"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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ON PAIRWISE s-COMPACT SPACES

AJOY MUKHARJEE, ARUP ROY CHOUDHURY AND M. K. BOSE

Abstract. We introduce and study the notion of pairwise s-compact spaces in bitopological spaces. The notion of pairwise s-compactness is stronger than the notion of pairwise compactness.

1. Introduction

Due to asymmetric nature of a quasi-metric space, there exists two topologies on a quasi-metric space which was first observed by Kelly [6]. On this observation, Kelly [6] introduced the notion of bitopological spaces. A set X endowed with two topologies $\mathscr{P}_1$ and $\mathscr{P}_2$ is called a bitopological space and it is denoted by $(X, \mathscr{P}_1, \mathscr{P}_2)$. Dochviri [4] introduced the notion of $(\mathscr{P}_i, \mathscr{P}_j)$semi-compactness in a bitopological space: a bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ is said to be $(\mathscr{P}_i, \mathscr{P}_j)$semi-compact $(i, j \in \{1, 2\}, i \neq j)$ if every $(\mathscr{P}_i, \mathscr{P}_j)$semi-open cover of X has a finite subcover. A cover of a bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ is $(\mathscr{P}_i, \mathscr{P}_j)$semi-open if each member of the cover is $(\mathscr{P}_i, \mathscr{P}_j)$semi-open. Balasubramanian [1] also studied the notion of $(\mathscr{P}_i, \mathscr{P}_j)$semi-compactness in the same fashion as of Dochviri [4].

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According to Balasubramanian [1], the bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ is pairwise semi-compact if it is both $(\mathscr{P}_1, \mathscr{P}_2)$semi-compact and $(\mathscr{P}_2, \mathscr{P}_1)$semi-compact. The notion of pairwise semi-compactness is defined by considering covers consisting of only $(\mathscr{P}_i, \mathscr{P}_j)$semi-open sets. So there can not exists a relation between pairwise compactness [5] and pairwise semi-compactness due to Dochviri [4] and Balasubramanian [1].

But the term 'pairwise semi-compactness' envisages that the pairwise semi-compactness should be implied by the notion of pairwise compactness. In this paper, we introduce a notion of pairwise s-compactness (Definition 1.13) in such a way that there exists a relation between pairwise compactness and pairwise s-compactness (see Fig. 1).

Unless otherwise mentioned, X stands for the bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ and Y stands for the bitopological space $(Y, \mathscr{Q}_1, \mathscr{Q}_2)$. $(\mathscr{T})\text{int}A$ (resp. $(\mathscr{T})\text{cl}A$) denotes the interior (resp. closure) of a set A in a topological space $(X, \mathscr{T})$. For a topological space $(X, \mathscr{T})$ and $A \subset X$, we write $(A, \mathscr{T}_A)$ to denote the subspace on A of $(X, \mathscr{T})$. So the relative bitopological space for $(X, \mathscr{P}_1, \mathscr{P}_2)$ corresponding to $A \subset X$ is $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$. Always $i, j \in \{1, 2\}$ and whenever i, j appear together, $j \neq i$. Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$, the set of real numbers.

To make the article as self-contained as possible, we recall the following known definitions.

Definition 1.1 (Fletcher, Hoyle III and Patty [5])**.** A collection $\mathscr{U}$ of X is pairwise open if $\mathscr{U} \subset \mathscr{P}_1 \cup \mathscr{P}_2$ and for each $i \in \{1, 2\}$, $\mathscr{U} \cap \mathscr{P}_i$ contains a nonempty set. $\mathscr{U}$ is said to be a pairwise open cover of X if $\mathscr{U}$ covers X.

Definition 1.2 (Fletcher, Hoyle III and Patty [5])**.** A bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ is pairwise compact if every pairwise open cover of X has a finite subcover.

Definition 1.3 (Mukharjee and Bose [8])**.** A bitopological space X is said to be nearly pairwise compact if for each pairwise open cover $\mathscr{U}$ of X there exists a finite subcollection $\mathscr{V} \subset \mathscr{U}$ such that $\{(\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}V) \mid V \in \mathscr{V} \cap \mathscr{P}_i,\ i \in \{1, 2\}\}$ covers X.

Definition 1.4 (Maheshwari et al. [7] and Bose [2])**.** A subset A of a bitopological space X is said to be $(\mathscr{P}_i, \mathscr{P}_j)$semi-open if there exists a $(\mathscr{P}_i)$open set G such that $G \subset A \subset (\mathscr{P}_j)\text{cl}G$.

