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A GENERAL FIXED POINT THEOREM FOR SELF MAPPINGS IN GP - METRIC SPACES

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Abstract. The purpose of this paper is to prove a general fixed point theorem in GP - metric spaces for mappings satisfying an implicit relation, which generalizes and improves Theorem 2.10 [6]. In the last part of the paper we prove that these mappings satisfy property (P) in GP - metric spaces and if GP - metric is symmetric, then the fixed point problems is well posed.

1. Introduction and Preliminaries

In [13], [14], Dhage introduced a new class of generalized metric spaces, named D - metric spaces. Mustafa and Sims [22], [23] proved that most of the claims concerning the fundamental structures on D - metric spaces are incorrect and introduced an appropriate notion of generalized metric space, named G - metric spaces. In fact, Mustafa, Sims and other authors [10], [18], [21], [26], [27], [28], [29], [38], [39] studied many fixed point results for self mappings in G - metric spaces under certain conditions.

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In 1994, Mathews [20] introduced the concept of partial metric space as a part of study of denotional semantics of dataflows and proved the Banach contraction principle in such spaces. Recently, in [1], [5], [9], [16], [17] and in other papers, some fixed point theorems under various contractive conditions in complete partial metric spaces are proved.

Quite recently, Zand and Nezhad introduced in [41] a generalization and unification of G - metric space and partial metric space, named GP - metric space. In [6], first, some fixed point theorems in GP - metric spaces are proved. Other results are obtained in [8] and [7].

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [31], [32] and in other papers. Recently, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, ultra - metric spaces, convex metric spaces, reflexive spaces, compact metric spaces, paracompact metric spaces, in two and three metric spaces, for single - valued mappings, hybrid pairs of mappings and set - valued mappings. Recently, the method is used in the study of fixed points for mappings satisfying contractive/extensive conditions of integral type, in fuzzy metric spaces, probabilistic metric spaces and intuitionistic metric spaces. Also, the method allows the study of local and global properties of fixed point structures.

The study of fixed points for mappings in G - metric spaces for mappings satisfying an implicit relation is initiated in [33], [34], [35].

The study of fixed point for mappings satisfying an implicit relation in partial metric spaces is initiated in [40].

Let T be a self mapping of a metric space (X, d) with nonempty fixed points set $F(T)$. Then T is said to satisfy property (P) is $F(T) = F(T^n)$ for each $n \in \mathbb{N}$.

An interesting fact about mappings satisfying property (P) is that they haven't trivial periodic points. Papers dealing with property (P) are [14], [15], [37] and other papers.

The notion of well posedness of fixed point problem has generates more interest to several mathematicians, for example [11], [19], [36].

In [2], [3], [4] and in other papers the authors studied well posedness of fixed point problem for mappings satisfying implicit relations.

The purpose of this paper is to prove a general fixed point theorems on GP - metric spaces for mappings satisfying an implicit relation, which generalizes and improves Theorem 2.10 [6]. In the last part of this paper we prove that these mappings satisfy property (P) and if the GP - metric is symmetric then the fixed point problem is well posed.

2. Preliminaries

Definition 2.1 ([41], [30]). Let X be a nonempty set. A function $G : X^3 \to [0, \infty)$ is called a GP - metric on X if the following conditions are satisfied:

(GP_1) : $x = y = z$ if $GP(x, y, z) = GP(x, x, x) = GP(y, y, y) = GP(z, z, z)$,

(GP_2) : $0 \leq GP(x, x, x) \leq GP(x, x, y) \leq GP(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$,

(GP_3) : $GP(x, y, z) = GP(y, z, x) = ...$ (symmetry in all three variables),

(GP_4) : $GP(x, y, z) \leq GP(x, a, a) + GP(a, y, z) - GP(a, a, a)$ for all $x, y, z, a \in X$.

The pair (X, GP) is called a GP - metric space.

Definition 2.2 ([41]). Let (X, GP) be a GP - metric space and $\{x_n\}$ a sequence in X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ or $x_n \to x$ if $\lim_{m,n\to\infty} GP(x, x_n, x_m) = GP(x, x, x)$.

Theorem 2.3 ([6]). *Let (X, GP) be a GP - metric space. Then, for any $\{x_n\} \in X$ and $x \in X$, the following conditions are equivalent:*

a) $\{x_n\}$ is GP - convergent to x,

b) $GP(x_n, x_n, x) \to GP(x, x, x)$ as $n \to \infty$,

c) $GP(x_n, x, x) \to GP(x, x, x)$ as $n \to \infty$.

Definition 2.4 ([6]). Let (X, GP) be a GP - metric space.

1) A sequence $\{x_n\}$ of X is called a 0 - GP - Cauchy sequence if and only if $\lim_{n,m\to\infty} GP(x_n, x_m, x_m) = 0$,

2) A GP - metric space is said to be 0 - GP - complete if and only if every 0 - GP - Cauchy sequence in X GP - converges to a point $x \in X$ such that $GP(x, x, x) = 0$.

Lemma 2.5 ([6]). *Let (X, GP) be a GP - metric space. Then:*

1) If $GP(x, y, z) = 0$ then $x = y = z$,

2) If $x \neq y$ then $GP(y, x, x) > 0$.

