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RHEONOMIC GENERAL RANDERS MECHANICAL SYSTEMS

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Abstract. This paper presents a special type of rheonomic Finslerian Mechanical System called rheonomic General Randers Mechanical System, establishing the canonical semispray, the local coefficients of the Lorentz connection and the expressions for the curvature and for the torsion.

1. Preliminary. Rheonomic Finsler Spaces

Let M be a real n-dimensional differentiable manifold and (TM, π, M) the tangent bundle of M.We consider $E = TM \times R$ a $2n+1$- dimensional real manifold. In a local chart $U \times (a, b)$ the points $u = (x, y, t) \in E$ have the local coordinates (x^i, y^i, t). Let $rF^n = (M, F(x, y, t))$ be a rheonomic Finsler space where $F : E \to R$ is the fundamental function and the Hessian of F given by $g_{ij}(x, y, t) = \frac{1}{2}\frac{\partial^2 F}{\partial y^i \partial y^j}$, called the fundamental tensor of rF^n, is positive defined.

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The Cartan nonlinear connection N has the coefficients $\left(N_j^i(x,y,t), N_j^0(x,y,t)\right)$ with

$$
\begin{aligned}
N_j^i(x,y,t) &= \tfrac{1}{2}\tfrac{\partial}{\partial y^j}\left(\gamma_{kh}^i(x,y,t)\,y^k y^h\right)\\
N_j^0(x,y,t) &= \tfrac{1}{2}\tfrac{\partial g_{jk}}{\partial t}y^k
\end{aligned}
\tag{1.1}
$$

and γ_{kj}^i are the Christoffel symbols of the fundamental tensor $g_{ij}(x,y,t)$. N determines the horizontal distribution on E which is supplementary to the vertical distribution. The adapted basis to these distribution is $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t}\right)$ with

$$
\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y,t)\frac{\partial}{\partial y^j} - N_i^0(x,y,t)\frac{\partial}{\partial t}. \tag{1.2}
$$

The dual adapted basis is $(dx^i, \delta y^i, dt)$ where

$$
\begin{aligned}
\delta y^i &= dy^i + N_j^i(x,y,t)\,dx^j\\
\delta t &= dt + N_j^0(x,y,t)\,dx^j.
\end{aligned}
\tag{1.3}
$$

2. Rheonomic General Randers Spaces

We consider $\beta(x,y,t) = b_i(x,t)\,y^i$ with $b_i(x,y)$ a covector field on $M\times R$ and the metric $L: E\to R$, $L(x,y,t) = F(x,y,t) + \beta(x,y,t)$. The pair $rGR^n = (M, L(x,y,t))$ is called a rheonomic General Randers space. The tensor field g_{ij} of rGR^n is

$$
\overset{L}{g}_{ij}(x,y,t) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i\partial y^j}. \tag{2.1}
$$

We denote

$$
F_{ij}(x,t) = \frac{\partial b_j}{\partial x^i} - \frac{\partial b_i}{\partial x^j} \tag{2.2}
$$

and we consider

$$
F_j^i(x,y,t) = g^{ik}(x,y,t)\,F_{kj}(x,t)\,. \tag{2.3}
$$

Let us consider in a rGR^n space the nonlinear connection N whose local coefficients are $\left(\overset{L}{N}{}_j^i(x,y,t), \overset{L}{N}{}_j^0(x,y,t)\right)$ with

$$\left\{\begin{array}{l} \overset{L}{N}{}^i_j = N^i_j - F^i_j \\ \overset{L}{N}{}^0_j = N^0_j - F^i_j. \end{array}\right. \tag{2.4}$$

N is called Lorentz nonlinear connection of the space rGR^n.

The local basis adapted to the Lorentz nonlinear connection is $\left(\frac{\overset{L}{\delta}}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t}\right)$, where

$$\frac{\overset{L}{\delta}}{\delta x^i} = \frac{\delta}{\delta x^i} + F^i_j \frac{\partial}{\partial y^j} + F^i_j \frac{\partial}{\partial t}. \tag{2.5}$$

The dual adapted basis is $\left(dx^i, \overset{L}{\delta} y^i, \overset{L}{\delta} t\right)$ with

$$\begin{array}{l} \overset{L}{\delta} y^i = \delta y^i - F^i_j dx^j \\ \overset{L}{\delta} t = \delta t - F^i_j dx^j. \end{array} \tag{2.6}$$

The weak torsion $\overset{L}{T}{}^i_{jk}$ of rGR^n space is

$$\overset{L}{T}{}^i_{jk} = \frac{\partial \overset{L}{N}{}^i_j}{\partial y^k} - \frac{\partial \overset{L}{N}{}^i_k}{\partial y^j} \tag{2.7}$$

And the curvature tensor $\overset{L}{R}{}^i_{jk}$ is

$$\overset{L}{R}{}^i_{jk} = \frac{\overset{L}{\delta} \overset{L}{N}{}^i_j}{\partial x^k} - \frac{\overset{L}{\delta} \overset{L}{N}{}^i_k}{\partial x^j}. \tag{2.8}$$

By a direct calculus we can state the following theorem:

Theorem 2.1. The torsion $\overset{L}{T}{}^i_{jk}$ of the Lorentz nonlinear connection of the rGR^n space vanishes and the curvature tensor is given by