The complement of a $(\mathscr{P}_i, \mathscr{P}_j)$semi-open set is called a $(\mathscr{P}_i, \mathscr{P}_j)$semi-closed set. In other words, a subset A of X is $(\mathscr{P}_i, \mathscr{P}_j)$semi-closed if and only there exists a $(\mathscr{P}_i)$closed set F such that $(\mathscr{P}_j)\text{int}F \subset A \subset F$.

Definition 1.5 (Romaguera and Marin [11], p. 237)**.** Let $(X, \mathscr{P}_1, \mathscr{P}_2)$ and $(Y, \mathscr{Q}_1, \mathscr{Q}_2)$ be two bitopological spaces. A function $f : (X, \mathscr{P}_1, \mathscr{P}_2) \to (Y, \mathscr{Q}_1, \mathscr{Q}_2)$ is said to be open if $f : (X, \mathscr{P}_1) \to (Y, \mathscr{Q}_1)$ and $f : (X, \mathscr{P}_2) \to (Y, \mathscr{Q}_2)$ are open.

Definition 1.6 (Swart[12], p. 136)**.** Let $(X, \mathscr{P}_1, \mathscr{P}_2)$ and $(Y, \mathscr{Q}_1, \mathscr{Q}_2)$ be two bitopological spaces. A function $f : (X, \mathscr{P}_1, \mathscr{P}_2) \to (Y, \mathscr{Q}_1, \mathscr{Q}_2)$ is said to be continuous if $f : (X, \mathscr{P}_1) \to (Y, \mathscr{Q}_1)$ and $f : (X, \mathscr{P}_2) \to (Y, \mathscr{Q}_2)$ are continuous.

Definition 1.7 (Bose [2])**.** Let $(X, \mathscr{P}_1, \mathscr{P}_2)$ and $(Y, \mathscr{Q}_1, \mathscr{Q}_2)$ be two bitopological spaces. A function $f : X \to Y$ is said to be semi-open if for each $(\mathscr{P}_i)$open set A in X, $f(A)$ is a $(\mathscr{Q}_i, \mathscr{Q}_j)$semi-open set in Y.

Definition 1.8 (Bose [2])**.** Let $(X, \mathscr{P}_1, \mathscr{P}_2)$ and $(Y, \mathscr{Q}_1, \mathscr{Q}_2)$ be two bitopological spaces. A function $f : X \to Y$ is said to be semi-continuous if for each $(\mathscr{Q}_i)$open set A in Y, $f^{-1}(A)$ is a $(\mathscr{P}_i, \mathscr{P}_j)$semi-open set in X.

In the sequel, we use the following theorems.

Theorem 1.9 (Bose [2])**.** *Let $(X, \mathscr{P}_1, \mathscr{P}_2)$ and $(Y, \mathscr{Q}_1, \mathscr{Q}_2)$ be two bitopological spaces. If $f : X \to Y$ is open and semi-continuous then the inverse image $f^{-1}(B)$ of each $(\mathscr{Q}_i, \mathscr{Q}_j)$semi-open set B in Y is a $(\mathscr{P}_i, \mathscr{P}_j)$semi-open set in X.*

Theorem 1.10 (Bose [2])**.** *Let $(X, \mathscr{P}_1, \mathscr{P}_2)$ and $(Y, \mathscr{Q}_1, \mathscr{Q}_2)$ be two bitopological spaces. If $f : X \to Y$ is continuous and semi-open then $f(B)$ is $(\mathscr{Q}_i, \mathscr{Q}_j)$semi-open in Y if B is $(\mathscr{P}_i, \mathscr{P}_j)$semi-open in X.*

We now introduce the following definitions.

Definition 1.11. A collection $\mathscr{V}$ of subsets of a bitopological space X is said to be pairwise s-open if each member of $\mathscr{V}$ is $(\mathscr{P}_i, \mathscr{P}_j)$semi-open for some $i \in \{1, 2\}$ and for each $i \in \{1, 2\}$, $\mathscr{V}$ contains a nonempty $(\mathscr{P}_i, \mathscr{P}_j)$semi-open set. $\mathscr{V}$ is called a pairwise s-cover of X if it covers X.

Definition 1.12. A collection $\mathscr{F}$ of subsets of a bitopological space X is said to be pairwise s-closed if $\{X - F \mid F \in \mathscr{F}\}$ is pairwise s-open.

Definition 1.13. A bitopological space X is said to be pairwise s-compact if each pairwise s-cover of X has a finite subcover.

Obviously, a pairwise s-compact space is a pairwise compact space.

