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A RELATED FIXED POINT THEOREM FOR THREE PAIRS OF MAPPINGS ON COMPLETE METRIC SPACES

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Abstract. In this paper we prove a related fixed point theorem for three pairs of mappings, on three complete metric spaces, satisfying rational type contractive conditions.

Dedicated to Professor Valeriu Popa on the Occasion of His 80th Birthday

1. Introduction and preliminaries

The first related fixed point theorem was proved in [2] for a pair of mappings on two complete metric spaces.

Fisher and Murthy [3] proved the following related fixed point theorem for two pairs of mappings on two complete metric spaces.

Theorem 1.1. *Let* (X, d_1) *and* (Y, d_2) *be complete metric spaces, let* A, B *be mappings of* X *into* Y *and let* S, T *be mappings of* Y *into* X *satisfying the inequalities*

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$$\begin{aligned} d_1(SAx, TBx') &\leq c\max\{d_1(x,x'), d_1(x,SAx), \\ &\qquad d_1(x',TBx'), d_2(Ax,Bx')\}, \\ d_2(BSy, ATy') &\leq c\max\{d_2(y,y'), d_2(y,BSy), \\ &\qquad d_2(y',ATy'), d_1(Sy,Ty')\} \end{aligned}$$

for all x, x' in X and y, y' in Y, where $0 \leq c < 1$. If one of the mappings A, B, S and T is continuous, then SA and TB have a unique common fixed point u in X and BS and AT have a unique fixed point v in Y. Further, $Au = Bu = v$ and $Sy = Ty = u$.

See also [1], [4], [5] and [6].

We now prove a related fixed point theorem for three pairs of mappings on three complete metric spaces. The mappings satisfy a system of three contractive conditions, where for each pair of mappings on the same metric space the distance between the images is bounded by an expression involving distances in all three spaces.

2. Main result

Theorem 2.1. *Let (X, d_1), (Y, d_2) and (Z, d_3) be complete metric spaces. Let A, B be mappings of X into Y, let C, D be mappings of Y into Z and let E, F be mappings of Z into X satisfying the inequalities*

$$d_1(ECAx, FDBx') \leq c\frac{f_1(x,x',y,y')}{g_1(x,x',y,y')}, \tag{1}$$

$$d_2(BECy, AFDy') \leq c\frac{f_2(y,y',z,z')}{g_2(y,y',z,z')}, \tag{2}$$

$$d_3(CAFz, DBEz') \leq c\frac{f_3(z,z',x,x')}{g_3(z,z',x,x')} \tag{3}$$

for all x, x' *in* X, y, y' *in* Y *and* z, z' *in* Z *such that* $g_1(x, x', y, y') \neq 0$, $g_2(y, y', z, z') \neq 0$ *and* $g_3(z, z', x, x') \neq 0$, *where* $0 \leq c < 1$ *and*

$$\begin{aligned}
f_1(x, x', y, y') &= \\
&= \max\{d_1(x, x')d_1(ECAx, FDBx'), d_1(x, ECAx)d_1(x', FDBx'), \\
&\qquad d_1(x, FDBx')d_1(x', ECAx), d_1(x, x')d_1(ECy, FDy'), \\
&\qquad d_1(x, ECy)d_1(x', FDy'), d_1(x, FDy')d_1(x', ECy)\}, \\
f_2(y, y', z, z') &= \\
&= \max\{d_2(y, y')d_2(BECy, AFDy'), d_2(y, BECy)d_2(y', AFDy'), \\
&\qquad d_2(y, AFDy')d_2(y', BECy), d_2(y, y')d_2(BEz, AFz'), \\
&\qquad d_2(y, BEz)d_2(y', AFz'), d_2(y, AFz')d_2(y', BEz)\}, \\
f_3(z, z', x, x') &= \\
&= \max\{d_3(z, z')d_3(DBEz, CAFz'), d_3(z, DBEz)d_3(z', CAFz'), \\
&\qquad d_3(z, CAFz')d_3(z', DBEz), d_3(z, z')d_3(CAx, DBx'), \\
&\qquad d_3(z, CAx)d_3(z', DBx'), d_3(z, DBx')d_3(z', CAx)\}, \\
g_1(x, x', y, y') &= \max\{d_1(ECAx, FDBx'), d_1(ECy, FDy'), \\
&\qquad d_2(BECy, AFDy'), d_3(CAx, DBx')\}, \\
g_2(y, y', z, z') &= \max\{d_2(BECy, AFDy'), d_2(BEz, AFz'), \\
&\qquad d_3(DBEz, CAFz'), d_1(ECy, FDy')\}, \\
g_3(z, z', x, x') &= \max\{d_3(DBEz, CAFz'), d_3(CAx, DBx'), \\
&\qquad d_1(ECAx, FDBx'), d_2(BEz, AFz')\}.
\end{aligned}$$