Definition 2.6 ([41]). A GP - metric on X is said to be symmetric if $GP(x, y, y) = GP(y, x, x)$.

In this case (X, GP) is said to be symmetric.

Lemma 2.7 ([6]). *Let (X, GP) be a GP - metric space and $\{x_n\}$ a sequence in X. Assume that $\{x_n\}$ is GP - convergent to a point $x \in X$*

with $GP(x,x,x)=0$. *Then* $\lim_{n\to\infty} GP(x_n,y,y)=GP(x,y,y)$ *for all* $y\in X$.

Moreover, $\lim_{n,m\to\infty} G_p(x_n,x_m,x)=0$.

The following theorem is proved in [6].

Theorem 2.8 (Theorem 2.10 [6])**.** *Let* (X,GP) *a* 0 - GP - *complete metric space and* $f:X\to X$ *a mapping on* X. *Assume that*

$$\frac{1}{3}GP\left(x,fx,fx\right)<GP\left(x,y,y\right) \tag{2.1}$$

implies

$$GP\left(fx,fy,fy\right)\leq \alpha GP\left(x,y,y\right)+\beta GP\left(x,fx,fx\right)+\gamma GP\left(y,fy,fy\right)$$

for all $x,y\in X$, *where* $\alpha,\beta,\gamma\geq 0$ *and* $\alpha+\beta+\gamma<1$. *Then* f *has a unique fixed point.*

3. Implicit relations

Definition 3.1. Let $\mathfrak{F}_{GP}$ be the set of all continuous functions $F(t_1,...,t_6):\mathbb{R}_+^6\to\mathbb{R}$ satisfying

(F_1) : F is nonincreasing in variables t_3,t_4,t_5,t_6,

(F_2) : There exists $h_1\in[0,1)$ such that for all $u,v\geq 0$, $F(u,v,v,u,u+v,v)\leq 0$ implies $u\leq h_1 v$,

(F_3) : There exists $h_2\in[0,1)$ such that for all $t,t'>0$, $F(t,t,t',t,t,t')\leq 0$ implies $t\leq h_2 t'$.

In the following examples, property (F_1) is obviously.

Example 3.2. $F(t_1,...,t_6)=t_1-at_2-bt_3-ct_4-dt_5-et_6$, *where* $a,b,c,d,e\geq 0$ *and* $0<a+b+c+2d+e<1$.

(F_2) : *Let* $u,v\geq 0$ *and* $F(u,v,v,u,u+v,v)=u-av-bv-cu-d(u+v)-ev\leq 0$. *Then* $u\leq h_1 v$, *where* $0\leq h_1=\frac{a+b+d+e}{1-(c+d)}<1$.

(F_3) : *Let* $t,t'>0$ *and* $F(t,t,t',t,t,t')=t-at-bt'-ct-dt-et'\leq 0$. *Then* $t\leq h_2 t'$, *where* $0<h_2=\frac{b+e}{1-(a+c+d)}<1$.

Example 3.3. $F(t_1,...,t_6)=t_1-k\max\{t_2,t_3,t_4,t_5,t_6\}$, *where* $k\in\left[0,\frac{1}{2}\right)$.

(F_2) : *Let* $u,v\geq 0$ *be and* $F(u,v,v,u,u+v,v)=u-k(u+v)\leq 0$, *which implies* $u\leq h_1 v$, *where* $0\leq h_1=k<1$.

(F_3) : *Let* $t,t'>0$ *be and* $F(t,t,t',t,t,t')=t-k\max\{t,t'\}\leq 0$. *If* $t>t'$, *then* $t(1-k)\leq 0$, *a contradiction. Hence,* $t\leq t'$, *which implies* $t\leq h_2 t'$, *where* $0<h_2=k<1$.

Example 3.4. *$F(t_1,...,t_6) = t_1 - k\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}$, where $k \in [0,1)$.*

(F_2) : Let $u,v \geq 0$ be and $F(u,v,v,u,u+v,v) = u - k\max\left\{u,v,\frac{u+v}{2}\right\} \leq 0$. If $u > v$, then $u(1-k) \leq 0$, a contradiction. Hence $u \leq v$ which implies $u \leq h_1 v$, where $0 \leq h_1 = k < 1$.

(F_3) : Let $t,t' > 0$ be and $F(t,t,t',t,t,t') = t - k\max\left\{t,t',\frac{t+t'}{2}\right\} \leq 0$, which implies $t \leq h_2 t'$, where $0 < h_2 = k < 1$.

Example 3.5. *$F(t_1,...,t_6) = t_1 - at_2 - bt_3 - c\max\{2t_4, t_5+t_6\}$, where $a,b,c \geq 0$ and $0 \leq a+b+3c < 1$.*

(F_2) : Let $u,v \geq 0$ be and $F(u,v,v,u,u+v,v) = u - av - bv - c\max\{2u, u+2v\} \leq 0$. If $u > v$, then $u[1-(a+b+3c)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq h_1 v$, where $0 \leq h_1 = a+b+3c < 1$.

(F_3) : Let $t,t' > 0$ be and $F(t,t,t',t,t,t') = t - at - bt' - c\max\{2t, t+t'\} \leq 0$. If $t > t'$ then $t[1-(a+b+3c)] \leq 0$, a contradiction. Hence $t \leq t'$, which implies $t \leq h_2 t'$, where $0 < h_2 = a+b+3c < 1$.

