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UPPER AND LOWER COMPLETELY CONTINUOUS MULTIFUNCTIONS

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Abstract. The notion of complete continuity of functions (Kyungpook Math. J. 14(1974), 131-143) is extended to the realm of multifunctions. Basic properties of upper (lower) completely continuous multifunctions are studied and their place in the hierarchy of variants of continuity of multifunctions is established. Examples are included to reflect upon the distinctiveness of upper (lower) complete continuity of multifunctions from that of other variants of continuity of multifunctions which already exist in the literature. Interplay between topological properties and completely continuous multifunctions is considered.

1. Introduction

Several weak and strong variants of continuity arise in diverse situations in mathematics and applications of mathematics which have been studied by host of authors. Strong variants of continuity with which we shall be dealing in this paper include strong continuity due to Levine [28], complete continuity initiated by Arya and Gupta [6], perfect continuity defined by Noiri [30] and further studied by Kohli, Singh and Arya [25], cl-supercontinuity studied by Singh [38] and almost complete continuity defined and studied by Kohli and Singh [22].

Keywords and phrases: upper/lower (almost) completely continuous multifunction, upper/lower (almost) cl-supercontinuous multifunction, upper/lower (almost) z-supercontinuous multifunction, upper/lower (almost) perfectly continuous multifunction, S-closed, almost regular, almost completely regular, nearly compact, nearly paracompact.
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Recently, there has been considerable interest in trying to extend the notions and results of variants of continuity of functions to the realm of multifunctions. For example see ([1], [2], [3], [4], [7], [11], [12], [13], [14], [15], [17], [27], [29], [31]). In this paper we extend the notion of complete continuity of functions to the realm of multifunctions and define the notions of upper and lower completely continuous multifunctions and elaborate on their place in the hierarchy of variants of continuity of multifunctions. It turns out that the class of upper (lower) completely continuous multifunctions properly contains the class of upper (lower) perfectly continuous multifunctions [13] and so includes all strongly continuous multifunctions [13] and is strictly contained in the class of upper (lower) supercontinuous multifunctions [1]. Organization of the paper is as follows. Section 2 is devoted to preliminaries and basic definitions. In Section 3 we define upper (lower) complete continuity of multifunctions and discuss its interrelations and interconnections with other variants of continuity of multifunctions which already exist in the mathematical literature. Examples are included to reflect upon the distinctivness of new notions from the existing ones. In Section 4 we investigate the properties of upper completely continuous multifunctions wherein, in particular, it is shown that (i) upper complete continuity of multifunctions is preserved under (a) compositions, (b) shrinking and expansion of range, (c) restrictions (with some mild hypothesis on subspaces); (ii) the image of a nearly compact set under an upper completely continuous multifunction is compact if point images are compact; (iii) if the graph multifunction of a multifunction is upper completely continuous, then the multifunction is upper completely continuous and every open set of domain is regular open; (iv) if a multifunction into a product space is upper completely continuous, then its composition with each projection map is upper completely continuous; (v) the graph of an upper supercontinuous (hence completely continuous) multifunction into a regular space with closed point images is δ-closed. Section 5 is devoted to the study of lower completely continuous multifunctions which have many properties similar to upper completely continuous multifunctions. Furthermore, a multifunction into a product space is shown to be lower completely continuous if and only if its composition with each projection map is lower completely continuous. Section 6 is devoted to study the interplay of topological properties and upper(lower) completely continuous multifunctions.

2. Basic Definitions and Preliminaries

Throughout the paper we essentially adopt the notation and terminology of Górniecowicz [10] pertaining to multifunctions.
Let X and Y be two spaces and assume that for every point $x \in X$ a nonempty subset $\varphi(x)$ of Y is given; in this case we say that $\varphi : X \multimap Y$ is a multifunction from X to Y. Let B be a subset of Y. Then the set $\varphi_+^{-1}(B) = \{x \in X : \varphi(x) \cap B \neq \emptyset\}$ is called ***large inverse image*** of B and the set $\varphi_-^{-1}(B) = \{x \in X : \varphi(x) \subset B\}$ is called ***small inverse image*** of B. The ***graph*** of a multifunction φ is given by $\Gamma\varphi = \{(x, y) \in X \times Y | y \in \varphi(x)\}$. Let A be a subset of X. Then $\varphi(A) = \cup\{\varphi(x) : x \in A\}$ is called the image of A. A multifunction $\varphi : X \multimap Y$ from a topological space X into a topological space Y is ***upper semicontinuous (lower semicontinuous)*** if $\varphi_-^{-1}(U)$ ($\varphi_+^{-1}(U)$) is an open set in X for every open set U in Y. A subset U of a topological space X is called a ***cl-open*** [38] if it can be expressed as the union of clopen sets. A subset A of a space X is said to be ***regular open*** if it is the interior of its closure, i.e. $A = (\bar{A})^\circ$. The complement of a regular open set is referred to as ***regular closed***. Any union of regular open sets is called δ***-open set***.

