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FIXED POINT THEOREMS IN ORDERED CONE B-METRIC SPACES

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Abstract. In the current work, some fixed point theorems are proved on ordered cone b-metric space motivated by [2], [14] and [15]. The proposed theorems extend and unify several well-known comparable results in the literature to ordered cone b-metric spaces. Some supporting examples are given.

1. Introduction

Existence of fixed point in partially ordered sets has been considered in [2], [14] and [15]. In [2], Altun and Durmaz proved some fixed point theorems on ordered cone metric space and discussed the existence of fixed points. Several other authors studied fixed point and common fixed point problems in ordered cone metric spaces; see [1, 3, 4, 5, 6, 7, 12, 13, 16, 17, 18].

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In 2005 Nieto and Lopez [14], proved the existence and uniqueness of the solution for a first order ordinary differential equation with boundary conditions of periodic type and allow only the existence of a lower solution. To this aim, they proved some fixed point theorems in partially ordered sets. Also, in [15] Ran and Reurings proved some fixed point theorems in partial ordered set and gave some applications to linear and nonlinear matrix equations. In [15], they proved the following theorem in ordered metric spaces.

Theorem 1.1. ([14],[15]) *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is complete metric space. Let $f : X \to X$ be a continuous and nondecreasing map with respect to $\preceq$ such that there exists $k \in [0, 1)$ with*

$$d(fx, fy) \leq kd(x, y), \text{ for all } x, y \in X \text{such that } y \preceq x,$$

and

$$\text{if there exists } x_0 \in X \text{ such that } x_0 \preceq fx_0,$$

then f has a fixed point in X.

Moreover Nieto and Lopez [14], proved the following theorem without assuming the continuity of the function f, but assuming an additional hypothesis on X.

Theorem 1.2. ([14]) *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is complete metric space. Assume that X satisfies " If a nondecreasing sequence (x_n) converges to some $x \in X$, then $x_n \preceq x$ for all n". Let $f : X \to X$ be a nondecreasing map with respect to $\preceq$, such that the following conditions hold:*
(i) there exists $k \in [0, 1)$ such that $d(fx, fy) \leq kd(x, y)$ for all $x, y \in X$;
(ii) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$.
Then f has a fixed point in X.

In Theorem 2.1 [15], Ran and Reurings assumed that the partial order set $(X, \preceq)$ satisfies the following condition:

"every pair of elements has a lower bound and an upper bound." (1)

Under the assumptions of Theorem 1.1, they proved that f has a unique fixed point $y \in X$. Moreover, for all $x \in X$, $\lim_{n\to\infty} f^n x = y$. If condition (1) fails, then it is possible to find examples of functions with more than one fixed point.

Example 1.3. *Let $Y = \mathbb{R}$ and consider the following relation on Y as follows:*

$x \preceq y \Longleftrightarrow (x = y)$ *or* $(x, y \in [0,1]$ *with* $x \leq y)$.

It is easy to see that $\preceq$ is a partial order on Y. Let d be the usual metric on Y. Now define a map on Y as follows:

$$f(x) = \begin{cases} 0, & x < 0; \\ kx, & if\, x \in [0,1]; \\ mx + n, & if\, x > 1, \end{cases}$$

where $k \in (0,1)$ and $m > 1, m + n = k$. Then f satisfies the assumptions of Theorem 1.1. The fixed points of f are $x_1 = 0$ and $x_2 = \frac{m-k}{m-1}$. Choosing $a \in [0,1]$ and $b > 1$ we can see that the pair a, b has no lower bound and no upper bound, so that Theorem 2.1 from [15] *is not applicable.*

In 2009, Altun and Durmaz [2, Theorem 4] proved the following result in ordered cone metric spaces, generalizing Theorem 1.1.