Example 3.6. *$F(t_1,...,t_6) = t_1 - at_2 - bt_3 - c\max\{t_4+t_5, 2t_6\}$, where $a,b,c \geq 0$ and $0 \leq 2a+b+3c < 1$.*

The proof is similar to the proof of Example 3.5.

Example 3.7. *$F(t_1,...,t_6) = t_1 - k\max\left\{t_2,t_3,t_4,\frac{2t_4+t_6}{3},\frac{2t_4+t_3}{3},\frac{t_5+t_6}{3}\right\}$, where $k \in [0,1)$.*

(F_2) : Let $u,v \geq 0$ be and $F(u,v,v,u,u+v,v) = u - k\max\left\{u,v,\frac{2u+v}{3},\frac{u+2v}{3}\right\} \leq 0$. If $u > v$, then $u(1-k) \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq h_1 v$, where $0 \leq h_1 = k < 1$.

(F_3) : Let $t,t' > 0$ be and $F(t,t,t',t,t,t') = t - k\max\left\{\frac{2t+t'}{3},\frac{t+t'}{3},t,t'\right\} \leq 0$. If $t > t'$ then $t(1-k) \leq 0$, a contradiction. Hence $t \leq t'$, which implies $t \leq h_2 t'$, where $0 < h_2 = k < 1$.

Example 3.8. *$F(t_1,...,t_6) = t_1 - at_2 - k\max\{t_3+2t_4, t_4+t_5+t_6\}$, where $a,k \geq 0$ and $0 \leq a+4k < 1$.*

(F_2) : Let $u,v \geq 0$ be and $F(u,v,v,u,u+v,v) = u - av - k\max\{2u+v, 2u+2v\} = u - av - k(2u+2v) \leq 0$, which implies $u \leq h_1 v$, where $0 \leq h_1 = a+4k < 1$.

(F_3) : Let $t,t' > 0$ be and $F(t,t,t',t,t,t') = t - at - k(2t+t') \leq 0$, which implies $t \leq h_2 t'$, where $0 < h_2 = \frac{k}{1-2k} < 1$.

Example 3.9. *$F(t_1,...,t_6) = t_1 - at_2 - bt_3 - c\max\{2t_4+t_3, t_1+t_4+t_5+t_6\}$, where $a,b,c \geq 0$ and $0 \leq a+b+5c < 1$.*

(F_2) : *Let* $u, v \geq 0$ *be and* $F(u, v, v, u, u + v, v) = u - av - bv - c\max\{2u + v, 3u + 2v\} \leq 0$, *which implies* $u \leq h_1 v$, *where* $0 \leq h_1 = \frac{a+b+2c}{1-3c} < 1$.

(F_3) : *Let* $t, t' > 0$ *be and* $F(t, t, t', t, t, t') = t - at - bt - c(3t + t') \leq 0$, *which implies* $t \leq h_2 t'$, *where* $0 < h_2 = \frac{b+c}{1-(a+3c)} < 1$.

Example 3.10. $F(t_1, ..., t_6) = t_1 - \max\{at_2, b(t_3 + 2t_4), b(t_4 + t_5 + t_6)\}$, *where* $a \in (0, 1)$ *and* $b \in \left(0, \frac{1}{4}\right)$.

(F_2) : *Let* $u, v \geq 0$ *be and* $F(u, v, v, u, u + v, v) = u - \max\{av, b(v + 2u), b(2u + 2v)\} \leq 0$. *If* $u > v$ *then* $u(1 - \max\{a, 4b\}) \leq 0$, *a contradiction. Hence* $u \leq b$ *which implies* $u \leq h_1 v$, *where* $0 \leq h_1 = \max\{a, 4b\} < 1$.

(F_3) : *Let* $t, t' > 0$ *be and* $F(t, t, t', t, t, t') = t - \max\{at, b(t' + 2t)\} \leq 0$. *If* $t > t'$, *then* $t(1 - \max\{a, 3b\}) \leq 0$, *a contradiction. Hence* $t \leq t'$ *which implies* $t \leq h_2 t'$, *where* $0 < h_2 = \max\{a, 3b\} < 1$.

Example 3.11. $F(t_1, ..., t_6) = t_1 - k\max\{t_2, t_3 + t_4, t_5 + t_6\}$, *where* $k \in \left[0, \frac{1}{3}\right)$.

(F_2) : *Let* $u, v \geq 0$ *be and* $F(u, v, v, u, u + v, v) = u - k\max\{v, u + v, u + 2v\} = u - k(u + 2v) \leq 0$, *which implies* $u \leq h_1 v$, *where* $0 \leq h_1 = \frac{2k}{1-k} < 1$.

(F_3) : *Let* $t, t' > 0$ *be and* $F(t, t, t', t, t, t') = t - \max\{t, t + t'\} = t - k(t + t') \leq 0$, *which implies* $t \leq h_2 t'$, *where* $0 < h_2 = \frac{k}{1-k} < 1$.