2.1. **Definitions.** A multifunction $\varphi : X \multimap Y$ from a topological space X into a topological space Y is said to be

(i) ***upper cl-supercontinuous*** [12]***(upper z-supercontinuous (***[2] [18]***))*** at $x \in X$ if for each open set V with $\varphi(x) \subset V$, there exists a clopen (cozero) set U containing x such that $\varphi(U) \subset V$;
(ii) ***lower cl-supercontinuous*** [12] ***(lower z-supercontinuous (***[2] [18]***))*** at $x \in X$ if for each open set V with $\varphi(x) \cap V \neq \emptyset$, there exists a clopen (cozero) set U containing x such that $\varphi(z) \cap V \neq \emptyset$ for each $z \in U$;
(iii) ***upper almost cl-supercontinuous (***[14] [22]***) (upper almost z-supercontinuous (***[5] [25]***))*** at $x \in X$ if for each regular open set V with $\varphi(x) \subset V$, there exists a clopen (cozero) set U containing x such that $\varphi(U) \subset V$;
(iv) ***lower almost cl-supercontinuous (***[14] [22]***) (lower almost z-supercontinuous (***[5] [25]***))*** at $x \in X$ if for each regular open set V with $\varphi(x) \cap V \neq \emptyset$, there exists a clopen (cozero) set U containing x such that $\varphi(z) \cap V \neq \emptyset$ for each $z \in U$;
(v) ***strongly continuous*** [13] if $\varphi_-^{-1}(B)$ (or $\varphi_+^{-1}(B)$) is clopen in X for every subset B of Y;

(vi) ***upper (lower) δ-perfectly continuous*** [23] if $\varphi_-^{-1}(U)(\varphi_+^{-1}(U))$ is clopen in X for every δ-open subset U of Y;
(vii) ***upper (almost) perfectly continuous*** [13] if $\varphi_-^{-1}(U)$ is clopen in X for every (regular) open subset U of Y;
(viii) ***lower (almost) perfectly continuous*** [13] if $\varphi_+^{-1}(U)$ is clopen in X for every (regular) open subset U of Y;
(ix) ***upper almost completely continuous*** [17] if $\varphi_-^{-1}(U)$ is a regular open set in X for every regular open subset U of Y;
(x) ***lower almost completely continuous*** [17] if $\varphi_+^{-1}(U)$ is a regular open set in X for every regular open subset U of Y;
(xi) ***upper supercontinuous*** [1] if $\varphi_-^{-1}(U)$ is a δ-open set in X for every open subset U of Y;
(xii) ***lower supercontinuous*** [1] if $\varphi_+^{-1}(U)$ is a δ-open set in X for every open subset U of Y.

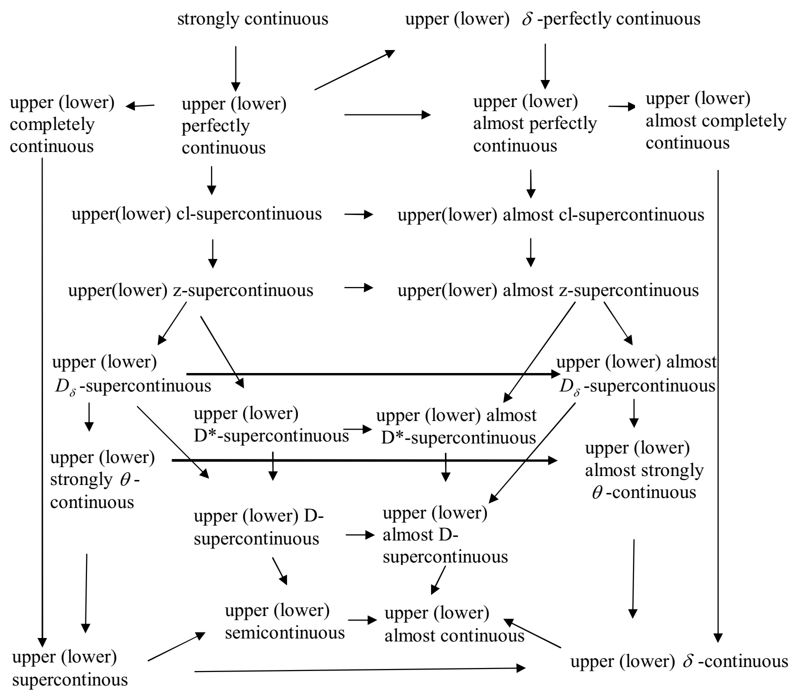
3. Upper(lower) Completely Continuous Multifunctions

3.1. **Definitions.** We say that a multifunction $\varphi : X \multimap Y$ is

(a) ***upper completely continuous*** if $\varphi_-^{-1}(V)$ is a regular open set in X for each open set V in Y; and
(b) ***lower completely continuous*** if $\varphi_+^{-1}(V)$ is a regular open set in X for each open set V in Y.

Since every clopen set is regular open, every upper(lower) perfectly continuous multifunction is upper(lower) completely continuous multifunction. Further since every regular open set is δ-open, every upper(lower) completely continuous multifunction is upper(lower) supercontinuous multifunction.
The following comprehensive diagram extends several existing diagrams in the literature, (see [13], [14], [15], [16]) and well illustrates the interrelations that exist among upper (lower) complete continuity of multifunctions and other variants of continuity of multifunctions which already exist in the literature and are related to the theme of the present paper.

3.2. **Example.** Let $X = \{a, b, c\}$ and let $\Im_X = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$. Let $Y = \{p, q, r\}$ and let $\Im_Y = \{\phi, Y, \{p, q\}\}$. Define a multifunction $\varphi : (X, \Im_X) \multimap (Y, \Im_Y)$ by $\varphi(a) = \{p, q\}$, $\varphi(b) = \{p, r\}$, $\varphi(c) = \{q, r\}$. Clearly φ is upper completely continuous but not upper perfectly continuous.



3.3. **Example.** Let $X = \{a, b, c\}$ and let $\Im_X = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$. Let $Y = \{p, q, r\}$ and let $\Im_Y = \{\phi, Y, \{p, q\}\}$. Define a multifunction $\varphi : (X, \Im_X) \multimap (Y, \Im_Y)$ by $\varphi(a) = \{p, q\}$, $\varphi(b) = \{r\}$, $\varphi(c) = \{r\}$. Clearly φ is lower completely continuous but not lower perfectly continuous.