Example 1.14. Let $a, b \in \mathbb{R}$ with $b > a + 1$. We define

$$\begin{aligned} \mathscr{P}_1 &= \{\emptyset, \mathbb{R}, (-\infty, a], (-\infty, b]\}, \\ \mathscr{P}_2 &= \{\emptyset, \mathbb{R}, [a, \infty), [b, \infty)\}. \end{aligned}$$

The bitopological space $(\mathbb{R}, \mathscr{P}_1, \mathscr{P}_2)$ is pairwise compact but the space is not pairwise s-compact.

Note. We define pairwise s-compactness (Definition 1.13) on considering pairwise s-covers of X. So it may appear that it should be called 'pairwise semi-compact' rather than 'pairwise s-compact'. Generally the word 'semi' is used as a prefix of a notion to mean a weaker notion e.g. 'semi-open sets' which are weaker than open sets. The notion of 'pairwise s-compactness' is stronger than the notion of 'pairwise compactness'. Using the concept of an s-cover which is a cover consisting of semi-open sets, Prasad and Yadav [10] introduced and studied a notion of s-compactness in topological spaces and we see that the notion of s-compactness is stronger than compactness. Hence follows the reasons of naming so the notion introduced in Definition 1.13.

The relevance and importance of the study of pairwise s-compactness follows from the implication relations epitomized in the following diagram of pairwise s-compactness with some other covering properties of bitopological settings. The implications are not reversible.

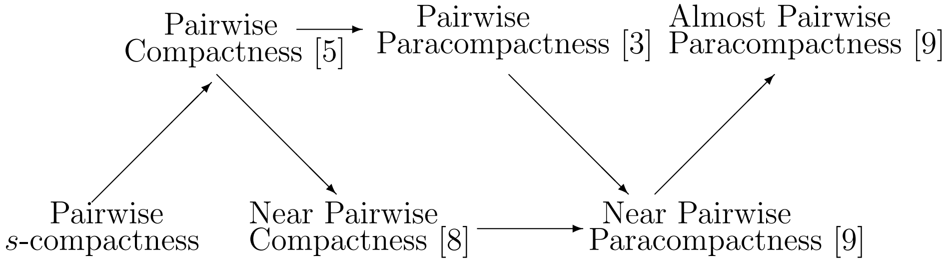

Fig. 1

Note: An arrow between two notions of a bitopological space stands to mean 'implies that'.

2. The Characterizations of Pairwise s-compactness

Theorem 2.1. *If X is pairwise s-compact and F is a proper $(\mathscr{P}_j, \mathscr{P}_i)$semi-closed subset of X, then each $(\mathscr{P}_i, \mathscr{P}_j)$semi-open cover of F has a finite subcover.*

Proof. Let $\mathscr{U} = \{U_\alpha \mid \alpha \in A\}$ be a $(\mathscr{P}_i, \mathscr{P}_j)$semi-open cover of F. Then $\mathscr{U} \cup \{X - F\}$ is a pairwise s-cover of X. Since X is pairwise s-compact, $\mathscr{U} \cup \{X - F\}$ has a finite subcover $\mathscr{V}$ for X. Now a finite subcover for F can be obtained from $\mathscr{V}$ easily. ∎

Lemma 2.2. Let A be $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$ in a bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$. If G is $(\mathscr{P}_{iA}, \mathscr{P}_{jA})$semi-open in $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$, then G is $(\mathscr{P}_i, \mathscr{P}_j)$semi-open in $(X, \mathscr{P}_1, \mathscr{P}_2)$.

Proof. Let G be $(\mathscr{P}_{iA}, \mathscr{P}_{jA})$semi-open in $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$. Then there exists a $(\mathscr{P}_{iA})$open set H such that $H \subset G \subset (\mathscr{P}_{jA})\mathrm{cl}H$. Since H is $(\mathscr{P}_{iA})$open in $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$ and A is $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$ in $(X, \mathscr{P}_1, \mathscr{P}_2)$, H is $(\mathscr{P}_i)$open in $(X, \mathscr{P}_1, \mathscr{P}_2)$. Also we have $(\mathscr{P}_{jA})\mathrm{cl}H = A \cap (\mathscr{P}_j)\mathrm{cl}H \subset (\mathscr{P}_j)\mathrm{cl}H$ and hence the result follows. ∎

Lemma 2.3. Let A be a $(\mathscr{P}_i)$open subset of a bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$. If G is $(\mathscr{P}_i, \mathscr{P}_j)$semi-open in $(X, \mathscr{P}_1, \mathscr{P}_2)$ then $A \cap G$ is $(\mathscr{P}_{iA}, \mathscr{P}_{jA})$semi-open in $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$.

