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STRONGLY GENERALIZED (WEAKLY) $\delta-$SUPPLEMENTED MODULES

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Abstract. In this paper, we introduce strongly generalized (weakly) $\delta-$supplemented modules. We call a module strongly generalized (weakly) $\delta-$supplemented (briefly $\delta-$SGS ($\delta-$SWGS)) if every submodule containing the $\delta-$radical has a (weak) δ-supplement. The first part of this paper investigates various properties of $\delta-$SGS modules. We prove that $\delta-$SGS modules are closed under factor modules and finite sums.Using these modules, we show that a ring R is $\delta-$semiperfect if and only if every left $R-$module is a $\delta-$SGS module. The second part of this paper establishes some properties of $\delta-$SWGS modules.

1. Introduction and preliminaries

Throughout this paper, R will be an associative ring with identity and all modules will be unital left $R-$modules unless otherwise specified. Let M be an $R-$module. By $N \subseteq M$ we mean that N is a submodule of M. Recall that a submodule $N \subseteq M$ is called *small*, denoted by $N \ll M$, if $N + L \neq M$ for all proper submodules L of M. Furthermore a submodule L of M is said to be *essential* in M, denoted by $L \trianglelefteq M$, if $L \cap K \neq 0$ for each nonzero submodule $K \subseteq M$. By $Rad\,(M)$ we denote the sum of all small submodule of M. A module M is said to be *singular* if $M \cong \frac{N}{L}$ for some module N and a submodule $L \subseteq N$ with $L \trianglelefteq N$.

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As a generalization of small submodules, $\delta-$small submodules were introduced in [13]. According to [13], a submodule L of M is called $\delta-$*small* in M, denoted by $L \ll_\delta M$, if for any submodule N of M with $\frac{M}{N}$ singular, $M = N + L$ implies $M = N$. The sum of all $\delta-$small submodules of a module M is denoted by $\delta(M)$. It is easy to see that every small submodule of a module M is $\delta-$small in M, so $Rad(M) \subseteq \delta(M)$ and $Rad(M) = \delta(M)$ if M is singular. Also any non-singular semisimple submodule of M is $\delta-$small in M and $\delta-$small submodules of a singular module are small submodules. For more detailed discussion on $\delta-$small submodules we refer to [13].

Let K, N be submodules of a module M. Then N is called a $\delta-$*supplement* of K in M, if $N+K = M$ and $N \cap K \ll_\delta N$. N is called a *weak* $\delta-$*supplement* of K in M, if $N + K = M$ and $N \cap K \ll_\delta M$. A module M is called $\delta-$*supplemented* if every submodule of M has a $\delta-$supplement in M. Also M is called *weakly* $\delta-$*supplemented* (briefly $\delta-$WS)if every submodule of M has a weak $\delta-$supplement in M [3, 11].

Let M be an $R-$module and let N and K be any submodules of M with $M = N+K$. If $N \cap K \leq \delta(N)$ ($N \cap K \leq \delta(M)$) then N is called a generalized (weak) $\delta-$supplement of K in M . Following [7], M is called a *generalized* $\delta-$*supplemented* module (briefly $\delta-GS$ module) if every submodule N of M has a generalized $\delta-$ supplemented K in M. In [7], an $R-$module M is called *generalized weakly* $\delta-$*supplemented* (briefly $\delta-GWS$ module) ($\delta-WGS$ module in [7]) if every submodule K of M has a generalized weak $\delta-$supplement N in M. Some properties of these modules were given in [5].

In [1], the authors studied strongly radical supplemented (briefly srs) modules. They were called a module *strongly radical supplemented* if every submodule containing the radical has a supplement. Motivated by this definition, we study strongly generalized (weakly) $\delta-$supplemented modules.

2. Main results

In this section, we will define strongly generalized $\delta-$*supplemented* modules as a generalization $\delta-GS$-modules and srs-modules by using Zhou's radical and investigate some properties of these modules.