3.4. **Example.** Let $X = \{a, b, c\}$ and let $\Im_X = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$. Let $Y = \{p, q, r\}$ and let $\Im_Y = \{\phi, Y, \{p\}, \{q\}, \{p, q\}\}$. Define a multifunction $\varphi : (X, \Im_X) \multimap (Y, \Im_Y)$ by $\varphi(a) = \{p\}$, $\varphi(b) = \{r\}$, $\varphi(c) = \{r\}$. Clearly φ is upper (lower) almost completely continuous but not upper (lower) completely continuous.

3.5. **Example.** Let $\mathbb{R}$ be the set of real numbers and $\mathfrak{U}$ be the usual topology on $\mathbb{R}$. Then every upper (lower) semicontinuous multifunction $\varphi : (\mathbb{R}, \mathfrak{U}) \multimap (\mathbb{R}, \mathfrak{U})$ is upper (lower) supercontinuous but not upper (lower) completely continuous.

3.6. **Example.** Let $\mathbb{R}$ be the set of real numbers and $\mathfrak{U}$ be the usual topology on $\mathbb{R}$. Then every upper (lower) almost continuous multifunction $\varphi : (\mathbb{R}, \mathfrak{U}) \multimap (\mathbb{R}, \mathfrak{U})$ is upper (lower) δ-continuous but not upper (lower) almost completely continuous.

4. Properties of Upper Completely Continuous Multifunctions

4.1. **Theorem.** A multifunction $\varphi : X \multimap Y$ is upper completely continuous if and only if for every closed set $A \subset Y$ the set $\varphi_+^{-1}(A)$ is a regular closed subset of X.

Proof. Suppose $\varphi : X \multimap Y$ is upper completely continuous multifunction. Let A be a closed subset of Y. Then $Y \setminus A$ is an open set in Y. Since φ is upper completely continuous, $\varphi_-^{-1}(Y \setminus A) = X \setminus \varphi_+^{-1}(A)$ is a regular open set in X and so $\varphi_+^{-1}(A)$ is a regular closed in X.
Conversely suppose that $\varphi_+^{-1}(A)$ is regular closed in X for every closed set $A \subset Y$. Let U be an open subset of Y. Then $Y \setminus U$ is a closed subset of Y. By hypothesis, $\varphi_+^{-1}(Y \setminus U) = X \setminus \varphi_-^{-1}(U)$ is regular closed in X and so $\varphi_-^{-1}(U)$ is a regular open in X. Thus φ is upper completely continuous.

4.2. **Theorem.** If $\varphi : X \multimap Y$ is upper completely continuous and $\psi : Y \multimap Z$ is upper semicontinuous, then $\psi \circ \varphi$ is upper completely continuous.

Proof. Let W be an open set in Z. Since ψ is upper semicontinuous, $\psi_-^{-1}(W)$ is an open set in Y. Again, since φ is upper completely continuous, $\varphi_-^{-1}(\psi_-^{-1}(W)) = (\psi o \varphi)_-^{-1}(W)$ is a regular open set in X and so $\psi o \varphi$ is upper completely continuous.

4.3. **Corollary.** If $\varphi : X \multimap Y$ and $\psi : Y \multimap Z$ are upper completely continuous multifunctions, then their composition $\psi \circ \varphi$ is upper completely continuous.
Next two results show that upper complete continuity is invariant under shrinking and expansion of co-domain of a multifunction.

4.4. **Theorem.** If $\varphi : X \multimap Y$ is upper completely continuous and $\varphi(X)$ is endowed with the subspace topology, then $\varphi : X \multimap \varphi(X)$ is upper completely continuous.

Proof. Since φ is upper completely continuous, for every open subset V of Y, $\varphi_-^{-1}(V \cap \varphi(X)) = \varphi_-^{-1}(V) \cap \varphi_-^{-1}(\varphi(X)) = \varphi_-^{-1}(V) \cap X = \varphi_-^{-1}(V)$ is a regular open set in X and so $\varphi : X \multimap \varphi(X)$ is upper completely continuous.

4.5. **Theorem.** Let $\varphi : X \multimap Y$ be an upper completely continuous multifunction. If Z is a space containing Y as a subspace, the $\psi : X \multimap Z$ define by $\psi(x) = \varphi(x)$ for each $x \in X$ is upper completely continuous.

Proof. Let W be an open set in Z. Then $W \cap Y$ is an open set in Y. Since $\varphi : X \multimap Y$ is upper completely continuous, $\varphi_{-}^{-1}(W \cap Y)$ is a regular open set in X. Since $\psi_{-}^{-1}(W) = \psi_{-}^{-1}(W \cap Y) = \varphi_{-}^{-1}(W \cap Y)$, $\psi : X \multimap Z$ is upper completely continuous.

4.6. **Theorem.** If $\varphi : X \multimap Y$ is upper completely continuous and A is a subset of X such that the intersection of every regular open set in X with A is regular open in A, then the restriction $\varphi_{|A} : A \multimap Y$ is upper completely continuous.

Proof. Let W be an open set in Y. Since $\varphi : X \multimap Y$ is upper completely continuous, $\varphi_{-}^{-1}(W)$ is a regular open set in X. Now $(\varphi_{|A})_{-}^{-1}(W) = \{x \in A : \varphi(x) \subset W\} = \{x \in A : \ x \in \varphi_{-}^{-1}(W)\} = A \cap \varphi_{-}^{-1}(W)$, which is regular open in A and so $\varphi_{|A}$ is upper completely continuous.