Proof. Let G be $(\mathscr{P}_i, \mathscr{P}_j)$semi-open in $(X, \mathscr{P}_1, \mathscr{P}_2)$. Then there exists a $(\mathscr{P}_i)$open set H such that $H \subset G \subset (\mathscr{P}_j)\mathrm{cl}H$. So $A \cap H \subset A \cap G \subset A \cap A \cap (\mathscr{P}_j)\mathrm{cl}H \subset A \cap (\mathscr{P}_j)\mathrm{cl}(A \cap (\mathscr{P}_j)\mathrm{cl}H) = A \cap (\mathscr{P}_j)\mathrm{cl}(A \cap H) = (\mathscr{P}_{jA})\mathrm{cl}(A \cap H)$. Since $A \cap H$ is $(\mathscr{P}_{iA})$open in $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$, it follows that $A \cap G$ is $(\mathscr{P}_{iA}, \mathscr{P}_{jA})$semi-open in $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$. ∎

Theorem 2.4. *If X is pairwise s-compact and $A \subset X$ is $(\mathscr{P}_i)$closed for some $i \in \{1, 2\}$ and $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$, then A is pairwise s-compact.*

Proof. Let $\mathscr{U}^{(A)} = \{U_\alpha^{(A)} \mid \alpha \in \Delta\}$ be a pairwise s-cover of $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$. Since A is $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$, by Lemma 2.2, $U_\alpha^{(A)}$ is $(\mathscr{P}_i, \mathscr{P}_j)$semi-open in $(X, \mathscr{P}_1, \mathscr{P}_2)$ if $U_\alpha^{(A)}$ is $(\mathscr{P}_{iA}, \mathscr{P}_{jA})$semi-open in $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$. So $\mathscr{U}^{(A)} \cup \{X - A\}$ is a pairwise s-cover of $(X, \mathscr{P}_1, \mathscr{P}_2)$. By pairwise s-compactness of X, we obtain a finite subcover $\mathscr{V}^{(X)}$ of $\mathscr{U}^{(A)} \cup \{X - A\}$. So $\mathscr{V}^{(X)} - \{X - A\}$ is a finite subcover of $\mathscr{U}^{(A)}$. ∎

Theorem 2.5. *Let $(X, \mathscr{P}_1, \mathscr{P}_2)$ be a bitopological space and $A \subset X$ be $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$. Then A is pairwise s-compact if and* *only if each pairwise s-cover of A with respect to* $(X, \mathscr{P}_1, \mathscr{P}_2)$ *has a finite subcover for A.*

Proof. Let A be pairwise s-compact and $\mathscr{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be a pairwise s-cover of A with respect to $(X, \mathscr{P}_1, \mathscr{P}_2)$. For definiteness, let U_α be $(\mathscr{P}_j, \mathscr{P}_i)$semi-open in X. Then by Lemma 2.3, $A \cap U_\alpha$ is $(\mathscr{P}_{jA}, \mathscr{P}_{iA})$semi-open in A. So $\mathscr{U}^{(A)} = \{A \cap U_\alpha \mid \alpha \in \Delta\}$ is a pairwise s-cover of A with respect to $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$. By pairwise s-compactness of A, we obtain a finite subcover $\mathscr{V}^{(A)}$ of $\mathscr{U}^{(A)}$. Let $\mathscr{V}^{(A)} = \{A \cap U_{\alpha_k} \mid k \in \{1, 2, \dots, n\}\}$. Then $A = \bigcup_{k=1}^{n}(A \cap U_{\alpha_k}) \subset \bigcup_{k=1}^{n} U_{\alpha_k}$.

Conversely, let $\mathscr{G} = \{G_\alpha \mid \alpha \in I\}$ be a pairwise s-cover of A with respect to $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$. Then by Lemma 2.2, U_α is $(\mathscr{P}_i, \mathscr{P}_j)$semi-open in X if U_α is $(\mathscr{P}_{iA}, \mathscr{P}_{jA})$semi-open in A. So we obtain a finite subcover $\mathscr{H} = \{G_{\alpha_k} \mid k \in \{1, 2, \dots, m\}\}$ of $\mathscr{G}$ for A. ∎

Let $\mathscr{U}$ be a pairwise s-cover of $(X, \mathscr{P}_1, \mathscr{P}_2)$. Then for each $(\mathscr{P}_i, \mathscr{P}_j)$semi-open set $U \in \mathscr{U}$, there exists a $(\mathscr{P}_i)$open set G such that $G \subset U \subset (\mathscr{P}_j)\text{cl}G$. So it follows that $\mathscr{G} = \{G \mid U \in \mathscr{U}, G \subset U \subset (\mathscr{P}_j)\text{cl}G, i, j \in \{1, 2\}, i \neq j\}$ is a pairwise open collection of $(X, \mathscr{P}_1, \mathscr{P}_2)$ but $\mathscr{G}$ may not be a pairwise open cover of X. We use the term 'pairwise open collection associated to $\mathscr{U}$' for $\mathscr{G}$.