Definition 2.1. *Let M be a module and N be a submodule of M which contains $\delta(M)$. If N has a $\delta-$supplement in M, then M is called strongly generalized $\delta-$supplemented ($\delta-SGS$) module.*

Proposition 2.1. *Every homomorphic image of a $\delta-$SGS module is a $\delta-$SGS module.*

Proof. Let M be a $\delta-$SGS module and $L \subseteq N \subseteq M$ with $\delta(M/L) \subseteq N/L$. By virtue of [11, Proposition 4.2], $(\delta(M)+L)/L \subseteq \delta(M/L)$ and $\delta(M) \subseteq N$. Since M is a $\delta-$SGS module and $N \subseteq M$, we have that N has a $\delta-$supplement K in M. Then $(K+L)/L$ is a $\delta-$supplement of N/L in M/L by [4, Proposition 2.7(4)]. Hence, M/L is a $\delta-$SGS module. ∎

Proposition 2.2. *If M is a $\delta-$SGS module, then $M/\delta(M)$ is semisimple.*

Proof. As a result of Proposition 2.1, we can conclude that $M/\delta(M)$ is a $\delta-$SGS module. Since $\delta(M/\delta(M)) = 0$, we get that $M/\delta(M)$ is $\delta-$supplemented. Because, every submodule of $M/\delta(M)$ is a direct summand, $M/\delta(M)$ is semisimple. ∎

To prove the finite sum of $\delta-$SGS modules is a $\delta-$SGS module, we need the following lemma.

Lemma 2.1. *Let M be a module, M_1 and N be a submodules of M with $\delta(M) \subseteq N$. If M_1 is a $\delta-$SGS module and M_1+N has a $\delta-$supplement in M, then N has a $\delta-$supplement in M.*

Proof. Let L be a $\delta-$supplement of M_1+N in M. Since $\delta(M_1) \subseteq \delta(M) \subseteq N$, we have $\delta(M_1) \subseteq (L+N)\cap M_1$. Then $(L+N)\cap M_1$ has a $\delta-$supplement K in M_1 because M_1 is a $\delta-$SGS module. Therefore, we have

$$M = M_1+N+L = K+[((L+N)\cap M_1)]+N+L = (K+N)+L.$$

Since $K+N \subseteq M_1+N$, we can conclude that L is also a $\delta-$supplement of $K+N$ in M. Therefore, according to [4, Proposition 2.7(1)], $K+L$ is a $\delta-$supplement of N in M. ∎

Proposition 2.3. *Let $M = M_1+M_2$, where M_1 and M_2 are $\delta-$SGS modules. Then M is a $\delta-$SGS module.*

Proof. Suppose that $N \subseteq M$ with $\delta(M) \subseteq N$. It is easy to see that M_1+M_2+N has the trivial $\delta-$supplement 0 in M. Therefore, M_1+N has a $\delta-$supplement in M by Lemma 2.1. Applying lemma once more, we obtain a $\delta-$supplement for N in M. ∎

Corollary 2.1. *Every finite sum of $\delta-$SGS modules is a $\delta-$SGS module.*

Recall that a module M is called *$\delta-$radical* if $M = \delta(M)$ and $P_\delta(M)$ denotes the sum of all $\delta-$radical submodules of M, i.e., $P_\delta(M) = \sum\{U \subseteq M \delta(U) = U\}$[8].

Lemma 2.2. *Let M be a module with $M = \delta(M)$. Then M is a $\delta-SGS$ module.*

Proof. Clearly, M has the trivial $\delta-$supplement 0 in M. Since $M = \delta(M)$ is the unique submodule containing the $\delta-$radical, we can conclude that M is a $\delta-$SGS module. ∎

Corollary 2.2. *$P_\delta(M)$ is a $\delta-SGS$ module for any module M.*

Proof. For any module M, it is well known that $\delta(P_\delta(M)) = P_\delta(M)$. Then, the result follows by Lemma 2.2. ∎

The examples below show that $\delta-$SGS modules need not to be $\delta-$ supplemented and supplemented.