The next result shows that the image of a nearly compact set under an upper completely continuous multifunction is compact in case point images are compact. We may recall that a space X is ***nearly compact*** [36] if every regular open cover of X has a finite subcover.

4.7. **Theorem.** Let $\varphi : X \multimap Y$ be an upper completely continuous multifunction such that $\varphi(x)$ is compact for each $x \in X$. If A is a nearly compact set in X, then its image $\varphi(A)$ is compact.

Proof. Let Ω be an open cover of $\varphi(A)$. Then Ω is also an open cover of $\varphi(a)$ for each $a \in A$. Since each $\varphi(a)$ is compact, there exists a finite subset $\beta_a \subset \Omega$ such that $\varphi(a) \subset \bigcup_{B \in \beta_a} B = V_a$(say). Since φ is upper completely continuous, $U_a = \varphi_{-}^{-1}(V_a)$ is a regular open set containing a. Let $Q = \{U_a \mid a \in A\}$. Then Q is a regular open cover of A. Since A is nearly compact, there exists a finite subset $\{a_1, - - -, a_n\}$ of A such that $A \subset \bigcup_{i=1}^{n} U_{a_i} = \bigcup_{i=1}^{n} \varphi_{-}^{-1}(V_{a_i})$. Therefore $\varphi(A) \subset \varphi(\bigcup_{i=1}^{n} \varphi_{-}^{-1}(V_{a_i})) = \bigcup_{i=1}^{n} \varphi(\varphi_{-}^{-1}(V_{a_i})) \subset \bigcup_{i=1}^{n} V_{a_i}$, where V_{a_i} is a finite union of members of Ω and so $\varphi(A)$ is compact.

4.8. **Theorem.** Let $\varphi : X \multimap Y$ be a multifunction and let $g : X \multimap X \times Y$ defined by $g(x) = \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$ for each $x \in X$ be the graph multifunction. If g is upper completely continuous, then φ is upper completely continuous and the space X has the property that every open set in X is regular open.

Proof. Suppose that the multifunction g is upper completely continuous. Then in view of Theorem 4.2 the multifunction $\varphi = p_y o g$ is upper completely continuous, where $p_y : X \times Y \multimap Y$ denotes the projection mapping. Now we show that every open set in X is regular open. Let U be an open set in X. Then $U \times Y$ is an open set in $X \times Y$. Since g is upper completely continuous, $g_-^{-1}(U \times Y) = U$ is a regular open set in X and hence X has the property that every open set in X is regular open.

4.9. **Theorem.** Let $\{\varphi_\alpha : X \multimap X_\alpha \mid \alpha \in \Lambda\}$ be a family of multifunctions. Let the multifunction $\varphi : X \multimap \prod_{\alpha\in\Lambda} X_\alpha$ be defined by $\varphi(x) = \prod_{\alpha\in\Lambda} \varphi_\alpha(x)$. If φ is upper completely continuous, then each φ_α is upper completely continuous.

Proof. Let $p_\beta : \prod_{\alpha\in\Lambda} X_\alpha \multimap X_\beta$ be the projection map onto X_β. Then p_β being single valued continuous function is upper (lower) semicontinuous. By Theorem 4.2 $\varphi_\beta = p_\beta o \varphi$ is upper completely continuous for each $\beta \in \Lambda$.

4.10. **Theorem.** Let $\varphi, \psi : X \multimap Y$ be upper supercontinuous multifunctions from a topological space X into a normal space Y such that $\varphi(x)$ and $\psi(x)$ are closed for each $x \in X$. Then the set $E = \{x \in X : \varphi(x) \cap \psi(x) \neq \emptyset\}$ is a δ-closed subset of X.

Proof. To prove that E is a δ-closed, we shall show that $X \setminus E$ is δ-open. To this end, let $x \in X \setminus E$. Then $\varphi(x) \cap \psi(x) = \emptyset$. Since Y is normal, there exist disjoint open sets U and V containing $\varphi(x)$ and $\psi(x)$ respectively. Since φ and ψ are upper supercontinuous multifunctions, $\varphi_-^{-1}(U)$ and $\psi_-^{-1}(V)$ are δ-open sets in X containing x. Let $G = \varphi_-^{-1}(U) \cap \psi_-^{-1}(V)$. Then G is a δ-open set containing x. Since U and V are disjoint, $G \subset X \setminus E$ and so $X \setminus E$ being union of δ-open sets is δ-open.

4.11. **Corollary.** Let $\varphi, \psi : X \multimap Y$ be upper completely continuous multifunctions from a topological space X into a normal space Y such that $\varphi(x)$ and $\psi(x)$ are closed for each $x \in X$. Then the set $E = \{x \in X : \varphi(x) \cap \psi(x) \neq \emptyset\}$ is a δ-closed subset of X.

4.12. **Definition.** The graph Γ_φ of a multifunction $\varphi : X \multimap Y$ is said to be ***δ-closed with respect to*** X if for each $(x, y) \notin \Gamma_\varphi$ there exist a regular open set U containing x and an open set V containing y such that $(U \times V) \cap \Gamma_\varphi = \emptyset$.

4.13. **Theorem.** If $\varphi : X \multimap Y$ is an upper supercontinuous multifunction, where X is a regular space and $\varphi(x)$ is closed for each $x \in X$, then the graph Γ_φ of φ is a δ-closed subset of $X \times Y$ with respect to X.