If A *and* C *are continuous or* B *and* D *are continuous, then* ECA *and* FDB *have a unique common fixed point* u *in* X, BEC *and* AFD *have a unique common fixed point* v *in* Y *and* CAF *and* DBE *have a unique common fixed point* w *in* Z. *Furthermore,* $Au = Bu = v$, $Cv = Dv = w$ *and* $Ew = Fw = u$.

Proof.

Existence of fixed points.

Note that the system of equalities

$$Au = Bu = v, \; Cv = Dv = w \text{ and } Ew = Fw = u$$

implies

$$\begin{aligned}
ECAu &= FDBu = u, \\
BECv &= AFDv = v \text{ and} \\
CAFw &= DBEw = w.
\end{aligned}$$

Let $x = x_0$ be an arbitrary point in X. We define the sequences $\{x_n\}$ in X, $\{y_n\}$ in Y and $\{z_n\}$ in Z inductively by

$$y_{2n-1} = Ax_{2n-2},\ z_{2n-1} = Cy_{2n-1},\ x_{2n-1} = Ez_{2n-1},$$

$$y_{2n} = Bx_{2n-1},\ z_{2n} = Dy_{2n},\ x_{2n} = Fz_{2n}$$

for $n = 1, 2, \ldots$.

First we take $x = x_{2n}$, $x' = x_{2n-1}$, $y = y_{2n-1}$, $y' = y_{2n}$, $z = z_{2n-1}$ and $z' = z_{2n}$ in inequalities (1), (2) and (3).

We have

$$\begin{aligned} CAx_{2n} &= z_{2n+1},\ DBx_{2n-1} = z_{2n}, \\ ECy_{2n-1} &= x_{2n-1},\ FDy_{2n} = x_{2n}, \\ BEz_{2n-1} &= y_{2n-1},\ AFz_{2n} = y_{2n+1}, \end{aligned}$$

hence

$$\begin{aligned} ECAx_{2n} &= x_{2n+1},\ FDBx_{2n-1} = x_{2n}, \\ BECy_{2n-1} &= y_{2n},\ AFDy_{2n} = y_{2n+1}, \\ DBEz_{2n-1} &= z_{2n},\ CAFz_{2n} = z_{2n+1}. \end{aligned}$$

It follows that

$$\begin{aligned} (4)\quad f_1(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) &= d_1(x_{2n-1}, x_{2n}) \max\{d_1(x_{2n}, x_{2n-1}), \\ &\qquad d_1(x_{2n}, x_{2n+1})\}, \\ f_2(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n}) &= d_2(y_{2n-1}, y_{2n}) d_2(y_{2n}, y_{2n+1}), \\ f_3(z_{2n-1}, z_{2n}, x_{2n}, x_{2n-1}) &= d_3(z_{2n-1}, z_{2n}) d_3(z_{2n}, z_{2n+1}), \end{aligned}$$

and

$$\begin{aligned} (5)\quad g_1(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) &= \max\{d_1(x_{2n-1}, x_{2n}), d_1(x_{2n}, x_{2n+1}), \\ &\qquad d_2(y_{2n}, y_{2n+1}), d_3(z_{2n}, z_{2n+1})\} \\ (6)\quad g_2(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n}) &= \max\{d_1(x_{2n-1}, x_{2n}), d_2(y_{2n}, y_{2n+1}), \\ &\qquad d_3(z_{2n}, z_{2n+1})\}, \\ (7)\quad g_3(z_{2n-1}, z_{2n}, x_{2n}, x_{2n-1}) &= \max\{d_1(x_{2n}, x_{2n+1}), d_2(y_{2n}, y_{2n+1}), \\ &\qquad d_3(z_{2n}, z_{2n+1})\}. \end{aligned}$$

If $g_1(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) \neq 0$, by inequality (1) we get

$$(8)\qquad d_1(x_{2n}, x_{2n+1}) \leq c d_1(x_{2n-1}, x_{2n}),$$

as $\max\{d_1(x_{2n}, x_{2n-1}),\ d_1(x_{2n}, x_{2n+1})\} \leq g_1(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})$.