Theorem 1.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that the cone metric space (X, d) is complete and P be a normal cone with normal constant K. Let $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\preceq$. Suppose that the following two assertions hold:*
(i) there exists $k \in (0,1)$ such that $d(fx, fy) \leq kd(x, y)$ for each x, y in X with $y \preceq x$;
(ii) there exists x_0 in X such that $x_0 \preceq fx_0$.
Moreover, the sequence $\{f^n x_0\}$ has a limit which is a fixed point of f.

The main purpose of this research paper is to extend Theorem 1.4 above and Theorem 2.1, 2.3 from [9] to ordered cone b-metric spaces.

2. Preliminaries

We will need the following definitions and results in this paper.
Let E always be a real Banach space and P be a subset of E. By θ we denotes the zero element of E. Then P is called cone if and only if:
(i) P is closed, nonempty and $P \neq \{\theta\}$,
(ii) $cx + dy \in P$ for all $x, y \in P$ and non-negative real numbers c, d.

(iii) $P \cap (-P) = \{\theta\}$.
The cone P is called solid cone if $intP \neq \emptyset$.

Definition 2.1. [8] *Let* P *be a cone in real Banach space* E*. We define a partial ordering* $\leq$ *with respect to* P *by* $a \leq b$ *if and only if* $b - a \in$ P*. We shall write* $a < b$ *to indicate that* $a \leq b$ *but* $a \neq b$*, while* $a \ll b$ *will stand for* $b - a \in int$P*, where* intP *denote interior of* P*.*

Definition 2.2. [8] *The cone* P *is called normal if there is number* $K > 0$ *such that for all* $p, q \in E$*,* $\theta \leq p \leq q$ *implies* $\|p\| \leq K\|q\|$*.*
The least positive number K *satisfying the above inequality is called the normal constant of cone.*

Definition 2.3. [10] *Let* Y *be a non-empty set and* $s \geq 1$ *be a given real number. A mapping* $d : Y \times Y \to E$ *is said to be cone b-metric if and only if, for all* a, b, c *in* Y*, the following conditions are satisfied:*
(i) $\theta < d(a, b)$ *for all* $a, b \in X$ *with* $a \neq b$*;*
(ii) $d(a, b) = \theta$ *if and only if* $a = b$*;*
(iii) $d(a, b) = d(b, a)$ *for all* $a, b \in Y$*;*
(iv) $d(a, b) \leq s[d(a, c) + d(c, b)]$ *for all* $a, b, c \in Y$*.*
Then d *is called a cone b-metric on* Y *and* (Y, d) *is called a cone b-metric space.*

Example 2.4. [8] *Suppose* $Y = \mathbb{R}$*,* $E = \mathbb{R}^2$*, and* $P = \{(a, b) \in E : a \geq 0, b \geq 0\}$*. Define* $d : Y \times Y \to E$ *by* $d(a, b) = (|a - b|^p, \alpha|a - b|^p)$*, where* $\alpha > 0$ *and* $p > 1$*. Then* (Y, d) *is a cone b-metric space, but not cone metric space.*

Definition 2.5. [10] *Let* (Y, d) *be a cone b-metric space,* $a \in Y$ *and* $\{a_n\}$ *be a sequence in* Y*. We say that*
(i) $\{a_n\}$ *converges to* a *whenever, for every* $c \in E$ *with* $\theta \ll c$*, there is a natural number* N *such that* $d(a_n, a) \ll c$ *for all* $n > N$*. We denote this by* $\lim_{n\to\infty} a_n = a$ *or* $a_n \to a$ *as* $n \to \infty$*.*
(ii) $\{a_n\}$ *is a Cauchy sequence whenever, for every* $c \in E$ *with* $\theta \ll c$*, there is a natural number* N *such that* $d(a_n, a_m) \ll c$ *for all* $n, m > N$*.*
(iii) (Y, d) *is a complete cone b-metric space if every Cauchy sequence is convergent.*
(iv) the function $f : Y \to Y$ *is continuous at the point* a_0 *if for any sequence* $\{a_n\}$ *that converges to* a_0 *the sequence* $\{fa_n\}$ *converges to* fa_0*.*

The following lemmas are useful in our work.