Proof. Let $(x, y) \notin \Gamma_\varphi$. Then $y \notin \varphi(x)$. Since Y is a regular space, there exist disjoint open sets V_y and $V_{\varphi(x)}$ containing y and $\varphi(x)$, respectively. Since φ is upper supercontinuous, $U_x = \varphi_-^{-1}(V_{\varphi(x)})$ is a δ-open set containing x. So $U_x = \cup U_\alpha$, where each U_α is regular open. It is easily verified that the sets V_y and $V_{\varphi(x)}$ may be chosen to be regular open. Since $x \in U_x$, there is an α_x such that $x \in U_{\alpha_x}$. We assert that $(U_{\alpha_x} \times V_y) \cap \Gamma_\varphi = \emptyset$. For, if $(h, k) \in (U_{\alpha_x} \times V_y) \cap \Gamma_\varphi$, then $h \in \varphi_-^{-1}(V_{\varphi(x)})$, $k \in V_y$ and $k \in \varphi(h)$. Hence $\varphi(h) \subset V_{\varphi(x)}$ and $k \in \varphi(h) \cap V_y$ which contradicts the fact that V_y and $V_{\varphi(x)}$ are disjoint. Thus the graph Γ_φ is a δ-closed set in $X \times Y$ with respect to X.

4.14. **Corollary.** If $\varphi : X \multimap Y$ is an upper completely continuous multifunction, where X is a regular space and $\varphi(x)$ is closed for each $x \in X$, then the graph Γ_φ of φ is a δ-closed subset of $X \times Y$ with respect to X.

4.15. **Theorem.** Let $\varphi : X \multimap Y$ be an upper completely continuous multifunction from a space X into a normal space Y such that $\varphi(x)$ is closed for each $x \in X$. Then the set $A = \{(x, y) \in X \times X : \varphi(x) \cap \varphi(y) \neq \emptyset\}$ is a δ-closed subset of $X \times X$.

Proof. Let $(x, y) \notin A$. Then $\varphi(x) \cap \varphi(y) = \emptyset$. Since Y is normal, there exist disjoint open sets U and V containing $\varphi(x)$ and $\varphi(y)$ respectively. Since φ is upper completely continuous, $\varphi_-^{-1}(U)$ and $\varphi_-^{-1}(V)$ are disjoint regular open sets containing x and y respectively. Let $G_1 = \varphi_-^{-1}(U)$ and $G_2 = \varphi_-^{-1}(V)$. Then $(G_1 \times G_2)$ is a regular open set in $X \times X$ containing (x, y). We claim that $(G_1 \times G_2) \cap A = \emptyset$. For if $(G_1 \times G_2) \cap A \neq \emptyset$. Then $(x_1, x_2) \in G_1 \times G_2$ and $(x_1, x_2) \in A$. This in turn implies that U and V are not disjoint which is a contradiction. Thus $(G_1 \times G_2) \cap A = \emptyset$ and so $(G_1 \times G_2) \subset (X \times X) \setminus A$. Hence $(X \times X) \setminus A$ being the union of regular open sets is δ-open and so A is a δ-closed subset of $X \times X$.

5. Properties of Lower Completely Continuous Multifunctions

5.1. **Theorem.** A multifunction $\varphi : X \multimap Y$ is lower completely continuous if and only if for every closed set $A \subset Y$ the set $\varphi_-^{-1}(A)$ is a regular closed subset of X.

Proof. Suppose the multifunction $\varphi : X \multimap Y$ is lower completely continuous and let A be a closed subset of Y. Then $Y \setminus A$ is an open set in Y. Since φ is lower completely continuous, $\varphi_+^{-1}(Y \setminus A) = X \setminus \varphi_-^{-1}(A)$ is a regular open set in X and so $\varphi_-^{-1}(A)$ is a regular closed set in X. Conversely suppose that $\varphi_-^{-1}(A)$ is a regular closed set in X for every closed set $A \subset Y$. Let U be an open subset of Y. Then $Y \setminus U$ is a closed subset of Y. By hypothesis, $\varphi_-^{-1}(Y \setminus U) = X \setminus \varphi_+^{-1}(U)$ is a regular closed set in X and so $\varphi_+^{-1}(U)$ is a regular open set in X. Thus the multifunctions φ is upper completely continuous.

5.2. **Theorem.** If $\varphi : X \multimap Y$ is a lower completely continuous multifunction and $\psi : Y \multimap Z$ is lower semicontinuous multifunction, then $\psi o \varphi$ is lower completely continuous.

Proof. Let W be an open set in Z. Since ψ is lower semicontinuous, $\psi_+^{-1}(W)$ is an open set in Y. Again since φ is lower completely continuous, $\varphi_+^{-1}(\psi_+^{-1}(W)) = (\psi o \varphi)_+^{-1}(W)$ is a regular open set in X. Thus $\psi o \varphi : X \multimap Y$ is a lower completely continuous multifunction.

5.3. **Corollary.** If the multifunctions $\varphi : X \multimap Y$ and $\psi : Y \multimap Z$ are lower completely continuous, then so is their composition $\psi o \varphi$.

5.4. **Theorem.** If a multifunction $\varphi : X \multimap Y$ is lower completely continuous and $\varphi(X)$ is endowed with the subspace topology, then $\varphi : X \multimap \varphi(X)$ is lower completely continuous.

Proof. Since φ is lower completely continuous, for every open set V of Y, $\varphi_+^{-1}(V \cap \varphi(X)) = \varphi_+^{-1}(V) \cap \varphi_+^{-1}(\varphi(X)) = \varphi_+^{-1}(V) \cap X = \varphi_+^{-1}(V)$ is a regular open set and so $\varphi : X \multimap \varphi(X)$ is upper completely continuous.