If $g_2(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n}) \neq 0$, by inequality (2) we obtain

$$d_2(y_{2n}, y_{2n+1}) \leq cd_2(y_{2n-1}, y_{2n}), \tag{9}$$

as $d_2(y_{2n}, y_{2n+1}) \leq g_2(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n})$.

Similarly, if $g_3(z_{2n-1}, z_{2n}, x_{2n}, x_{2n-1}) \neq 0$, then inequality (3) implies

$$d_3(z_{2n}, z_{2n+1}) \leq cd_3(z_{2n-1}, z_{2n}). \tag{10}$$

Now we take $x = x_{2n}$, $x' = x_{2n-1}$, $y = y_{2n-1}$, $y' = y_{2n}$, $z = z_{2n-1}$ and $z' = z_{2n}$ in inequalities (1), (2) and (3). We have

$$\begin{aligned} f_1(x_{2n}, x_{2n+1}, y_{2n+1}, y_{2n}) &= d_1(x_{2n}, x_{2n+1}) \max\{d_1(x_{2n}, x_{2n+1}), \\ &\qquad d_1(x_{2n+1}, x_{2n+2})\}, \\ f_2(y_{2n+1}, y_{2n}, z_{2n+1}, z_{2n}) &= d_2(y_{2n}, y_{2n+1}) d_2(y_{2n+1}, y_{2n+2}), \\ f_3(z_{2n-1}, z_{2n}, x_{2n}, x_{2n-1}) &= d_3(z_{2n}, z_{2n+1}) d_3(z_{2n+1}, z_{2n+2}), \end{aligned}$$

and

$$\begin{aligned} g_1(x_{2n}, x_{2n+1}, y_{2n+1}, y_{2n}) = \max\{&d_1(x_{2n}, x_{2n+1}), d_1(x_{2n+1}, x_{2n+2}), \\ &d_2(y_{2n+1}, y_{2n+2}), d_3(z_{2n+1}, z_{2n+2})\} \end{aligned} \tag{11}$$

$$\begin{aligned} g_2(y_{2n+1}, y_{2n}, z_{2n+1}, z_{2n}) = \max\{&d_1(x_{2n}, x_{2n+1}), d_2(y_{2n+1}, y_{2n+2}), \\ &d_3(z_{2n+1}, z_{2n+2})\}, \end{aligned} \tag{12}$$

$$\begin{aligned} g_3(z_{2n+1}, z_{2n}, x_{2n}, x_{2n+1}) = \max\{&d_1(x_{2n+1}, x_{2n+2}), d_2(y_{2n+1}, y_{2n+2}), \\ &d_3(z_{2n+1}, z_{2n+2})\}. \end{aligned} \tag{13}$$

If $g_1(x_{2n}, x_{2n+1}, y_{2n+1}, y_{2n}) \neq 0$, by inequality (1) we get

$$d_1\left(x_{2n+1}, x_{2n+2}\right) \leq cd_1\left(x_{2n}, x_{2n+1}\right). \tag{14}$$

If $g_2(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n}) \neq 0$, by inequality (2) we obtain

$$d_2(y_{2n+1}, y_{2n+2}) \leq cd_2(y_{2n}, y_{2n+1}). \tag{15}$$

Finally, if $g_3(z_{2n-1}, z_{2n}, x_{2n}, x_{2n-1}) \neq 0$, inequality (3) implies

$$d_3(z_{2n+1}, z_{2n+2}) \leq cd_3(z_{2n}, z_{2n+1}). \tag{16}$$

Case 1.