Lemma 2.6. [11] *If* E *is a real Banach space with a cone* P *and* $a \leq ka$ *where* $a \in P$ *and* $k \in [0, 1)$*, then* $a = \theta$*.*

Lemma 2.7. [11] *Let P be a cone in E. If $c \in intP$, $\theta \leq a_n$ and $a_n \to \theta$ in E, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.*

Lemma 2.8. [11] *Let $a, b, c \in E$. If $a \leq b$ and $b \ll c$, then $a \ll c$.*

Lemma 2.9. [11] *If $\theta \leq u \ll c$ for each $\theta \ll c$, than $u = \theta$.*

3. Main Results

In this section, we prove some fixed point theorems on ordered cone b-metric spaces.

Theorem 3.1. *Let $(Y, \preceq)$ be a partially ordered set. Suppose that there exists a cone b-metric $d : Y \times Y \to E$ such that the cone b-metric space (Y, d) is complete with the coefficient $s \geq 1$ relative to a solid cone P in E. Let $H : Y \to Y$ be a continuous and nondecreasing mapping with respect to $\preceq$. Assume that the following two conditions hold:*
(i) there exists $\kappa \in [0, 1/s)$ such that $d(Ha, Hb) \leq \kappa\, d(a, b)$ for all $a, b \in Y$ with $a \succeq b$,
(ii) there exists $a_0 \in Y$ such that $a_0 \preceq Ha_0$.
Then H has a fixed point a^* in Y.

Proof. If $Ha_0 = a_0$, then the proof is finished. Assume that $Ha_0 \neq a_0$. Since $a_0 \preceq Ha_0$ and H is nondecreasing with respect to $\preceq$, we get by induction that

$$a_0 \preceq Ha_0 \preceq H^2 a_0 \preceq \cdots \preceq H^n a_0 \preceq H^{n+1} a_0 \preceq \ldots. \tag{1}$$

Now,

$$\begin{aligned}
d(H^{n+1}a_0, H^n a_0) &\leq \kappa d(H^n a_0, H^{n-1} a_0) \\
&\leq \kappa^2 d(H^{n-1} a_0, H^{n-2} a_0) \\
&\leq \kappa^3 d(H^{n-2} a_0, H^{n-3} a_0) \\
&\;\;\vdots \\
d(H^{n+1}a_0, H^n a_0) &\leq \kappa^n d(Ha_0, a_0),
\end{aligned}$$

for every $n \geq 1$. For $n, q \geq 1$, using repeatedly Definition 2.3 (iv) we get

$$\begin{aligned} d(H^{n+q}a_0, H^n a_0) &\leq sd(H^n a_0, H^{n+1}a_0) + sd(H^{n+1}a_0, H^{n+q}a_0) \\ &\leq sd(H^n a_0, H^{n+1}a_0) + s^2 d(H^{n+1}a_0, H^{n+2}a_0) \\ &\quad + s^2 d(H^{n+2}a_0, H^{n+q}a_0) \\ &\leq sd(H^n a_0, H^{n+1}a_0) + s^2 d(H^{n+1}a_0, H^{n+2}a_0) \\ &\quad + s^3 d(H^{n+2}a_0, H^{n+3}a_0) + s^3 d(H^{n+3}a_0, H^{n+q}a_0). \end{aligned}$$

Using induction over q we prove that

$$(2) d(H^{n+q}a_0, H^n a_0) \leq \sum_{j=1}^{q} s^j d(H^{n+j-1}a_0, H^{n+j}a_0) \text{ for all } n, q \geq 1.$$

Note that

$$d(H^{n+q+1}a_0, H^n a_0) \leq sd(H^n a_0, H^{n+1}a_0) + sd(H^{n+1}a_0, H^{n+1+q}a_0).$$