(2.9)

$$\overset{L}{R}{}^i_{jk} = \left(\frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}\right) - \left(\frac{\delta F^i_j}{\delta x^k} - \frac{\delta F^i_k}{\delta x^j}\right) + \left(F^i_j \frac{\partial N^i_j}{\partial y^k} - F^i_k \frac{\partial N^i_k}{\partial y^j}\right)$$
$$- \left(F^i_j \frac{\partial F^i_j}{\partial y^k} - F^i_k \frac{\partial F^i_k}{\partial y^j}\right) + \left(F^i_j \frac{\partial N^i_j}{\partial t} - F^i_k \frac{\partial N^i_k}{\partial t}\right) - \left(F^i_j \frac{\partial F^i_j}{\partial t} - F^i_k \frac{\partial F^i_k}{\partial t}\right).$$

3. Rheonomic General Randers Mechanical Systems

Definition 3.1. A Rheonomic General Randers Mechanical System (*rhGRMS*) is a triple $\sum = (M, L^2(x,y,t), \sigma_i(x,y,t))$, where:

i) M is an n-dimensional, real, differentiable manifold;

ii) $L(x,y,t)$ is the fundamental function of the rheonomic general CityplaceRanders space rGR^n;

iii) $\sigma_i(x,y,t)$ is a d-covector field called the external force of $\sum$.

The evolution equations of the *rhGRMS* $\sum$ are given by the system of second order differential equations:

$$\frac{d^2x^i}{dt} + 2\overset{M}{G^i}(x,y,t) + \overset{M}{N^i_0}(x,y,t) = \frac{1}{2}\sigma^i(x,y,t) \tag{3.1}$$

where

$$\sigma^i = g^{ij}\sigma_j \tag{3.2}$$

and

$$\overset{M}{G^i} = \frac{1}{2}\gamma^i_{rs}(x,y,t)\, y^r y^s. \tag{3.3}$$

The equations (3.1) determine a semispray $\overset{M}{S}$, or a dynamical system on $TM \times R$.

We can prove

Theorem 3.1. a) The semispray $\overset{M}{S}$ on $TM \times R$, is given by

(3.3)
$$\overset{M}{S} = y^i\frac{\partial}{\partial x^i} - \left(2\overset{M}{G^i}(x,y,t) - \frac{1}{2}\sigma^i(x,y,t) + \overset{M}{N^i_0}(x,y,t)\right)\frac{\partial}{\partial y^i} + \frac{\partial}{\partial t}.$$

b) $\overset{M}{S}$ is a dynamical system on $\widetilde{TM} \times R$ depending only of the *rhGRMS* $\sum$. We call this semispray the evolution semispray of the *rhGRMS* $\sum$.

c) The integral curves of $\overset{M}{S}$ are the evolution curves of $\sum$ given by (3.1).

The evolution semispray (3.3) determines a nonlinear connection $\overset{M}{N}$ depending only by the *rhGRMS* $\sum$. The local coefficients are $\left(\overset{M}{N^i_j}, \overset{M}{N^0_j}\right)$, where

$$
\begin{aligned}
\overset{M}{N^i_j} &= \overset{L}{N^i_j} - \tfrac{1}{4}\tfrac{\partial\sigma^i}{\partial y^j} \\
\overset{M}{N^0_j} &= \overset{L}{N^0_j},
\end{aligned}
\tag{3.4}
$$

with $\overset{L}{N^i_j}$ the local coefficients of the Lorenz nonlinear connection of the rheonomic General Randers space. So,

$$
\begin{aligned}
\overset{M}{N^i_j} &= N^i_j - \left(F^i_j + \tfrac{1}{4}\tfrac{\partial\sigma^i}{\partial y^j}\right) \\
\overset{M}{N^0_j} &= N^0_j - F^i_j.
\end{aligned}
\tag{3.5}
$$

$\overset{M}{N}$ is called the Lorentz nonlinear connection of the *rhGRMS* $\sum$and determines a direct decomposition on $TM \times R$ into horizontal and vertical subspaces. The adapted basis to this decomposition is $\left(\frac{\overset{M}{\delta}}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t}\right)$ where

$$
\frac{\overset{M}{\delta}}{\delta x^i} = \frac{\delta}{\delta x^i} + \left(F^i_j + \frac{1}{4}\frac{\partial\sigma^j}{\partial y^i}\right)\frac{\partial}{\partial y^j} + F^i_j\frac{\partial}{\partial t}.
\tag{3.6}
$$

The dual basis is $\left(dx^i, \overset{M}{\delta^i}, \overset{M}{\delta t}\right)$ where

$$
\begin{aligned}
\overset{M}{\delta}\, y^i &= \delta y^i - \left(F^i_j + \tfrac{1}{4}\tfrac{\partial\sigma^i}{\partial y^j}\right) dx^j \\
\overset{M}{\delta}\, t &= \delta\, t - F^i_j dx^j.
\end{aligned}
\tag{3.7}
$$

Theorem 3.2. The torsion $\overset{M}{T^i_{jk}}$ of the Lorentz nonlinear connection $\overset{M}{N}$ of the *rhGRMS* $\sum$ vanishes and the curvature tensor $\overset{M}{R^i_{jk}}$ is given by

$$
\overset{M}{R^i_{jk}} = \frac{\overset{M}{\delta}\, \overset{M}{N^i_j}}{\delta x^k} - \frac{\overset{M}{\delta}\, \overset{M}{N^i_k}}{\delta x^j}.
\tag{3.8}
$$

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