Theorem 2.6. *A bitopological space X is pairwise s-compact if X is pairwise compact and if for each pairwise s-cover* $\mathscr{U}$ *of X there exists a pairwise open collection associated to* $\mathscr{U}$ *that is a cover of X.*

Proof. Let $\mathscr{U}$ be a pairwise s-cover of X. By hypothesis, we have a pairwise open cover $\mathscr{G}$ associated to $\mathscr{U}$. So for each $(\mathscr{P}_i, \mathscr{P}_j)$semi-open set $U \in \mathscr{U}$, there exists a $(\mathscr{P}_i)$open set $G \in \mathscr{G}$ such that $G \subset U \subset (\mathscr{P}_j)\text{cl}G$. Since X is pairwise compact, $\mathscr{G}$ has a finite subcover and hence $\mathscr{U}$ has a finite subcover. Thus X is pairwise s-compact. ∎

Theorem 2.7. *Let X be pairwise s-compact. Then for each pairwise open collection* $\mathscr{G}$ *associated to a pairwise s-cover of X there exists a finite subcollection* $\mathscr{H} \subset \mathscr{G}$ *such that* $\{(\mathscr{P}_j)clH \mid H \in \mathscr{H} \cap \mathscr{P}_i, i \in \{1, 2\}, i \neq j\}$ *covers X.*

Proof. Let $\mathscr{U}$ be a pairwise s-cover of X and $\mathscr{G}$ be a pairwise open collection associated to $\mathscr{U}$. Since X is pairwise s-compact, there exists a finite subcover $\mathscr{V} = \{V_k \mid k \in \{1, 2, \dots, n\}\}$ of $\mathscr{U}$. Now for each $(\mathscr{P}_i, \mathscr{P}_j)$semi-open set $V_k \in \mathscr{V}$, there exists a $(\mathscr{P}_i)$open set $G_k \in \mathscr{G}$ such that $G_k \subset V_k \subset (\mathscr{P}_j)\text{cl}G_k$. So $\mathscr{G}_n = \{G_k \mid k \in \{1, 2, \dots, n\}\}$ is a finite subcollection of $\mathscr{G}$. Since $\mathscr{V}$ is a subcover of $\mathscr{U}$, it follows that $\{(\mathscr{P}_j)\mathrm{cl}G_k \mid G_k \in \mathscr{G}_n \cap \mathscr{P}_i, i \in \{1,2\}\}$ covers X. ∎

Definition 2.8. Let A be a $(\mathscr{P}_i, \mathscr{P}_j)$semi-open set in X. A is said to be $(\mathscr{P}_i)$covered if there exist $(\mathscr{P}_i)$open sets G and H such that $G \subset A \subset H \subset (\mathscr{P}_j)\mathrm{cl}G$. A is said to be $(\mathscr{P}_i)$uncovered if $G \subset A \subset (\mathscr{P}_j)\mathrm{cl}G$ for some $(\mathscr{P}_i)$open set G, then $G \subset (\mathscr{P}_i)\mathrm{int}((\mathscr{P}_j)\mathrm{cl}G) \subset A \subset (\mathscr{P}_j)\mathrm{cl}G$.

In Example 1.14, (c, ∞) where $a < c < b$ is a $(\mathscr{P}_2, \mathscr{P}_1)$semi-open set and it is $(\mathscr{P}_2)$uncovered.

Example 2.9. For $b \in \mathbb{R}$, we define

$$\begin{aligned}\mathscr{P}_1 &= \{\emptyset, \mathbb{R}, \mathbb{R} - \{b\}, (-\infty, b), (b, \infty,)\},\\ \mathscr{P}_2 &= \{\emptyset, \mathbb{R}, (b, \infty)\} \bigcup \left\{ \left(b + \frac{1}{n}, \infty\right) \mid n \in \mathbb{N} \right\}.\end{aligned}$$

In the bitopological space $(\mathbb{R}, \mathscr{P}_1, \mathscr{P}_2)$, $\left[b + \frac{1}{n}, \infty\right)$ is a $(\mathscr{P}_2, \mathscr{P}_1)$semi-open set covered by a $(\mathscr{P}_2)$open set.