By the inductive hypothesis,

$$d(H^{n+1}a_0, H^{n+1+q}a_0) \leq \sum_{j=1}^{q} s^j d(H^{n+j}a_0, H^{n+1+j}a_0), \text{ for all } n.$$

The last two inequalities imply

$$\begin{aligned} d(H^{n+q+1}a_0, H^n a_0) &\leq sd(H^n a_0, H^{n+1}a_0) + \sum_{j=1}^{q} s^{j+1} d(H^{n+j}a_0, H^{n+1+j}a_0) \\ &= \sum_{i=1}^{q+1} s^i d(H^{n+i-1}a_0, H^{n+i}a_0). \end{aligned}$$

Since

$$d(H^{n+j-1}a_0, H^{n+j}a_0) \leq k^{n+j-1} d(Ha_0, a_0),$$

inequality (2) implies

$$\begin{aligned} d(H^{n+q}a_0, H^n a_0) &\leq \left[k^{n-1} \sum_{j=1}^{q} (sk)^j\right] d(Ha_0, a_0) \\ &\leq \left[k^{n-1} \sum_{j=1}^{\infty} (sk)^j\right] d(Ha_0, a_0) \\ &= \frac{sk^n}{1 - sk} d(Ha_0, a_0). \end{aligned}$$

Let $\theta \ll c$ be given. Notice that

$$\frac{sk^n}{1-sk}\, d(Ha_0, a_0) \to \theta \, \text{as} \, n \to \infty.$$

By using Lemma 2.6, we can find $n_0 \in \mathbb{N}$ such that

$$\frac{sk^n}{1-sk}\, d(Ha_0, a_0) \ll c,$$

for all $n > n_0$. Thus

$$d(H^{n+q}a_0, H^n a_0) \leq \frac{sk^n}{1-sk}\, d(Ha_0, a_0) \ll c,$$

for all $n > n_0$ and for any q. So, by Lemma 2.8 and Definition 2.5 (ii) $\{H^n a_0\}$ is a Cauchy sequence. As (Y, d) is a complete cone b-metric space, there exists $a^* \in Y$ such that $H^n a_0 \to a^*$ as $n \to \infty$. Finally as H is continuous, this implies that a^* is a fixed point of H, since $Ha^* = H\big(\lim_{n\to\infty} H^n a_0\big) = \lim_{n\to\infty} H^{n+1} a_0 = a^*$. The proof is completed.

Remark 3.2. *Theorem* 3.1 *generalizes Theorem* 1.4 [2] *and Theorem* 2.1 [9]. *The proof of Theorem 3.1 follows the lines from* [9]. *The contractive condition from the statement of Theorem 2.1 is assumed to hold for all $a, b \in Y$, while here we used a restricted case $a \succeq b$.*

Example 3.3. *Assume that $Y = [0,1]$ endowed with the standard order and $E = \mathbb{R}^2$, $P = \{(a,b) : a, b \geq 0\}$. Define $d : Y \times Y \to E$ as in Example 2.4, with $\alpha = 1$, $p = 3$ and $H : Y \to Y$ by $H(a) = \frac{a^2}{8}$. Then the mapping H is nondecreasing and continuous with respect to $\preceq$. We have*

$$\begin{aligned} d\,(Ha, Hb) &= d\left(\frac{a^2}{8}, \frac{b^2}{8}\right) \\ &= \left(\left|\frac{a^2}{8} - \frac{b^2}{8}\right|^3, \left|\frac{a^2}{8} - \frac{b^2}{8}\right|^3\right) \\ &= \frac{1}{512}\,|a+b|^3(|a-b|^3, |a-b|^3) \\ &\leq \frac{8}{512}\,(|a-b|^3, |a-b|^3) \\ &\leq \frac{1}{64}\, d(a,b), \end{aligned}$$

where $\kappa = \frac{1}{64} \in [0,1)$*. We can see clearly that conditions of Theorem 3.1 are satisfied. Therefore,* H *has a fixed point* $a = 0$*.*