Example 2.1. *Let $R = \mathbb{Z}$ and $M = \bigoplus\limits_{i=1}^{\infty} M_i$ with each $M_i = \mathbb{Z}_{p^\infty}$ (the Prüfer group), where p is a prime number. Then M is a δ-SGS module because $\delta(M) = \bigoplus\limits_{i=1}^{\infty} \delta(M_i) = \bigoplus\limits_{i=1}^{\infty} M_i = M$. On the other hand, M is not $\delta-$supplemented as shown in Example 2.14 [3].*

Example 2.2. *Consider the $\mathbb{Z}-$module $\mathbb{Q}$. Since $\delta(\mathbb{Q}) = \mathbb{Q}$, $\mathbb{Q}$ is a δ-SGS module but $\mathbb{Q}$ is not supplemented by [14, Theorem 3.1]*

Proposition 2.4. *Let M be a module with $\delta(M) \ll_\delta M$. In this case, M is $\delta-$supplemented if and only if M is a $\delta-SGS$ module.*

Proof. In one direction, the statement is obvious. Suppose that M is a $\delta-$SGS module and N a submodule of M. Then $N + \delta(M)$ has a $\delta-$supplement L in M. Hence $M = N + \delta(M) + L$ and $(N + \delta(M)) \cap L \ll_\delta L$. Since $\delta(M) \ll_\delta M$, we have $M = N + L$. If we consider Lemma 1.3(a) in [13], then we obtain that $N \cap L \subseteq (N + \delta(M)) \cap L \ll_\delta L$, i.e. $N \cap L \ll_\delta L$. Therefore N has a $\delta-$supplement L in M and M is $\delta-$supplemented. ∎

Proposition 2.5. *If M is a $\delta-SGS$ module and $\delta(M)$ is $\delta-$ supplemented, then M is $\delta-$supplemented.*

Proof. Let N be a submodule of M. Being a $\delta-$SGS module of M implies that, $\delta(M) + N$ has a $\delta-$supplement in M. Since $\delta(M)$ is $\delta-$supplemented, N has a $\delta-$supplement in M by virtue of [11, Lemma 3.4]. Hence M is a $\delta-$supplemented. ∎

Proposition 2.6. *Let M be a module and $U, V \subseteq M$. If V is a $\delta-$supplement of U in M and $\delta(V) \subseteq U$, then $\delta(V) \ll_\delta V$.*

Proof. Suppose that $T+\delta(V) = V$ for some $T \subseteq V$ with V/T singular. Then $M = U+V = U+\delta(V)+T = U+T$. Since V is a $\delta-$supplement of U in M, we have $T = V$ by Lemma 2.1 of [4]. Therefore $\delta(V) \ll_\delta V$. ∎

Proposition 2.7. *Let M be a module and $\delta(M) \subseteq U \subseteq M$. If V is a $\delta-$supplement of U in M, then $\delta(V) \ll_\delta V$.*

Proof. Since $\delta(M) \subseteq U$, we have $\delta(V) \subseteq U$. Then $\delta(V) \ll_\delta V$ by Proposition 2.6. ∎

Corollary 2.3. *Let M be a module and let $N \subseteq M$ be such that $\delta(M) \subseteq N$. Suppose that $M = N + L$ for some $L \subseteq M$. In this case, L is a $\delta-$supplement of N in M if and only if L is a generalized $\delta-$supplement of N and $\delta(L) \ll_\delta L$.*

A submodule N of a module M is said to be *cofinite* if M/N is finitely generated. An $R-$module M is called *cofinitely $\delta-$supplemented*, if each cofinite submodule of M has a $\delta-$supplement in M. By this small note, we can write the following which shows the relation between $\delta-$SGS modules and cofinitely $\delta-$supplement modules.

Proposition 2.8. *Let M be a module and $M/\delta(M)$ is finitely generated. In this case, if M is cofinitely $\delta-$supplemented, then M is a $\delta-$SGS module.*

Proof. Let N be a submodule of M with $\delta(M) \subseteq N$. Note that $[M/\delta(M)]/[N/\delta(M)] \cong M/N$ is finitely generated, and so N is a cofinite submodule of M. Since M is cofinitely $\delta-$supplemented, N has a $\delta-$supplement in M. Therefore M is a $\delta-$SGS module. ∎

Now we characterize the rings over which all(finitely generated) modules are $\delta-$SGS modules.