Assume that for every $n \geq 1$ all six numbers $g_1(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})$, $g_1(x_{2n}, x_{2n+1}, y_{2n+1}, y_{2n})$, $g_2(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n})$, $g_2(y_{2n+1}, y_{2n}, z_{2n+1}, z_{2n})$, $g_3(z_{2n-1}, z_{2n}, x_{2n}, x_{2n-1})$ and $g_3(z_{2n+1}, z_{2n}, x_{2n}, x_{2n+1})$ are not zero.

By (8) and (14), $d_1(x_{n+1}, x_{n+2}) \leq cd_1(x_n, x_{n+1})$ for every $n \geq 1$. We get inductively $d_1(x_n, x_{n+1}) \leq c^{n-1} d_1(x_1, x_2)$ for every $n \geq 1$, hence

$$d_1(x_n, x_{n+p}) \leq c^{n-1}\left(1 + c + .. + c^{p-1}\right) \leq \frac{c^{n-1}}{1-c} \to 0,$$

since $0 \leq c < 1$. Then $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d_1), therefore it is convergent in X. Let $u := \lim_{n\to\infty} x_n$.

Similarly, using (9) and (15), then using (10) and (16) we conclude that $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences in the complete metric spaces (Y, d_2) and (Z, d_3), respectively, therefore are convergent. Let $v := \lim_{n\to\infty} y_n$ and $w := \lim_{n\to\infty} z_n$.

Subcase 1.1. Now we suppose that A and C are continuous. Then

$$\begin{aligned} v &= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Ax_{2n} = Au, \\ w &= \lim_{n\to\infty} z_{2n-1} = \lim_{n\to\infty} Cy_{2n-1} = Cv \end{aligned} \tag{17}$$

and therefore

$$\lim_{n\to\infty} f_1(u, x_{2n-1}, v, y_{2n}) = 0, \tag{18}$$

$$\lim_{n\to\infty} f_2(v, y_{2n}, w, z_{2n}) = 0, \tag{19}$$

$$\lim_{n\to\infty} f_3(w, z_{2n}, u, x_{2n-1}) = 0. \tag{20}$$

Moreover,

$$\lim_{n\to\infty} g_1(u, x_{2n-1}, v, y_{2n}) = \max\{d_1(Ew, u), d_2(BEw, v)\}, \tag{21}$$

$$\begin{aligned} \lim_{n\to\infty} g_2(v, y_{2n}, w, z_{2n}) &= \lim_{n\to\infty} g_3(w, z_{2n}, u, x_{2n+1}) = \\ &\max\{d_1(Ew, u), d_2(BEw, v), \\ &d_3(DBEw, w)\} \end{aligned} \tag{22}$$

Assume that at least one of the distances $d_1(Ew, u)$ and $d_2(BEw, v)$ is positive. Then by (21) and (22) it follows that $\lim_{n\to\infty} g_1(u, x_{2n-1}, v, y_{2n}) \neq 0$ and $\lim_{n\to\infty} g_2(v, y_{2n}, w, z_{2n}) = \lim_{n\to\infty} g_3(w, z_{2n}, u, x_{2n+1}) \neq 0$, respectively. By inequality (1) and by equations (17) and (18), we obtain

$$d_1(Ew, u) = \lim_{n\to\infty} d_1(ECAu, FDBx_{2n-1}) = 0$$

which gives $Ew = u$ and $ECAu = u$.

Similarly, by inequality (2) and by equations (17) and (19) we obtain

$$d_2(Bu, v) = \lim_{n\to\infty} d_2(BECv, AFDy_{2n}) = 0,$$

which gives $Bu = v$ and $BECv = v$.

So, assuming that $d_1(Ew, u) > 0$ or $d_2(BEw, v) > 0$ we obtain $Ew = u$ and $BEw = v$, a contradiction. It follows that $d_1(Ew, u) = 0$ and $d_2(BEw, v) = 0$, hence

$$Ew = u \text{ and } Bu = v. \tag{23}$$

Furthermore,

$$ECAu = u \text{ and } BECv = v.$$

If $d_3(DBEw, w) > 0$, then applying inequality (3) and equations (17) and (20) we get

$$d_3(Dv, w) = \lim_{n\to\infty} d_3(DBEw, CAFz_{2n}) = 0,$$

which gives $Dv = w$ and $DBEw = w$, a contradiction. It follows that $d_3(DBEw, w) = 0$ and hence

$$DBEw = w, \text{ i.e. } Dv = w. \tag{24}$$

It remains to prove that $Fw = u$ and $FDBu = u$, $AFDv = v$, $CAFw = w$. Using (17), (23) and (24), we see that $Fw = u$ implies $FDBu = FDv = Fw = u$, $AFDv = AFw = Au = v$ and $CAFw = CAu = Cv = w$. So, it suffices to prove that $Fw = u$.