Theorem 3.4. *Let* $(Y, \preceq)$ *be a partially ordered set. Suppose that there exists a cone b-metric* $d : Y \times Y \to E$ *such that the cone b-metric space* (Y, d) *is complete with the coefficient* $s \geq 1$ *relative to a solid cone* P *in* E*. Let* $H : Y \to Y$ *be a continuous and nondecreasing mapping with respect to* $\preceq$*. Assume that the following two conditions hold:*
(i) there exists $\kappa_j \in [0,1)$*,* $j = 1,2,3,4$*, such that* $(\kappa_1 + \kappa_2) + s(\kappa_3 + \kappa_4) < \frac{2}{s+1}$ *with* $\sum_{j=1}^{4} \kappa_j < 1$ *and*

$$d(Ha, Hb) \leq \kappa_1 d(a, Ha) + \kappa_2 d(b, Hb) + \kappa_3 d(a, Hb) + \kappa_4 d(b, Ha)$$

for all $a, b \in Y$ *with* $a \succeq b$*,*
(ii) there exists $a_0 \in Y$ *such that* $a_0 \preceq Ha_0$*.*
Then H *has a fixed point* a^* *in* Y*, which is the limit of the sequence* $(H^n a_0)$*.*

Proof. If $Ha_0 = a_0$, then the proof is finished. Assume that $Ha_0 \neq a_0$. Since $a_0 \preceq Ha_0$ and H is nondecreasing with respect to $\preceq$, we get by induction that

$$a_0 \preceq Ha_0 \preceq H^2 a_0 \preceq \cdots \preceq H^n a_0 \preceq H^{n+1} a_0 \preceq \dots. \tag{3}$$

Denote $a_n := H^n a_0$ for $n \in \mathbb{N}$. Then

$$\begin{aligned} d(a_{n+1}, a_n) &= d\big(H(H^n a_0), H(H^{n-1} a_0)\big) \\ &\leq \kappa_1 d(H^n a_0, H^{n+1} a_0) + \kappa_2 d(H^{n-1} a_0, H^n a_0) \\ &\quad + \kappa_3 d(H^n a_0, H^n a_0) + \kappa_4 d(H^{n-1} a_0, H^{n+1} a_0) \\ &= \kappa_1 d(a_n, a_{n+1}) + \kappa_2 d(a_{n-1}, a_n) \\ &\quad + \kappa_3 d(a_n, a_n) + \kappa_4 d(a_{n-1}, a_{n+1}) \\ &\leq \kappa_1 d(a_n, a_{n+1}) + \kappa_2 d(a_{n-1}, a_n) \\ &\quad + s\kappa_4 [d(a_{n-1}, a_n) + d(a_n, a_{n+1})] \\ &= (\kappa_1 + s\kappa_4) d(a_n, a_{n+1}) + (\kappa_2 + s\kappa_4) d(a_{n-1}, a_n). \end{aligned}$$

This implies that

$$(1 - \kappa_1 - s\kappa_4) d(a_{n+1}, a_n) \leq (\kappa_2 + s\kappa_4) d(a_{n-1}, a_n). \tag{4}$$

Secondly, we have

$$\begin{aligned}
d(a_{n+1}, a_n) &= d(H^{n+1}a_0, H^n a_0) \\
&\leq \kappa_1 d(H^{n-1}a_0, H^n a_0) + \kappa_2 d(H^n a_0, H^{n+1}a_0) \\
&\quad + \kappa_3 d(H^{n-1}a_0, H^{n+1}a_0) + \kappa_4 d(H^n a_0, H^n a_0) \\
&\leq \kappa_1 d(a_{n-1}, a_n) + \kappa_2 d(a_n, a_{n+1}) \\
&\quad + \kappa_3 d(a_{n-1}, a_{n-1}) + \kappa_4 d(a_n, a_n) \\
&\leq \kappa_1 d(a_{n-1}, a_n) + \kappa_2 d(a_n, a_{n+1}) \\
&\quad + s\kappa_3[d(a_{n-1}, a_n) + d(a_n, a_{n+1})] \\
&= (\kappa_2 + s\kappa_3) d(a_n, a_{n+1}) + (\kappa_1 + s\kappa_3) d(a_{n-1}, a_n).
\end{aligned}$$