Corollary 2.4. *For a ring R, the following statements are equivalent.*

(i) R is $\delta-$semiperfect.
(ii) ${}_RR$ is a $\delta-$SGS module.
(iii) Every finitely generated left $R-$module is a $\delta-$SGS module.

Proof. For every finitely generated module M, we have $\delta(M) \ll_\delta M$ [9, Lemma 2.1]. On the other hand, according to [3, Theorem 3.3], R

is $\delta-$semiperfect if and only if every finitely generated $R-$module is $\delta-$supplemented. In view of this fact and Proposition 2.4, the implications (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) are obvious. ∎

Example 2.3. *Let F be a field, $I = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and*

$$R = \{(x_1, x_2, ..., x_n, x, x, ...) \,|n \in \mathbb{N},\ x_i \in M_2(F),\ x \in I\},$$

with component-wise operations, R is a ring. By example 4.3 in [13], R is a $\delta-$perfect and so $\delta-$semiperfect. By implication, ${}_RR$ is a $\delta-SGS-$module.

It is clear that every $srs-$module is a $\delta-$SGS$-$module. But the following example shows that the converse is not true in general.

Example 2.4. *Let $Q = \Pi_{i=1}^{\infty} F_i$, where each $F_i = \mathbb{Z}_2$. Let R be the subring of Q generated by $\oplus_{i=1}^{\infty} F_i$ and 1_Q.Then R is $\delta-$semiperfect and so ${}_RR$ is a $\delta-SGS$ module. If we consider Corollary 2.5 in [1], then ${}_RR$ is not a $srs-$module because R is not semiperfect.[13, Example 4.1]*

Proposition 2.9. *Let M be a module and K be a submodule of M. If K and M/K are $\delta-SGS$ modules and K has a $\delta-$supplement L in P for every submodule with $K \subseteq P \subseteq M$,then M is a $\delta-SGS$ module.*

Proof. Let N be a submodule of M with $\delta(M) \subseteq N$. It is easy to see that $\delta(M/K) = (\delta(M) + K)/K \subseteq (N + K)/K$. Since M/K is a $\delta-$SGS module, we can conclude that $(N + K)/K$ has a $\delta-$supplement in M/K. This means that there exists a submodule V/K of M/K such that $(N + K)/K + V/K = M/K$ and $[(N + K)/K] \cap [V/K] \ll_\delta V/K$. Since $K \subseteq V$, we can say K has a $\delta-$supplement in V. Therefore $V = K + L$ and $K \cap L \ll_\delta L$ for some $L \subseteq V$. Now we have

$$M = N + V = N + (K + L) = (N + K) + L.$$

Suppose that $M = (N + K) + L'$ for some $L' \subseteq L$. Then $M/K = (N + K)/K + \left(L' + K\right)/K$. However V/K is a $\delta-$supplement of $(N + K)/K$ in M/K and $\left(L' + K\right)/K \subseteq V/K$. By Lemma 2.1 of [4], we get $\left(L' + K\right)/K = V/K$ and so $L' + K = V$. Since L is a $\delta-$supplement of K in V, we have $L' = L$ by Lemma 2.1 in [4]. Therefore, L is a $\delta-$supplement of $N + K$ in M and N has a $\delta-$supplement in M by the same lemma. As a result, M is a $\delta-$SGS module. ∎

2.1. **Strongly Generalized Weakly $\delta-$Supplemented Modules.**

Definition 2.2. *Let M be a module and N be a submodule of M which contains $\delta(M)$. If N has a weak $\delta-$supplement in M, then M is called strongly generalized weakly $\delta-$supplemented ($\delta-SGWS$) module.*

Proposition 2.10. *Let M be a $\delta-SGWS$ module which contains its $\delta-$radical as a $\delta-$small submodule. Then, M is $\delta-WS$ module.*