4. Main results

Theorem 4.1. *Let* (X, GP) *be a* GP *- metric space and let* $T : X \to X$ *such that:*

$$\begin{aligned} &F(GP(Tx, Ty, Ty), GP(x, y, y), GP(x, Tx, Tx), \\ &GP(y, Ty, Ty), GP(x, Ty, Ty), GP(y, Tx, Tx)) \leq 0 \end{aligned} \tag{4.1}$$

for all $x, y \in X$, *where* F *satisfy property* (F_3). *Then,* T *has at most a fixed point.*

Proof. Suppose that T has two distinct fixed points u and v. Then, by (4.1) we have successively

$$\begin{aligned} &F(GP(Tu, Tv, Tv), GP(u, v, v), GP(u, Tu, Tu), \\ &GP(v, Tv, Tv), GP(u, Tv, Tv), GP(v, Tu, Tu)) \leq 0, \end{aligned}$$

$$\begin{aligned} &F(GP(u, v, v), GP(u, v, v), GP(u, u, u), \\ &GP(v, v, v), GP(u, v, v), GP(v, u, u)) \leq 0. \end{aligned}$$

By (GP_2),

$$GP(u, u, u) \leq GP(v, u, u)$$

and

$$GP(v,v,v) \leq GP(u,v,v).$$

By (F_1) we obtain

$$F(GP(u,v,v), GP(u,v,v), GP(v,u,u),$$
$$GP(u,v,v), GP(u,v,v), GP(v,u,u)) \leq 0.$$

By (F_3) we have

$$GP(u,v,v) \leq h_2 GP(v,u,u).$$

Similarly, we obtain

$$GP(v,u,u) \leq h_2 GP(u,v,v).$$

Hence

$$GP(u,v,v)(1-h_2^2) \leq 0,$$

a contradiction.

Therefore, $u = v$. □

Theorem 4.2. *Let (X, GP) be a 0 - GP - complete metric space and let $T : X \to X$ satisfying inequality (4.1), for all $x, y \in X$ and $F \in \mathfrak{F}_{GP}$. Then, T has a unique fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point of X. We define $x_n = Tx_{n-1}, n = 1, 2, \ldots$. Then by (4.1) we have successively

$$F(GP(Tx_{n-1}, Tx_n, Tx_n), GP(x_{n-1}, x_n, x_n), GP(x_{n-1}, Tx_{n-1}, Tx_{n-1}),$$
$$GP(x_n, Tx_n, Tx_n), GP(x_{n-1}, Tx_n, Tx_n), GP(x_n, Tx_{n-1}, Tx_{n-1})) \leq 0,$$

$$F(GP(x_n, x_{n+1}, x_{n+1}), GP(x_{n-1}, x_n, x_n), GP(x_{n-1}, x_n, x_n),$$
$$GP(x_n, x_{n+1}, x_{n+1}), GP(x_{n-1}, x_{n+1}, x_{n+1}), GP(x_n, x_n, x_n)) \leq 0.$$

By (GP_4)

$$GP(x_{n-1}, x_{n+1}, x_{n+1}) \leq GP(x_{n-1}, x_n, x_n) + GP(x_n, x_{n+1}, x_{n+1})$$

and by (GP_2)

$$GP(x_n, x_n, x_n) \leq GP(x_{n-1}, x_n, x_n).$$

By (F_1) we obtain

$$F(GP(x_n, x_{n+1}, x_{n+1}), GP(x_{n-1}, x_n, x_n), GP(x_{n-1}, x_n, x_n), GP(x_n, x_{n+1}, x_{n+1}),$$
$$GP(x_{n-1}, x_n, x_n) + GP(x_n, x_{n+1}, x_{n+1}), GP(x_{n-1}, x_n, x_n)) \leq 0,$$

which implies by (F_2) that

$$GP(x_n, x_{n+1}, x_{n+1}) \leq h_1 GP(x_{n-1}, x_n, x_n)$$

for $n = 1, 2, ...$ Then,
(4.2)
$$GP(x_n, x_{n+1}, x_{n+1}) \leq h_1 GP(x_{n-1}, x_n, x_n) \leq ... \leq h_1^n GP(x_0, x_1, x_1).$$

By (4.2) and (GP_4) we obtain for $m > n$ that

$$\begin{aligned} GP(x_n, x_m, x_m) &\leq GP(x_n, x_{n+1}, x_{n+1}) + GP(x_{n+1}, x_{n+2}, x_{n+2}) + \\ &+ ... + GP(x_{m-1}, x_m, x_m) \\ &\leq h_1^n(1 + h_1 + ... + h_1^{m-1}) GP(x_0, x_1, x_1) \\ &\leq \frac{h_1^n}{1 - h_1} GP(x_0, x_1, x_1). \end{aligned}$$

It implies that,

$$\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0.$$

That is $\{x_n\}$ is a 0 - GP - Cauchy sequence. Since X is 0 - GP - complete, $\{x_n\}$ converges to some point z in X with $GP(z, z, z) = 0$. Then

$$\lim_{n\to\infty} GP(x_n, z, z) = \lim_{n\to\infty} GP(z, x_n, x_n) = GP(z, z, z) = 0. \tag{4.3}$$

By (4.1) we obtain successively

$$\begin{gathered} F(GP(Tx_n, Tz, Tz), GP(x_n, z, z), GP(x_n, Tx_n, Tx_n), \\ GP(z, Tz, Tz), GP(x_n, Tz, Tz), GP(z, Tx_n, Tx_n)) \leq 0, \end{gathered}$$

$$\begin{gathered} F(GP(x_{n+1}, Tz, Tz), GP(x_n, z, z), GP(x_n, x_{n+1}, x_{n+1}), \\ GP(z, Tz, Tz), GP(x_n, Tz, Tz), GP(z, x_{n+1}, x_{n+1})) \leq 0. \end{gathered}$$