5.5. **Theorem.** Let $\varphi : X \multimap Y$ be a lower completely continuous multifunction. If Z is a space containing Y as a subspace, then $\psi : X \multimap Z$ defined by $\psi(x) = \varphi(x)$ for $x \in X$ is lower completely continuous.

Proof. Let W be an open set in Z. Then $W \cap Y$ is an open set in Y. Since $\varphi : X \multimap Y$ is lower completely continuous, $\varphi_+^{-1}(W \cap Y)$ is a regular open set in X. Now $\psi_+^{-1}(W) = \psi_+^{-1}(W \cap Y) = \varphi_+^{-1}(W \cap Y)$. Thus $\psi : X \multimap Z$ is lower completely continuous.

5.6. **Theorem.** Let $\varphi : X \multimap Y$ be lower completely continuous and let A be subspace of X such that intersection of every regular open set in X with A is regular open in A, then $\varphi_{|A} : A \multimap Y$ is lower completely continuous.

Proof. Let W be an open set in Y. Since $\varphi : X \multimap Y$ is lower completely continuous, $\varphi_+^{-1}(W)$ is a regular open set in X. Now $(\varphi_{|A})_+^{-1}(W) = \{x \in A \mid \varphi(x) \cap W \neq \emptyset\} = \{x \in A \mid x \in \varphi_+^{-1}(W)\} = A \cap \varphi_+^{-1}(W)$, which is a regular open set in X and so $\varphi_{|A}$ is lower completely continuous.

5.7. **Theorem.** Let $\varphi : X \multimap Y$ be a multifunction and let $g : X \multimap X \times Y$ defined by $g(x) = \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$ for each $x \in X$ be the graph multifunction. Then g is lower completely continuous, if and only if φ is lower completely continuous and the space X has property that every open set in X is regular open in X.

Proof. Suppose that g is lower completely continuous. By Theorem 5.2 the multifunction $\varphi = p_y o g$ is lower completely continuous. Let U be an open set in X. Then $U \times Y$ is an open set in $X \times Y$. Since g is lower completely continuous, $g_+^{-1}(U \times Y) = U$ is regular open in X. Hence every open set in X is regular open.
Conversely Suppose that φ is lower completely continuous. Let W be an open set in $X \times Y$. Then there exist open sets U_α in X and V_α in Y such that $W = \bigcup_\alpha (U_\alpha \times V_\alpha)$. Then $g_+^{-1}(W) = \bigcup_\alpha g_+^{-1}(U_\alpha \times V_\alpha) = \bigcup_\alpha (U_\alpha \cap \varphi_+^{-1}(V_\alpha))$ being open in X, is a regular open set in X, which proves that g is lower completely continuous.

5.8. **Theorem.** Let $\{\varphi_\alpha : X \multimap X_\alpha \mid \alpha \in \Lambda\}$ be a family of multifunctions and let the multifunction $\varphi : X \multimap \prod_{\alpha \in \Lambda} X_\alpha$ be defined by $\varphi(x) = \prod_{\alpha \in \Lambda} \varphi_\alpha(x)$ for each $x \in X$. Then φ is lower completely continuous if and only if each $\varphi_\alpha : X \multimap X_\alpha$ is lower completely continuous.

Proof. Let $\varphi : X \multimap \prod_{\alpha \in \Lambda} X_\alpha$ be a lower completely continuous multifunction. Let $p_\beta : \prod_{\alpha \in \Lambda} X_\alpha \multimap X_\beta$ be the projection map onto X_β. Then p_β being a single valued continuous function is lower (upper) semicontinuous. By Theorem5.2 $\varphi_\beta = p_\beta o \varphi : X \multimap X_\beta$ is lower completely continuous for each $\beta \in \Lambda$.
Conversely, suppose that $\varphi_\alpha : X \multimap X_\alpha$ is lower completely continuous for each $\alpha \in \Lambda$. Since the finite intersection and arbitrary union of regular open sets is δ-open, therefore, it suffices to prove that $\varphi_+^{-1}(S)$ is a regular open set for every subbasic open set S in the product space $\prod_{\alpha \in \Lambda} X_\alpha$. Let $U_\beta \times \prod_{\alpha \neq \beta} X_\alpha$ be a sub basic open set in $\prod_{\alpha \in \Lambda} X_\alpha$. Then $\varphi_+^{-1}(U_\beta \times \prod_{\alpha \neq \beta} X_\alpha) = \varphi_+^{-1}(p_\beta^{-1}(U_\beta)) = (p_\beta o \varphi)_+^{-1}(U_\beta) = (\varphi_\beta)_+^{-1}(U_\beta)$ is regular open. Thus φ is lower supercontinuous.

6. Interplay Between Topological Properties and Completely Continuous Multifunctions

6.1. **Definitions.** A topological space X is said to be

(a) S-***closed*** [40] if every semi open cover of X has a finite subcollection whose closures cover X or equivalently, every regular closed cover of X has a finite subcover(see [8])
(b) ***almost regular*** [34] if every regular closed set and a point outside it are contained in disjoint open sets.
(c) ***almost completely regular*** [35] if for every regular closed set F in X and a point $x \notin F$ there exists a continuous functions $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = 1$.
(d) ***mildly normal*** [37] if every pair of disjoint regular closed sets are contained in disjoint open sets
(e) ***nearly paracompact*** [33] if every regular open cover of X has a locally finite open refinement.

6.2. **Theorem.** Let $\varphi : X \multimap Y$ be an upper supercontinuous multifunction such that $\varphi(x) \cap \varphi(y) = \emptyset$ for each $x \neq y$ in X and $\varphi(x)$ is closed for each $x \in X$. If Y is a normal space, then X is a Hausdorff space.