Theorem 2.10. *Let $(X, \mathscr{P}_1, \mathscr{P}_2)$ be a bitopological space and each $(\mathscr{P}_i, \mathscr{P}_j)$semi-open set $(i, j \in \{1,2\}, i \neq j)$ in X is $(\mathscr{P}_i)$uncovered. Then X is pairwise s-compact if for each pairwise s-cover $\mathscr{A}$ of X there exists a pairwise open cover associated to $\mathscr{A}$ and X is nearly pairwise compact.*

Proof. Let $\mathscr{A} = \{A_\alpha \mid \alpha \in \Delta\}$ be a pairwise s-cover of X and let $\mathscr{G} = \{G_\beta \mid \beta \in B\}$ be a pairwise open cover associated to $\mathscr{A}$. Since each $(\mathscr{P}_i, \mathscr{P}_j)$semi-open set $A_\alpha \in \mathscr{A}$ is $(\mathscr{P}_i)$uncovered, there exists a $(\mathscr{P}_i)$open set $G_{\beta(\alpha)} \in \mathscr{G}$ such that $G_{\beta(\alpha)} \subset (\mathscr{P}_i)\mathrm{int}((\mathscr{P}_j)\mathrm{cl}G_{\beta(\alpha)}) \subset A_\alpha \subset (\mathscr{P}_j)\mathrm{cl}G_{\beta(\alpha)}$. Since $\mathscr{G}$ is a pairwise open cover of X and X is nearly pairwise compact, there exists a finite subcollection $\mathscr{G}_m = \{G_{\beta_k} \mid k \in \{1, 2, \ldots, m\}, \beta_k \in B\}$ such that $\{(\mathscr{P}_i)\mathrm{int}((\mathscr{P}_j)\mathrm{cl}G_{\beta_k}) \mid G_{\beta_k} \in \mathscr{G}_m \cap \mathscr{P}_i, i \in \{1,2\}\}$ covers X. For each $G_{\beta_k} \in \mathscr{G}_m \cap \mathscr{P}_i$, there exists a $(\mathscr{P}_i, \mathscr{P}_j)$semi-open set $A_{\alpha_k} \in \mathscr{A}$ such that $(\mathscr{P}_i)\mathrm{int}((\mathscr{P}_j)\mathrm{cl}G_{\beta_k}) \subset A_{\alpha_k}$. So $\{A_{\alpha_k} \mid k \in \{1, 2, \ldots, m\}, \alpha_k \in \Delta\}$ is a finite subcover of $\mathscr{A}$. ∎

Definition 2.11. A bitopological space X is said to be natural if there exist a $(\mathscr{P}_1)$open set $G \neq X$ and a $(\mathscr{P}_2)$open set $H \neq X$ such that $(\mathscr{P}_2)\mathrm{cl}G \cup (\mathscr{P}_1)\mathrm{cl}H = X$.

The bitopological space of Example 2.9 is natural.

Theorem 2.12. *A pairwise s-compact space is natural if there exists a pairwise open collection associated to some pairwise s-cover.*

Proof. Let $(X, \mathscr{P}_1, \mathscr{P}_2)$ be pairwise s-compact and $\mathscr{U}$ be a pairwise s-cover of X. Suppose $\mathscr{G}$ is a pairwise open collection associated to $\mathscr{U}$ and $\mathscr{G}$ is not a cover of X. For each $G \in \mathscr{G} \cap \mathscr{P}_i$, it follows that $G \neq X$, and there exists a $U \in \mathscr{U}$ such that $G \subset U \subset (\mathscr{P}_j)\mathrm{cl}G$. Since X is pairwise s-compact, there exists a finite subcover $\mathscr{U}_n = \{U_1, U_2, \ldots, U_n\}$ of $\mathscr{U}$. As $\mathscr{G}_n = \{G_1, G_2, \ldots, G_n\}$ is a subcollection of $\mathscr{G}$, $\mathscr{G}_n$ is not a cover of X. But $\{(\mathscr{P}_j)\mathrm{cl}G_k \mid G_k \in \mathscr{G}_n \cap \mathscr{P}_i, i \in \{1,2\}\}$ is a cover of X. We put $A = \bigcup\{G_k \mid G_k \in \mathscr{G}_n \cap \mathscr{P}_1\}$ and $B = \bigcup\{G_k \mid G_k \in \mathscr{G}_n \cap \mathscr{P}_2\}$. Obviously, $A \in \mathscr{P}_1, B \in \mathscr{P}_2$ and $A, B \neq X$. Also $(\mathscr{P}_2)\mathrm{cl}A = (\mathscr{P}_2)\mathrm{cl}(\bigcup\{G_k \mid G_k \in \mathscr{G}_n \cap \mathscr{P}_1\}) = \bigcup\{(\mathscr{P}_2)\mathrm{cl}G_k \mid G_k \in \mathscr{G}_n \cap \mathscr{P}_1\}$ and $(\mathscr{P}_1)\mathrm{cl}B = \bigcup\{(\mathscr{P}_1)\mathrm{cl}G_k \mid G_k \in \mathscr{G}_n \cap \mathscr{P}_2\}$. So we have $(\mathscr{P}_2)\mathrm{cl}A \cup (\mathscr{P}_1)\mathrm{cl}B = X$. ∎