By Lemma 2.7, (4.2) and (4.3), letting n tends to infinity we obtain

$$F(GP(z, Tz, Tz), 0, 0, GP(z, Tz, Tz), GP(z, Tz, Tz), 0) \leq 0.$$

By (F_2) we obtain $GP(z, Tz, Tz) = 0$. By Lemma 2.5 (a), we obtain $z = Tz$. Hence T has a fixed point. By Theorem 4.1, z is the unique fixed point of T. □

Corollary 4.3. *Let* (X, GP) *be a* 0 - GP - *complete metric space and* $T : X \to X$ *such that*

$$\begin{gathered} GP(Tx, Ty, Ty) \leq aGP(x, y, y) + bGP(x, Tx, Tx) + \\ + cGP(y, Ty, Ty) + dGP(x, Ty, Ty) + eGP(y, Tx, Tx)\}, \end{gathered} \tag{4.4}$$

where $a, b, c, d, e \geq 0$ *and* $0 < a + b + c + 2d + e < 1$, *for all* $x, y \in X$. *Then* T *has a unique fixed point.*

Proof. The proof it follows by Theorem 4.2 and Example 3.2. □

Remark 4.4. *1) If in Example 3.2, $d = e = 0$, then by Corollary 4.3 we obtain a new form of Theorem 2.8, without the condition (2.1).*

2) By Theorem 4.1 and Examples 3.3 - 3.11.we obtain new particular results, which generalize some results from G - metric spaces.

5. Property P in GP - metric spaces

Theorem 5.1. *Under the conditions of Theorem 4.1, T has property P.*

Proof. From Theorem 4.2, T has an unique fixed point, therefore $F(T^n) \neq \emptyset$ for each $n \in \mathbb{N}$. Fix $n > 1$ and assume that $q \in F(T)$. Using (4.1) we have

$$F(GP\left(T^nq, T^{n+1}q, T^{n+1}q\right), GP\left(T^{n-1}q, T^nq, T^nq\right), GP\left(T^{n-1}q, T^nq, T^nq\right),$$
$$GP\left(T^nq, T^{n+1}q, T^{n+1}q\right), GP\left(T^{n-1}q, T^{n+1}q, T^{n+1}q\right), GP\left(T^nq, T^nq, T^nq\right)) \leq 0.$$

By (GP_4),

$$GP(T^{n-1}q, T^{n+1}q, T^{n+1}q) \leq GP(T^{n-1}q, T^nq, T^nq) + GP(T^nq, T^{n+1}q, T^{n+1}q)$$

and by (GP_2)

$$GP(T^nq, T^nq, T^nq) \leq GP(T^{n-1}q, T^nq, T^nq).$$

By (F_1) we obtain

$$F(GP(T^nq, T^{n+1}q, T^{n+1}q), GP(T^{n-1}q, T^nq, T^nq),$$
$$GP(T^{n-1}q, T^nq, T^nq), GP(T^nq, T^{n+1}q, T^{n+1}q),$$
$$GP(T^{n-1}q, T^nq, T^nq) + GP(T^nq, T^{n+1}q, T^{n+1}q),$$
$$GP(T^{n-1}q, T^nq, T^nq)) \leq 0.$$

By (F_2) we obtain

$$GP(T^nq, T^{n+1}q, T^{n+1}q) \leq h_1 GP(T^{n-1}q, T^nq, T^nq) \leq ... \leq h_1^n GP(q, Tq, Tq).$$

Since $q \in F(T^n)$, then

$$GP(q, Tq, Tq) = GP(T^nq, T^{n+1}q, T^{n+1}q).$$

Therefore

$$GP(q, Tq, Tq) \leq h_1^n GP(q, Tq, Tq),$$

which implies $GP(q, Tq, Tq) = 0$. By Lemma 2.5 (a), $q = Tq$ and T has property P. □

6. Well posedness problem of fixed point in GP - metric spaces

Definition 6.1 ([36])**.** Let (X, d) be a metric space and $f : (X, d) \to (X, d)$ be a mapping. The fixed point problem of f is said to be well posed if:

1) f has a unique fixed point x_0,

2) for any sequence $\{x_n\} \in X$ with $\lim_{n\to\infty} d(x_n, fx_n) = 0$ we have $\lim_{n\to\infty} d(x_n, x_0) = 0$.

Definition 6.2. Let (X, GP) be a GP - metric space and let $T : X \to X$ be a self mapping. The fixed point problem of T is said to be well posed if:

1) T has a unique fixed point x_0,

2) for any sequence $\{x_n\} \in X$ with $\lim_{n\to\infty} GP(x_n, Tx_n, Tx_n) = 0$ we have $\lim_{n\to\infty} GP(x_0, x_n, x_n) = 0$.

Definition 6.3. A function $F : \mathbb{R}_+^6 \to \mathbb{R}$ has property (F_p) if for all $u, v, w \geq 0$ and $F(u, v, 0, w, u, v) \leq 0$, there exists $p \in (0, 1)$ such that $u \leq p \max\{v, w\}$.