Proof. Let $U \subseteq M$. By the hypothesis, $\delta(M) + U$ has a weak $\delta-$supplement V in M. Then, $M = (\delta(M) + U) + V$ and $(\delta(M) + U) \cap V \ll_\delta M$. Since $\delta(M) \ll_\delta M$, we obtain that $M = U + V$. Clearly $U \cap V \subseteq (\delta(M) + U) \cap V$. Applying [13, Lemma1.3(a)] we get the result $U \cap V \ll_\delta M$. Therefore, V is a weak $\delta-$supplement of U in M. Hence M is weakly $\delta-$supplemented. ∎

A module M is said to be $\delta-$*coatomic* if every proper submodule K of M is contained in a maximal submodule N with M/N singular. Every $\delta-$coatomic module has a $\delta-$small radical [Lemma 2.3(2), 2].

Corollary 2.5. *Let M be a $\delta-$coatomic module. Then, M is a $\delta-SGWS$ module if and only if it is weakly $\delta-$supplemented.*

Recall that a module M over an arbitrary ring is said to be $\delta-$local if $\delta(M) \ll_\delta M$ and $\delta(M)$ is a maximal submodule of M [10].

Corollary 2.6. *Let M be a $\delta-$local module. Then, M is a $\delta-SGWS$ module if and only if it is $\delta-WS$ module.*

We will call a module M is *cofinitely weak $\delta-$ supplemented* (or briefly $\delta-CWS$-module) if every cofinite submodule of M has a weak $\delta-$supplement.

Proposition 2.11. *Let M be a $\delta-CWS$ module with cofinite radical. Then M is a $\delta-SGWS$ module.*

Proof. Let U be a submodule of M with $\delta(M) \subseteq U$. Note that

$$[M/\delta(M)] / [U/\delta(M)] \cong M/U$$

is finitely generated, and so U is a cofinite submodule of M. Applying our assumption, we conclude that M is a $\delta-$SGWS module. ∎

Proposition 2.12. *Every homomorphic image of a $\delta-SGWS$ module is a $\delta-SGWS$ module.*

Proof. Let $f : M \to N$ be a homomorphism and L be a submodule of $f(M)$ with $\delta(f(M)) \subseteq L$. Then $\delta(M) \subseteq f^{-1}(L)$. By our assumption, $f^{-1}(L)$ has a weak $\delta-$supplement K in M. Therefore $f^{-1}(L) + K = M$ and $f^{-1}(L) \cap K \ll_\delta M$. It follows that $L + f(K) = f(M)$. Note that $f(f^{-1}(L) \cap K) = L \cap f(K) \ll_\delta f(M)$. This means that L has a weak $\delta-$supplement in M. This completes the proof. ∎

To prove that a finite sum of $\delta-$SGWS modules is a $\delta-$SGWS module, we use the following lemma.

Lemma 2.3. *Let M be a module and M_1, N be submodules of M with $\delta(M) \subseteq N$. If $M_1 + N$ has a weak $\delta-$supplement L in M and $M_1 \cap (N + L)$ has a weak $\delta-$supplement V in M_1, then $V + L$ is a weak $\delta-$supplement of N in M.*

Proof. By the hypothesis, we have $M = (M_1 + N) + L$ and $(M_1 + N) \cap L \ll_\delta M$.Since V is a weak supplement of $M_1 \cap (N + L)$ in M_1, we can write $M_1 = [M_1 \cap (N + L)] + V$ and $V \cap (N + L) \ll_\delta M_1$.Then

$$M = (M_1 + N) + L = [(M_1 \cap (N + L) + V) + N] + L = N + (V + L)$$

and by [13, Lemma 1.3 (a), (b)]

$$\begin{aligned} N \cap (V + L) &\subseteq (N + V) \cap L + (N + L) \cap V \\ &\subseteq (M_1 + N) \cap L + (N + L) \cap V \ll_\delta M. \end{aligned}$$

Hence $V + L$ is a weak $\delta-$supplement of N in M. ∎

Proposition 2.13. *Let $M = \sum_{i=1}^{n} M_i$, where each M_i is a $\delta-$SGWS module. Then M is a $\delta-$SGWS module .*

