end

The complexity of α *and* ω*is O(n(n+m)).* The fact that *V(F)* is clique reads as follows: If there is a vertex v in *V(F)* with the degree in F less than |V(F)|-1 (thus we determine the grades of the vertices in F), then *V(F)* is not the clique. This way, the algorithm is *O(n(n+m))* (as long as the weak decomposition has the complexity *O(n+m)*).

4. Conclusions

We give a characterization of hereditary dually chordal graphs using weak decomposition. We also give recognition algorithms for hereditary dually chordal graphs in *O(n(n+m))* time. Finally, we determine the combinatorial optimization numbers in *O(n(n+m))* time.

Acknowledgment. The research of Mihai Talmaciu was supported by the project entitled Decomposition of graphs and solving discrete optimization problems, "AR-FRBCF", 2014-2015, a Bilateral Cooperation Research Project, involving Romanian Academy ("Vasile Alecsandri" University of Bacau is partner), National Academy of Sciences of Belarus, Belarusian Republican Foundation for Fundamental Research.

References

[1] C. Berge, **Graphs**, Nort-Holland, Amsterdam, (1985).
[2] A. Brandstdt, F. Dragan, V. Chepoi and V. Voloshin, **Dually chordal graphs**, SIAM J.Discrete Math. **11**(1988), 437-455.
[3] E. Dahlhaus, P.D. Manuel, M. Miller, **A characterization of strongly chordal graphs**, Discrete Mathematics **187**(1998), 269-271.
[4] G.A. Dirac, **On rigid circuit graphs**, Abh. Math. Sem. Univ. Hamburg **25**(1961), 71-76.

[5] M. Farber, **Characterizations of Strongly Chordal Graphs**, Discrete Mathematics **43**(1983), 173-189.
[6] Fulkerson, D.R. and Gross, O.A. **Incidence matrices and interval graphs**, Pacific J. Math. **15**(1965), 835-855, MR32#3881.
[7] T. A. McKee, **A new characterization of strongly chordal graphs**, Discrete Mathematics **205**(1999), 245-247.
[8] Mihai Talmaciu, Cornelius Croitoru, **Structural Graph Search**, Stud.Cercet.Stiint., Ser.Mat., **16**(2006) , Supplement, Proceedings of ICMI 45, Bacau, Sept.18-20, 2006, pp.573-588
[9] Talmaciu, M., Nechita, E., **Recognition algorithm for diamond-free graphs**, Informatica, ISSN (print:0868-4952; online:1822-8844), **18**(2007), 3, 457-462.
[10] Mihai Talmaciu, **Decomposition Problems in Graph Theory, with Applications in Combinatorial Optimization**, Thesis, Iaşi, (2002).
[11] Mihai Talmaciu, Elena Nechita, Barna Iantovics, **Recognition and Combinatorial Optimization Algorithms for Bipartite Chain Graphs**, Computing and Informatics, ISSN 1335-9150, **32**(2013), No 2 , pp. 313-329.
[12] Talmaciu, M., Nechita, E., Crisan, G.C. - **A Recognition Algorithm for a Class of Partitionable Graphs that Satisfies the Normal Graph Conjecture**, Studies In Informatics And Control, **18**(2009), 4, 349-354.
[13] http://www.graphclasses.org/classes/gc_185.html.

Department of Mathematics, Informatics and Educational Sciences,
Faculty of Sciences, "Vasile Alecsandri" University of Bacău,
157 Calea Mărăşeşti, Bacău, 600115, Romania
e-mail : mtalmaciu@ub.ro , mihaitalmaciu@yahoo.com