Example 6.4. *$F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, where $a, b, c, d, e \geq 0$ and $a + b + c + d + e < 1$.*

(F_p) : Let $u, v, w \geq 0$ be such that $F(u, v, 0, w, u, v) = u - av - cw - du - ev \leq 0$. If $u > \max\{v, w\}$, then $u[1 - (a + c + d + e)] \leq 0$, a contradiction. Hence $u \leq \max\{v, w\}$, which implies $u \leq p \max\{v, w\}$, where $0 < p = a + c + d + e < 1$.

Example 6.5. *$F(t_1, ..., t_6) = t_1 - k \max\{t_2, ..., t_6\}$, where $k \in \left[0, \frac{1}{2}\right)$.*

(F_p) : Let $u, v, w \geq 0$ be such that $F(u, v, 0, w, u, v) = u - k \max\{v, w\} \leq 0$. If $u > \max\{v, w\}$, then $u (1 - k) \leq 0$, a contradiction. Hence $u \leq \max\{v, w\}$, which implies $u \leq p \max\{v, w\}$, where $0 < p = k < 1$.

Example 6.6. *$F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\}$, where $k \in [0, 1)$..*

(F_p) : Let $u, v, w \geq 0$ be such that $F(u, v, 0, w, u, v) = u - k \max\{u, v, w\} \leq 0$. If $u > \max\{v, w\}$, then $u (1 - k) \leq 0$, a contradiction. Hence $u \leq \max\{v, w\}$, which implies $u \leq p \max\{v, w\}$, where $0 < p = k < 1$.

Example 6.7. *$F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - c \max\{2t_4, t_5 + t_6\}$, where $a, b, c \geq 0$ and $0 < a + b + 2c < 1$.*

(F_p) : Let $u, v, w \geq 0$ be such that $F(u, v, 0, w, u, v) = u - av - c \max\{2w, u + v\} \leq 0$. If $u > \max\{v, w\}$, then $u[1 - (a + 2c)] \leq 0$, a

contradiction. Hence $u \leq \max\{v, w\}$*, which implies* $u \leq p \max\{v, w\}$*, where* $0 < p = a + 2c < 1$.

Example 6.8. $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - c\max\{t_4 + t_5, 2t_6\}$*, where* $a, b, c \geq 0$ *and* $0 < a + b + 2c < 1$.

As in Example 6.7, $u \leq p \max\{v, w\}$*, where* $0 < p = a + 2c < 1$.

Example 6.9. $F(t_1, ..., t_6) = t_1 - k\max\left\{t_2, t_3, t_4, \frac{2t_4+t_3}{3}, \frac{2t_4+t_6}{3}, \frac{t_5+t_6}{3}\right\}$*, where* $k \in [0, 1)$.

(F_p) : *Let* $u, v, w \geq 0$ *be such that* $F(u, v, 0, w, u, v) = u - k\max\{v, w, \frac{2w+v}{3}, \frac{u+v}{3}\} \leq 0$*. If* $u > \max\{v, w\}$*, then* $u(1 - k) \leq 0$*, a contradiction. Hence* $u \leq \max\{v, w\}$*, which implies* $u \leq p \max\{v, w\}$*, where* $0 < p = k < 1$.

Example 6.10. $F(t_1, ..., t_6) = t_1 - at_2 - k\max\{t_3 + 2t_4, t_4 + t_5 + t_6\}$*, where* $a, k \geq 0$ *and* $0 < a + 4k < 1$.

(F_p) : *Let* $u, v, w \geq 0$ *be such that* $F(u, v, 0, w, u, v) = u - av - k\max\{2w, u + v + w\} \leq 0$*. If* $u > \max\{v, w\}$*, then* $u[1 - (a + 3k)] \leq 0$*, a contradiction. Hence* $u \leq \max\{v, w\}$*, which implies* $u \leq p \max\{v, w\}$*, where* $0 < p = a + 3k < 1$.

Example 6.11. $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - k\max\{2t_4 + t_5, t_1 + t_4 + t_5 + t_6\}$*, where* $a, b, c \geq 0$ *and* $0 < a + b + 5k < 1$.

Similar, as in Example 6.10 we obtain $u \leq p \max\{v, w\}$*, where* $0 < p = a + 4k < 1$.

Example 6.12. $F(t_1, ..., t_6) = t_1 - \max\{at_2, b(t_3 + t_4), b(t_4 + t_5 + t_6)\}$*, where* $a \in (0, 1)$ *and* $b \in \left(0, \frac{1}{4}\right)$.

(F_p) : *Let* $u, v, w \geq 0$ *be such that* $F(u, v, 0, w, u, v) = u - \max\{av, 2bw, u + v + w\} \leq 0$*. If* $u > \max\{v, w\}$*, then* $u(1 - \max\{a, 3b\}) \leq 0$*, a contradiction. Hence* $u \leq \max\{v, w\}$*, which implies* $u \leq p \max\{v, w\}$*, where* $0 < p = \max\{a, 3b\} < 1$.

Example 6.13. $F(t_1, ..., t_6) = t_1 - k\max\{t_2, t_3 + t_4, t_5 + t_6\}$*, where* $k \in \left[0, \frac{1}{3}\right)$.