Proof. Suppose that $n = 2$, that is $M = M_1 + M_2$. Let $\delta(M) \subseteq N \subseteq M$. Then $M_1 + M_2 + N$ has the trivial weak $\delta-$supplement 0 in M. Since $\delta(M_2) \subseteq \delta(M) \subseteq N$, we have $\delta(M_2) \subseteq M_2 \cap (M_1 + N)$. It follows from hypothesis that $M_2 \cap (M_1 + N)$ has a weak $\delta-$supplement L in M_2. By Lemma 2.3, L is also weak $\delta-$supplement of $M_1 + N$ in M. Note that $\delta(M_1) \subseteq M_1 \cap (N + L)$. Since $M_1 \cap (N + L)$ has a weak $\delta-$supplement V in M_1. Again applying the Lemma 2.3, $V + L$ is a weak $\delta-$supplement of N in M. The proof is completed by induction on n. ∎

Lemma 2.4. *Let M be a module. Suppose that K is a $\delta-$small submodule of M.Then, M is a $\delta-$SGWS module if and only if M/K is a $\delta-$SGWS module.*

Proof. Necessity follows from Proposition 2.12. Conversely suppose that M/K is a $\delta-$SGWS module. Let $\delta(M) \subseteq N \subseteq M$. Since K is a $\delta-$small submodule of M, $K \subseteq \delta(M)$ and so $K \subseteq N$. By assumption, $M/K = (N/K)+(L/K)$ and $(N/K)\cap(L/K) = (N \cap L)/K \ll_\delta M/K$ for some submodule L/K of M/K. Then we get $M = N + L$. Since $K \ll_\delta M$, by [13, Lemma 1.3(a)], $N \cap L \ll_\delta M$. Thus M is a $\delta-$SGWS module. ∎

Let M and N be $R-$modules. An epimorphism $f : M \to N$ is called a $\delta-$*cover* if $Kerf \ll_\delta M$ [11]. Recall that an epimorphism $f : M \to N$ is called a *generalized* $\delta-$*cover* if $Kerf \leq \delta(M)$ and M is called a generalized $\delta-$cover of N with an epimorphism $f : M \to N$. Using Lemma 2.4, we obtain the following result.

Corollary 2.7. *Every generalized $\delta-$cover of a $\delta-SGWS$ module is a $\delta-SGWS$.*

Proposition 2.14. *Let $0 \to K \to M \to M/K \to 0$ be a short exact sequence. If K and M/K are $\delta-SGWS$ modules and K has a weak $\delta-$supplement in M, then M is a $\delta-SGWS$ module.*

Proof. Without restriction of generality, we will assume that $K \subseteq M$. Let L be a weak $\delta-$supplement of K in M, i.e., $M = K + L$ and $K \cap L \ll_\delta M$. Then we get the decomposition $M/(K \cap L) = K/(K \cap L) \oplus L/(K \cap L)$. By Lemma 2.4, it suffices to prove that $M/(K \cap L)$ is $\delta-$SGWS. $K/(K \cap L)$ is a $\delta-$SWGS module as a homomorphic image of K. On the other hand $L/(K \cap L) \cong M/K$ is $\delta-$SWGS. Thus $M/(K \cap L)$ is $\delta-$SGWS module according to Proposition 2.13. ∎

Next we consider linearly compact modules. Let M be *a* module. A *coset* of M is subset of the form $m + N = \{m + x : x \in N\}$, for some $m \in M$ and submodule N of M. A non-empty collection $\{C_i : i \in I\}$ of cosets of M has the *finite intersection property* if $\bigcap_{i \in F} C_i$ is non-empty for every finite subset F of I. The module M is called *linearly compact* if $\bigcap_{i \in I} C_i$ is non-empty for every non-empty collection $\{C_i : i \in I\}$ of cosets the finite intersection property [6].

Corollary 2.8. *Let M be a module and K be a linearly compact submodule of M.Then, M is a $\delta-SGWS$ module if and only if M/K is a $\delta-SGWS$ module.*

Proof. By [12, 41.10(1)], K has a weak $\delta-$supplement in every extension. Since every weak supplement is weak $\delta-$supplement. Applying Proposition 2.14, the proof follows. ∎

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