We consider inequality (1) with $x = x' = u$ and $y = v$, $y' = y_{2n}$. We have

$$\lim_{n\to\infty} f_1(u, u, v, y_{2n}) = 0.$$

and

$$\begin{aligned} g_1(u, u, v, y_{2n}) &= \max\{d_1(ECAu, FDBu), d_1(ECv, FDy_{2n}), \\ &\quad d_2(BECv, AFDy_{2n}), d_3(CAu, DBu)\} = \\ &= \max\{d_1(u, Fw), d_1(u, x_{2n}), \\ &\quad d_2(v, y_{2n+1}), d_3(w, w)\}. \end{aligned}$$

Then $\lim_{n\to\infty} g_1(u, u, v, y_{2n}) = d_1(u, Fw)$. If we assume that $d_1(u, Fw) > 0$, applying inequality (1) we get $d_1(u, Fw) = d_1(ECAu, FDBu) \le 0$, a contradiction. It follows that $d_1(u, Fw) = 0$, as required.

Subcase 1.2. Suppose that B and D are continuous. The proof is analogous to that from Subcase 1.1.

Case 2.

Now we analyse the case where there exists $n \geq 1$ such that at least one of the six numbers $g_1(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})$, $g_1(x_{2n}, x_{2n+1}, y_{2n+1}, y_{2n})$, $g_2(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n})$, $g_2(y_{2n+1}, y_{2n}, z_{2n+1}, z_{2n})$, $g_3(z_{2n-1}, z_{2n}, x_{2n}, x_{2n-1})$ and $g_3(z_{2n+1}, z_{2n}, x_{2n}, x_{2n+1})$ is zero.

Subcase 2.1.a. Suppose that for some n we have $g_1(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) = 0$. By (5),

$$x_{2n-1} = x_{2n} = x_{2n+1} =: u, \quad y_{2n} = y_{2n+1} =: v, \quad z_{2n} = z_{2n+1} =: w.$$

Using the definitions of the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ we see that

$$Au = Bu = v, Cv = Dv = w \text{ and } Ew = Fw = u. \tag{25}$$

Then

$$\begin{aligned} ECAu &= FDBu = u, \\ BECv &= AFDv = v, \\ CAFw &= DBEw = w \end{aligned} \tag{26}$$

and the claim follows, except for the uniqueness.

Subcase 2.1.b. Suppose that for some n we have $g_1(x_{2n}, x_{2n+1}, y_{2n+1}, y_{2n}) = 0$. This is similar to Subcase 2.1.a. Using (11) we get

$$x_{2n} = x_{2n+1} = x_{2n+2} =: u, \quad y_{2n+1} = y_{2n+2} : v \quad , z_{2n+1} = z_{2n+2} =: w.$$

Using the definitions of the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ we obtain (25) and then (26) is a consequence of (25).

Subcase 2.2.a. Suppose that for some n we have $g_2(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n}) = 0$. By (6),

$$x_{2n-1} = x_{2n} =: u, \quad y_{2n} = y_{2n+1} =: v \quad , z_{2n} = z_{2n+1} =: w.$$

According to these equalities and to (4) and (5), $f_1(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) = 0$ and $g_1(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) = d_1(x_{2n}, x_{2n+1})$, respectively. If $d_1(x_{2n}, x_{2n+1}) > 0$, applying inequality (1) with $x = x_{2n-1}$, $x' = x_{2n}$ and $y = y_{2n-1}$, $y' = y_{2n}$ we get

$$d_1(x_{2n+1}, x_{2n}) = d_1(ECAx_{2n}, FDBx_{2n-1}) \leq 0,$$

a contradiction. It follows that $x_{2n} = x_{2n+1}$, which allows us to go back to Subcase 2.1.a.