Proof. Let $x, y \in X$, $x \neq y$. Then $\varphi(x) \cap \varphi(y) = \emptyset$. Since Y is normal, there exist disjoint open sets U and V containing $\varphi(x)$ and $\varphi(y)$ respectively. Since φ is upper supercontinuous, $\varphi_-^{-1}(U)$ and $\varphi_-^{-1}(V)$ are disjoint δ-open sets containing x and y respectively and so X is Hausdorff.

6.3. **Corollary.** Let $\varphi : X \multimap Y$ be an upper conpletely continuous multifunction such that $\varphi(x) \cap \varphi(y) = \emptyset$ for each $x \neq y$ in X and $\varphi(x)$ is closed for each $x \in X$. If Y is a normal space, then X is a Hausdorff space.

6.4. **Theorem.** Every upper(lower) completely continuous multifunction $\varphi : X \multimap Y$ defined on a Hausdorff S-closed space X into a space Y is upper(lower) perfectly continuous.

Proof. Let V be an open subset of Y. since φ is upper completely continuous, $\varphi_-^{-1}(V)(\varphi_+^{-1}(V))$ is a regular open set in X. Now since a Hausdorff S-closed space is extremally disconnected and since every regular open set is clopen in an extremally disconnected space, $\varphi_-^{-1}(V)(\varphi_+^{-1}(V))$ is a clopen set in X and so φ is upper(lower) perfectly continuous.

6.5. **Definition.** [41] A multifunction $\varphi : X \multimap Y$ is said to have ***nonmingled*** point images provided that for $x, y \in X$ with $x \neq y$, the image sets $\varphi(x)$ and $\varphi(y)$ are either disjoint or identical.

6.6. **Theorem.** Let $\varphi : X \multimap Y$ be an upper completely continuous, closed nonmingled multifunction from an almost regular space X onto a topological space Y. If either φ is open or $\varphi_-^{-1}(y)$ is compact for each $y \in Y$, then Y is a regular space.

Proof. Let F be a closed set in Y such that $y \notin F$. Then $\varphi_-^{-1}(y) \cap \varphi_-^{-1}(F) = \emptyset$. In view of upper complete continuity of φ, $\varphi_-^{-1}(F)$ is regular closed. Let $x \in \varphi_-^{-1}(y)$. Since X is almost regular, there exist disjoint open sets U and V containing x and $\varphi_-^{-1}(F)$ respectively.

Case I: φ is open. Since here φ is both open and closed, $\varphi(U)$ and $Y \setminus \varphi(X \setminus V)$ are open sets. Again since φ is nonmingled and onto, $\varphi(U)$ and $Y \setminus \varphi(X \setminus V)$ are disjoint and contain y and F, respectively So Y is a regular space.

Case II: $\varphi_-^{-1}(y)$ is compact for each $y \in Y$. Let $U_1 = Y \setminus \varphi(X \setminus U)$ and $V_1 = Y \setminus \varphi(X \setminus V)$. Since φ is closed, U_1 and V_1 are open sets. It suffices to show that U_1 and V_1 are disjoint containing y and F respectively. Since $\varphi(U) \subset \varphi(X \setminus V)$ and $\varphi(V) \subset \varphi(X \setminus U)$, $\varphi(U) \cap (Y \setminus \varphi(X \setminus V)) = \emptyset$ and $\varphi(V) \cap (Y \setminus \varphi(X \setminus U)) = \emptyset$. Again since φ is onto, it is easily verified that U_1 and V_1 are disjoint. Now since $\varphi_-^{-1}(y) \subset U$, and so $\varphi_-^{-1}(y) \cap (X \setminus U) = \emptyset$, $\varphi(\varphi_-^{-1}(y)) \cap \varphi(X \setminus U) = \emptyset$. Again since $\varphi(\varphi_-^{-1}(y)) = y$, $y \in Y \setminus \varphi(X \setminus U) = U_1$. Now since $\varphi_-^{-1}(F) \subset V$, $\varphi_-^{-1}(F) \cap (X \setminus V) = \emptyset$, and moreover since φ is nonmingled, $\varphi(\varphi_-^{-1}(F)) \cap \varphi(X \setminus V) = \emptyset$, and so $F \subset Y \setminus \varphi(X \setminus V) = V_1$. Thus U_1 and V_1 are disjoint open sets containing y and F respectively.

6.7. **Theorem.** Let $\varphi : X \multimap Y$ be a lower completely continuous, closed multifunction from an almost regular space X onto a topological space Y. If either φ is open or $\varphi_-^{-1}(y)$ is compact for each $y \in Y$, then Y is a regular space.

We omit proof of Theorem 6.7 which is similar to that of Theorem 6.6 except for obvious requisite modifications.

6.8. **Theorem.** Let $\varphi : X \multimap Y$ be an upper completely continuous, open and closed multifunction from an almost completely regular space X onto a topological space Y. Then Y is a completely regular space.

Proof. Let $F \subset Y$ be a closed set and $y_0 \notin F$. Then $\varphi_-^{-1}(F)$ is a regular closed set and $\varphi_-^{-1}(y_0) \cap \varphi_-^{-1}(F) = \emptyset$. Take $x_0 \in \varphi_-^{-1}(y_0)$. Since X is almost completely regular, there exists a continuous function $f : X \to [0,1]$ such that $f(\varphi_-^{-1}(F)) = 0$, $f(x_0) = 1$. Define $\hat{f} : Y \to [0,1]$ by $\hat{f}(y) = sup\{f(x) : x \in \varphi_-^{-1}(y)\}$ for each $y \in Y$. Then $\hat{f}(y_0) = 1$ and $\hat{f}(F) = 0$ and by ([9], p.96, Exercise 16) $\hat{f}$ is continuous. Thus Y is completely regular.