(F_p) : *Let* $u, v, w \geq 0$ *be such that* $F(u, v, 0, w, u, v) = u - k\max\{v, w, u + v\} \leq 0$*. If* $u > \max\{v, w\}$*, then* $u(1 - 2k) \leq 0$*, a contradiction. Hence* $u \leq \max\{v, w\}$*, which implies* $u \leq p \max\{v, w\}$*, where* $0 < p = 2k < 1$.

Theorem 6.14. *Let* (X, GP) *be a* GP *- symmetric space and* $T : X \to X$ *a function satisfying the conditions from Theorem 4.2 and* T *having property* (F_p)*. Then the fixed point problem of* T *is well posed.*

Proof. By Theorem 4.2, T has an unique fixed point x_0 with $GP(x_0, x_0, x_0) = 0$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} GP(x_n, Tx_n, Tx_n) = 0$. By (4.1) we have successively

$$F(GP(Tx_0, Tx_n, Tx_n), GP(x_0, x_n, x_n), GP(x_0, Tx_0, Tx_0),\\ GP(x_n, Tx_n, Tx_n), GP(x_0, Tx_n, Tx_n), GP(x_n, Tx_0, Tx_0)) \le 0,$$

$$F(GP(x_0, Tx_n, Tx_n), GP(x_0, x_n, x_n), 0,\\ GP(x_n, Tx_n, Tx_n), GP(x_0, Tx_n, Tx_n), GP(x_n, x_0, x_0)) \le 0.$$

Since the space (X, GP) is symmetric, $G(x_0, x_0, x_n) = G(x_0, x_n, x_n)$. By (F_p) we have

$$GP(x_0, Tx_n, Tx_n) \le p \max\{GP(x_0, x_n, x_n), GP(x_n, Tx_n, Tx_n)\}\\ \le p[GP(x_0, x_n, x_n) + GP(x_n, Tx_n, Tx_n)].$$

By (GP_4):

$$\begin{aligned} GP(x_0, x_n, x_n) &\le GP(x_0, Tx_n, Tx_n) + GP(Tx_n, x_n, x_n)\\ &\le p[GP(x_0, x_n, x_n) + GP(x_n, Tx_n, Tx_n)] + GP(x_n, Tx_n, Tx_n), \end{aligned}$$

which implies

$$GP(x_0, x_n, x_n) \le \frac{1+p}{1-p} GP(x_n, Tx_n, Tx_n).$$

Hence,

$$\lim_{n\to\infty} GP(x_0, x_n, x_n) = 0$$

and the fixed point problem of T is well posed. □

Remark 6.15. *By Examples 6.4 - 6.13 we obtain new particular results.*

References

[1] T. Abdeljawad, E. Karapinar and K. Tas, **Existence and uniqueness of common fixed points on partial metric spaces**, Appl. Math. Lett., 24 (11) (2011), 1900 - 1904.

[2] M. Akkouchi and V. Popa, **Well posedness of common fixed point problem for three mappings under strict contractive conditions**, Bul., Univ. Petrol-Gaze Ploiesti, Ser. Mat. Inform. Fiz., 61 (2) (2009), 1 - 10.

[3] M. Akkouchi and V. Popa, **Well posedness of a fixed point problem for mappings satisfying an implicit relation**, Demonstr. Math., 43 (4) (2010), 923 - 929.

[4] M. Akkouchi and V. Popa, **Well posedness of fixed point problem for hybrid pairs of mappings**, Fasc. Math., 46 (2011), 5 - 16.

[5] I. Altun, F. Sola and H. Simsek, **Generalized contractive principle on partial metric spaces**, Topology Appl., 157 (18) (2010), 2778 - 2785.

[6] H. Aydi, E. Karapinar and P. Salimi, **Some fixed point results in** GP **- metric spaces**, J. Appl. Math., (2012), Article ID 891713, 15 pages, DOI:10.1155/2012/891713.

[7] M. A. Barakat and A. M. Zidad, **A common fixed point theorem for weak contractive maps in** GP **- metric spaces**, J. Egypt. Math. Soc., (2014), DOI: 10.1016/j.joems.2014.06.008.

[8] N. Bilgili, E. Karapinar and P. Salimi, **Fixed point theorems for generalized contractions on** GP **- metric spaces**, J. Inequal. Appl., (2013), 2013:39.

[9] R. Chi, E. Karapinar and T. D. Than, **A generalized contraction principle in partial metric spaces**, Math. Comput. Modelling, 53 (2012), 1673 - 1681.

[10] R. Chugh, T. Kadian, A. Rani and B. E. Rhoades, **Property** (P) **in** G **- metric spaces**, Fixed Point Theory Appl., Volume 2010, Article ID 461184.

[11] F. S. De Blasi and J. Myjak, **Sur la porosité de l'ensemble des contractions sans point fixe**, C. R. Acad. Sci., Paris, Sr. I, Math., 308 (1989), 51 - 54.

[12] B. C. Dhage, **Generalized metric spaces and mappings with fixed point**, Bull. Calcutta Math. Soc., 84 (1992), 329 - 336.

[13] B. C. Dhage, **Generalized metric spaces and topological structures, I**, An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Mat., 46 (2000), 3 - 24.