It follows that

$$(1 - \kappa_2 - s\kappa_3) d(a_{n+1}, a_n) \leq (\kappa_1 + s\kappa_3) d(a_{n-1}, a_n). \tag{5}$$

Adding (4) and (5) we get,

$$d(a_n, a_{n+1}) \leq \frac{\kappa_1 + \kappa_2 + s(\kappa_3 + \kappa_4)}{2 - \kappa_1 - \kappa_2 - s(\kappa_3 + \kappa_4)} d(a_n, a_{n-1}).$$

Now put $\kappa = \frac{\kappa_1+\kappa_2+s(\kappa_3+\kappa_4)}{2-\kappa_1-\kappa_2-s(\kappa_3+\kappa_4)}$, we can see easily that $\kappa \in [0, 1)$. Thus,

$$d(a_{n+1}, a_n) \leq \kappa d(a_n, a_{n-1}) \leq \cdots \leq \kappa^n d(a_1, a_0).$$

Following the argument from Theorem 3.1, we see that $\{a_n\}$ is a Cauchy sequence. Due to completeness, there exists $a^* \in Y$ such that $a_n \to a^*$. Since H is continuous. Thus

$$\begin{aligned}
Ha^* &= H\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} H(a_n) \\
&= \lim_{n\to\infty} H(H^n a_0) = \lim_{n\to\infty} H^{n+1}a_0 \\
&= \lim_{n\to\infty} a_{n+1} = a^*.
\end{aligned}$$

Therefore, a^* is a fixed point of H.

Remark 3.5. *Theorem* 3.2 *is similar to Theorem* 2.3 *from* [9] *Huang and Xu, where the continuity of the mapping was not imposed but the contractive condition was assumed to hold for all elements in the cone b-metric space.*

Now we give the following result by removing the continuity of H in Theorem 3.4.

Theorem 3.6. *Assume that the hypothesis of Theorem 3.4 are satisfied, except the continuity of H. In addition, assume that Y satisfies the following condition: "If a nondecreasing sequence (y_n) in Y converges to some $y \in Y$, then $y_n \preceq y$ for all n". Then H has a fixed point in Y, which is* $\lim_{n\to\infty} H^n a_0$.

Proof. Proceeding as in the proof of inequalities (2.3) and (2.4) from [9], we get for all $n \geq 1$.

$$(2-s\kappa_1-s\kappa_2-s^2\kappa_3-s^2\kappa_4)\, d(a^*, Ha^*) \leq s^2 d(a_n, a^*)+(s^2+2s)d(a_{n+1}, a^*).$$

Denote $\lambda := 2-s\kappa_1-s\kappa_2-s^2\kappa_3-s^2\kappa_4$. Since $(\kappa_1+\kappa_2)+s(\kappa_3+\kappa_4) < \frac{2}{s}$, we have $\lambda > 0$. Then

$$d(a^*, Ha^*) \leq \frac{1}{\lambda}\Big[s^2 d(a_n, a^*) + (s^2 + 2s)d(a_{n+1}, a^*)\Big]$$

for all $n \geq 1$. Let $c \gg \theta$. Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = a^*$ in Y, there exists $N \geq 1$ such that

$$d(a_n, a^*) \ll \frac{\lambda}{2s^2 + 2s}\, c,$$

for all $n \geq N$. The last two inequalities imply $d(a^*, Ha^*) \ll c$, by Lemma 2.8. Since $c \gg \theta$, using Lemma 2.6 we get $Ha^* = a^*$, therefore a^* is a fixed point of H. The proof is completed.

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