6.9. **Theorem.** Let $\varphi : X \multimap Y$ be a lower completely continuous, open and closed multifunction from an almost completely regular space X onto a topological space Y. Then Y is a completely regular space.

We omit proof of Theorem 6.9 which is essentially similar to the proof of Theorem 6.8.

6.10. **Theorem.** Let $\varphi : X \multimap Y$ be an upper completely continuous, closed multifunction from a mildly normal space X onto a topological space Y. Then Y is a normal space.

Proof. Let A and B be any disjoint closed subsets of Y. In view of upper complete continuity of φ, $\varphi_+^{-1}(A)$ and $\varphi_+^{-1}(B)$ are disjoint regular closed subsets of X. Since X is mildly normal, there exist disjoint open sets U and V containing $\varphi_+^{-1}(A)$ and $\varphi_+^{-1}(B)$ respectively. Again since φ is closed, the sets $\varphi(X \setminus U)$ and $\varphi(X \setminus V)$ are closed sets. Since φ is onto and nonmingled, it is easily verified that the sets $Y \setminus \varphi(X \setminus U)$ and $Y \setminus \varphi(X \setminus V)$ are disjoint open sets containing A and B respectively and so Y is a normal space.

6.11. **Theorem.** Let $\varphi : X \multimap Y$ be a lower completely continuous, closed nonmingled multifunction from a mildly normal space X onto a topological space Y. Then Y is a normal space.

Proof of Theorem 6.11 is essentially similar to the case of upper complete continuity except for obvious necessary modifications and hence omitted.

6.12. **Theorem.** Let $\varphi : X \multimap Y$ be an upper completely continuous, open multifunction from a topological space X into a topological space Y such that $\varphi(x) \cap \varphi(y) = \emptyset$ for $x \neq y \in X$ and $\varphi(x)$ is paracompact for each $x \in X$. If A is nearly paracompact, then $\varphi(A)$ is paracompact. In particular, if X is nearly paracompact and φ is onto, then Y is paracompact.

Proof. Let $\Psi = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. Then Ψ is also an open cover of $\varphi(x)$ for each $x \in A$. Since $\varphi(x)$ is paracompact, Ψ has a locally finite open refinement ψ_x such that $\varphi(x) \subset \cup_{V \in \psi_x} V = V_x$

(say). Since φ is upper completely continuous, $\varphi_-^{-1}(V_x) = U_x$ is a regular open set containing x. Now $\mathbf{u} = \{U_x \mid x \in A\}$is a regular open cover of A. Since A is nearly paracompact, $\mathbf{u}$ has a locally finite regular open refinement $\Omega = \{W_\beta \mid \beta \in \Sigma\}$ such that $A \subset \cup_{\beta\in\Sigma} W_\beta$. So for each $\beta \in \Sigma$ there exists a $x_\beta \in A$ such that $W_\beta \subset U_{x_\beta}$ and hence $\varphi(W_\beta) \subset \varphi(U_{x_\beta}) \subset \cup_{V\in\psi_{x_\beta}} V$. Let $\Re_\beta = \{\varphi(W_\beta) \cap V \mid V \in \psi_{x_\beta}\}$ and let $\Re = \{R \mid R \in \Re_\beta, \beta \in \Sigma\}$. We shall show that $\Re$ is a locally finite open refinement of Ψ. Since φ is open, $\varphi(W_\beta)$ is open and so each $R \in \Re$ is open. Let $R \in \Re$. Then $R \in \Re_\beta$ for some $\beta \in \Sigma$, i.e. $R = \varphi(W_\beta) \cap V \subset V \subset U$ for some $U \in \Psi$. This shows that $\Re$ is an open refinement of Ψ. To show that $\Re$ is locally finite, let $y \in \varphi(A)$. Then $y \in \varphi(x)$ for some $x \in A$. Since Ω is locally finite, for each $x \in A$ we can choose an open neighborhood G_x of x which intersects only finitely many members $W_{\beta_1}, W_{\beta_2}, ...W_{\beta_n}$ of Ψ. Since $\varphi(x) \cap \varphi(y) = \emptyset$ for each $x \neq y \in X$, it follows that $H_0 = \varphi(G_x)$ is an open neighborhood of y which intersects only finitely many members $\varphi(W_{\beta_1}), \varphi(W_{\beta_2}))...\varphi(W_{\beta_n})$ of the family $\{\varphi(W_\beta) \mid \beta \in \Sigma\}$. Furthermore each $\Re_{\beta_k}$ $(k = 1, ..., n)$ is locally finite, hence there exists an open neighborhood H_k $(k = 1, ..., n)$ of y which intersects only finitely many members of $\Re_{\beta_k}$ $(k = 1, ..., n)$. Finally let $H = \cap_{k=1}^n H_k$. Clearly H is an open neighborhood of y which intersects at most finitely many member of $\Re$, and so $\Re$ is locally finite. Moreover, $\varphi(A) \subset \varphi(\cup_{\beta\in\Sigma} W_\beta) = \cup_{\beta\in\Sigma}\varphi(W_\beta) \subset \cup_{\beta\in\Sigma}(\cup\Re_\beta) = \cup\{R : R \in \Re\}$. Hence $\Re$ is a locally finite open refinement of Ψ that covers $\varphi(A)$. Thus $\varphi(A)$ is paracompact.

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