[14] G. S. Jeong, **More maps for which** $F(T) = F(T^n)$, Demonstr. Math., 40 (3) (2007), 671 - 680.

[15] G. S. Jeong and B. E. Rhoades, **Maps for which** $F(T) = F(T^n)$, Fixed Point Theory Appl., 6 (2007), 71 - 105.

[16] Z. Kadelburg, H. K. Nashine and S. Radanović, **Fixed point results under various contractive conditions in partial metric spaces**, Rev. R. Acad. Cienc. Exactas Fs. Nat., Ser. A Mat., RACSAM, 10 (2013), 241 - 256.

[17] E. Karapinar and I. M. Erhan, **Fixed point theorems for operators on partial metric spaces**, Appl. Math. Lett., 24 (11) (2011), 1894 - 1899.

[18] D. S. Kaushal and S. S. Pagey, **Some results of fixed point theorems on complete** G *- metric spaces*, South Asian J. Math., 2 (4) (2014), 318 - 324.

[19] B. K. Lahiri and P. Das, **Well-posedness and porosity of certain classes of operators**, Demonstr. Math., 38 (2005), 170 - 176.

[20] S. Matthews, **Partial metric topology and applications**, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci., 728 (1994), 183 - 197.

[21] S. K. Mohanta, **Some fixed point theorems in** G **- metric spaces**, An. Ştiinţ. Univ. Ovidius Constanţa, Ser. Mat., 20 (1) (2012), 285 - 306.

[22] Z. Mustafa and B. Sims, **Some remarks concerning** D **- metric spaces**, Conf. Fixed Point Theory and Applications, Yokohama, (2004), 184 - 198.

[23] Z. Mustafa and B. Sims, **A new approach to generalized metric spaces**, J. Nonlinear Convex Anal., 7 (2) (2006), 289 - 297.

[24] Z. Mustafa, H. Obiedat, and F. Awadeh, **Some fixed point theorem for mapping on complete** G **- metric spaces**, Fixed Point Theory Appl., (2008), Article ID 189870.

[25] Z. Mustafa, W. Shatanawi and M. Bataineh, **Fixed point theorems on uncomplete G - metric spaces**, J. Math. Stat., 4 (4) (2008), 196 - 201.

[26] Z. Mustafa and B. Sims, **Fixed point theorems for contractive mappings in complete G - metric spaces**, Fixed Point Theory Appl., (2009), Article ID 917175.

[27] Z. Mustafa, W. Shatanawi and M. Bataineh, **Existence of fixed point results in G - metric spaces**, Int. J. Math. Math. Sci., (2009), Article ID 283028.

[28] Z. Mustafa and H. Obiedat, **A fixed point theorem of Reich in G - metric spaces**, Cubo, 12 (2010), 83 - 93.

[29] Z. Mustafa, M. Khandagji and W. Shatanawi, **Fixed point results on complete G - metric spaces**, Stud. Sci. Math. Hung., 48 (3) (2011), 304 - 319.

[30] V. Parvaneh, J. R. Roshan and Z. Kadelburg, **On generalized weakly GP-contractive mappings in ordered GP-metric spaces**, Gulf J. Math., 1 (2013), 78 - 97.

[31] V. Popa, **Fixed point theorems for implicit contractive mappings**, Stud. Cercet. Ştiinţ., Ser. Mat., Univ. Bacău, 7 (1997), 129 - 133.

[32] V. Popa, **Some fixed point theorems for compatible mappings satisfying an implicit relation**, Demonstr. Math., 32 (1) (1999), 157 - 163.

[33] V. Popa, **A general fixed point theorem for several mappings in G - metric spaces**, Sci. Stud. Res., Ser. Math. Inform., 21 (1) (2011), 205 - 214.

[34] V. Popa and A.-M. Patriciu, **A general fixed point theorem for mappings satisfying an ϕ - implicit relation in complete G - metric spaces**, GUJS, 25 (2) (2012), 403 - 408.

[35] V. Popa and A.-M. Patriciu, **A general fixed point theorem for pair of weakly compatible mappings in G - metric spaces**, J. Nonlinear Sci. Appl., 5 (2) (2012), 151 - 160.

[36] S. Reich, A. J. Zaslavski, **Well posedness of fixed point problem**, Far East J. Math. Sci., Special Volume, Part III, (2001), 393 - 401.

[37] B. E. Rhoades and M. Abbas, **Mappings satisfying contractive conditions of integral type for which $F(T) = F(T^n)$**, Int. J. Pure Appl. Math. Sci., 45 (2008), 225 - 231.

[38] W. Shatanawi, **Fixed point theory for contractive mappings satisfying ϕ - maps in G - metric spaces**, Fixed Point Theory Appl., (2010), Article ID 181650.

[39] R. Srivastava, S. Agrawal, R. Bhardwaj and R. Vardava, **Fixed point theorems in complete G - metric spaces**, South Asian J. Math., 2 (2) (2013), 167 - 174.

[40] C. Vetro and F. Vetro, **Common fixed points of mappings satisfying implicit relations in partial metric spaces**, J. Nonlinear Sci. Appl., 6 (2013), 152 - 160.

[41] M. R. A. Zand and A. N. Nezhad, **A generalization of partial metric spaces**, J. Contemp. Appl. Math., 24 (2010), 86 - 93.

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