Theorem 2.13. *A bitopological space X is pairwise s-compact if and only if each pairwise s-closed collection of subsets of X with finite intersection property has a nonempty intersection.*

Proof. Firstly, let X be pairwise s-compact and $\mathscr{F} = \{F\alpha \mid \alpha \in A\}$ be a pairwise s-closed collection of subsets of X with finite intersection property i.e. for each finite subcollection $\mathscr{E}$ of $\mathscr{F}$, $\bigcap\{E \mid E \in \mathscr{E}\} \neq \emptyset$. If possible, let $\bigcap\{F \mid F \in \mathscr{F}\} = \emptyset$. Then $\mathscr{G} = \{X - F\alpha \mid \alpha \in A\}$ is a pairwise s-cover of X. So there exists a finite subcover $\mathscr{G}_n = \{X - F_{\alpha_k} \mid \alpha_k \in \Delta, k \in \{1, 2, \ldots, n\}\}$ of $\mathscr{G}$. Thus we get $X - \bigcup\{X - F_{\alpha_k} \mid \alpha_k \in \Delta, k \in \{1, 2, \ldots, n\}\} = \emptyset$ which in turn implies that $\bigcap\{F_{\alpha_k} \mid \alpha_k \in \Delta, k \in \{1, 2, \ldots, n\}\} = \emptyset$, a contradiction.

Conversely, let $\mathscr{U} = \{U\beta \mid \beta \in B\}$ be a pairwise s-cover of X. If possible, let $\mathscr{U}$ does not have a finite subcover. So for any finite subcollection $\mathscr{V}$ of $\mathscr{U}$, we have $\bigcup\{G \mid G \in \mathscr{V}\} \neq X$ which in turn implies that $\bigcap\{X - G \mid G \in \mathscr{V}\} \neq \emptyset$. Hence $\mathscr{F} = \{X - U\beta \mid \beta \in B\}$ is a pairwise s-closed collection of subsets of X with finite intersection property. Accordingly, we have $\bigcap\{X - U\beta \mid \beta \in B\} \neq \emptyset \Rightarrow X - \bigcap\{X - U\beta \mid \beta \in B\} \neq X \Rightarrow \bigcup\{U\beta \mid \beta \in B\} \neq X$, a contradiction. ∎

3. Preservation Theorems on Pairwise s-compactness

Theorem 3.1. *Pairwise s-compactness is preserved under semi-continuous, open and onto mappings.*

Proof. Let $(X, \mathscr{P}_1, \mathscr{P}_2)$ and $(Y, \mathscr{Q}_1, \mathscr{Q}_2)$ be two bitopological spaces, and let $f : X \to Y$ be a semi-continuous, open and onto mapping.

Suppose $\mathscr{U}^{(Y)} = \{U_\alpha \mid \alpha \in A\}$ is a pairwise s-cover of Y. By Theorem 1.9, $\mathscr{U}^{(X)} = \{f^{-1}(U\alpha) \mid \alpha \in A\}$ is a pairwise s-cover of X. Since X is pairwise s-compact, there exists a finite subcover $\mathscr{V}^{(X)} = \{f^{-1}(U_{\alpha_k}) \mid \alpha_k \in A,\ k = 1, 2, \ldots, n\}$ of $\mathscr{U}^{(X)}$ for X. Now we have

$$\begin{aligned} Y &= f(X) \\ &= f\left(\bigcup\nolimits_{k=1}^{n} \{f^{-1}(U_{\alpha_k}) \mid k \in \{1, 2, \ldots, n\}\}\right) \\ &= \bigcup\nolimits_{k=1}^{n} \{f\left(f^{-1}(U_{\alpha_k})\right) \mid k \in \{1, 2, \ldots, n\}\} \\ &= \bigcup\nolimits_{k=1}^{n} \{U_{\alpha_k} \mid k \in \{1, 2, \ldots, n\}\} \quad \text{(since } f \text{ is onto).} \end{aligned}$$

Therefore Y is pairwise s-compact. ∎

Theorem 3.2. *Assume that there exists a continuous and semi-open mapping $f : X \to Y$ such that $f(X) = Y$. Then X is pairwise s-compact if Y is pairwise s-compact.*

Proof. Let $\mathscr{U}^{(X)} = \{U\alpha \mid \alpha \in \Delta\}$ be a pairwise s-cover of X. By Theorem 1.10, $f(U_\alpha)$ is $(\mathscr{Q}_i, \mathscr{Q}_j)$semi-open in Y if U_α is $(\mathscr{P}_i, \mathscr{P}_j)$semi-open in X. So $\mathscr{U}^{(Y)} = \{f(U\alpha) \mid \alpha \in \Delta\}$ is a pairwise s-cover of Y. Since Y is pairwise s-compact, we obtain a finite subcover $\mathscr{V}^{(Y)} = \{f(U_{\alpha_k}) \mid \alpha_k \in \Delta, k \in \{1, 2, \ldots, n\}\}$ of $\mathscr{U}^{(Y)}$ for Y. Since $Y = f(X)$, it follows that $\mathscr{V}^{(X)} = \{U_{\alpha_k} \mid \alpha_k \in \Delta, k \in \{1, 2, \ldots, n\}\}$ is a finite subcover of $\mathscr{U}^{(X)}$ for X. ∎

Theorem 3.3. *Assume that there exists a semi-open mapping $f : X \to Y$ such that $f(X) = Y$. Then X is pairwise compact if Y is pairwise s-compact.*

Proof. Similar to the proof of Theorem 3.2. ∎

Definition 3.4. Let $(X, \mathscr{P}_1, \mathscr{P}_2)$ and $(Y, \mathscr{Q}_1, \mathscr{Q}_2)$ be two bitopological spaces. A function $f : (X, \mathscr{P}_1, \mathscr{P}_2) \to (Y, \mathscr{Q}_1, \mathscr{Q}_2)$ is said to be $(*)$open if for each $(\mathscr{P}_i, \mathscr{P}_j)$semi-open set A in X, $f(A)$ is $(\mathscr{Q}_i)$open in Y.

It follows that every $(*)$open function is an open function.

Definition 3.5. Let $(X, \mathscr{P}_1, \mathscr{P}_2)$ and $(Y, \mathscr{Q}_1, \mathscr{Q}_2)$ be two bitopological spaces. A function $f : (X, \mathscr{P}_1, \mathscr{P}_2) \to (Y, \mathscr{Q}_1, \mathscr{Q}_2)$ is said to be $(*)$continuous if the inverse image $f^{-1}(A)$ of each $(\mathscr{Q}_i, \mathscr{Q}_j)$semi-open set A in Y is $(\mathscr{P}_i)$open in X.

It follows that every $(*)$continuous function is a continuous function.

We consider the bitopological space of Example 2.9. Let $f : \mathbb{R} \to \mathbb{R}$ be the identity mapping. Then the function is both open and continuous. But the function is neither $(*)$open nor $(*)$continuous.

Theorem 3.6. *Assume that there exists a* $(*)$*open mapping* $f : X \to Y$ *such that* $f(X) = Y$. *Then* X *is pairwise s-compact if* Y *is pairwise compact.*

Proof. Similar to the proof of Theorem 3.2. ∎

Theorem 3.7. *Assume that there exists a* $(*)$*continuous mapping* $f : X \to Y$ *such that* $f(X) = Y$. *Then* Y *is pairwise s-compact if* X *is pairwise compact.*

Proof. Similar to the proof of Theorem 3.1. ∎

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Ajoy Mukharjee, Department of Mathematics, St. Joseph's College, Darjeeling, W. Bengal- 734 104, INDIA
E-mail address: ajoyjee@gmail.com

Arup Roy Choudhury, Department of Mathematics, Malda College, Malda, W. Bengal- 732 101, INDIA
E-mail address: roychoudhuryarup@yahoo.co.in

M. K. Bose, Department of Mathematics, University of North Bengal, Siliguri, W. Bengal- 734 013, INDIA
E-mail address: manojkumarbose@yahoo.com