Subcase 2.2.b. Suppose that for some n we have $g_2(y_{2n+1}, y_{2n}, z_{2n+1}, z_{2n}) = 0$. Using (12) we get

$$x_{2n} = x_{2n+1} =: u, \quad y_{2n+1} = y_{2n+2} =: v, \quad z_{2n+1} = z_{2n+2} =: w.$$

This is similar to Subcase 2.2.a. Note that (4) and (5) show that $f_1(x_{2n}, x_{2n+1}, y_{2n+1}, y_{2n}) = 0$ and $g_1(x_{2n}, x_{2n+1}, y_{2n+1}, y_{2n}) = d_1(x_{2n+1}, x_{2n+2})$, respectively. If $d_1(x_{2n+1}, x_{2n+2}) > 0$, applying inequality (1) with $x = x_{2n}$, $x' = x_{2n+1}$ and $y = y_{2n+1}$, $y' = y_{2n}$ we get

$$d_1(x_{2n+1}, x_{2n+2}) = d_1(ECAx_{2n}, FDBx_{2n+1}) \leq 0,$$

a contradiction. It follows that $x_{2n} = x_{2n+1}$, which allows us to go back to Subcase 2.1.b.

Subcase 2.3.a. Suppose that for some n we have $g_3(z_{2n-1}, z_{2n}, x_{2n}, x_{2n-1}) = 0$. By (7),

$$x_{2n} = x_{2n+1} =: u, \quad y_{2n} = y_{2n+1} =: v, \quad z_{2n} = z_{2n+1} =: w.$$

Using the definitions of the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ we see that

$$Au = v,\ Cv = Dv = w \text{ and } Ew = Fw = u.$$

Then $ECAu = u$, $AFDv = v$ and $CAFw = w$.

If suffices to prove that $Bu = v$, since this and the above equalities imply $FDBu = u$, $BECv = v$ and $DBEw = w$.

Note that $Bu = Bx_{2n+1} = y_{2n+2}$ and $v = y_{2n+2}$.

Recall that, by (12), we have

$$g_2(y_{2n+1}, y_{2n}, z_{2n+1}, z_{2n}) = \max\{d_1(x_{2n}, x_{2n+1}), d_2(y_{2n+1}, y_{2n+2}), d_3(z_{2n+1}, z_{2n+2})\}.$$

But $f_2(y_{2n+1}, y_{2n}, z_{2n+1}, z_{2n}) = d_2(y_{2n}, y_{2n+1})d_2(y_{2n+1}, y_{2n+2})$ and hence

$$f_2(y_{2n+1}, y_{2n}, z_{2n+1}, z_{2n}) = 0.$$

If $d_2(y_{2n+1}, y_{2n+2}) > 0$, then $g_2(y_{2n+1}, y_{2n}, z_{2n+1}, z_{2n}) \neq 0$ and by inequality (2) we obtain $d_2(y_{2n+2}, y_{2n+1}) = d_2(BECy_{2n+1}, AFDy_{2n}) \leq 0$, a contradiction. Then $y_{2n+2} = y_{2n+1}$, i.e. $Bu = v$.

Subcase 2.3.b. Suppose that for some n we have $g_3(z_{2n+1}, z_{2n}, x_{2n}, x_{2n+1}) = 0$. By (13),

$$x_{2n+1} = x_{2n+2} =: u, \quad y_{2n+1} = y_{2n+2} =: v, \quad z_{2n+1} = z_{2n+2} =: w.$$

As in Subcase 2.3.a., using the definitions of the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ we see that

$$Bu = v,\ Cv = Dv = w \text{ and } Ew = Fw = u.$$

Then $FDBu = u$, $BECv = v$ and $DBEw = w$.

If suffices to prove that $Au = v$, since this and the above equalities imply $ECAu = u$, $AFDv = v$ and $CAFw = w$.

Note that $Au = Ax_{2n+2} = y_{2n+3}$ and $v = y_{2n+2}$.

As a consequence of (6), we get

$$g_2(y_{2n+1}, y_{2n+2}, z_{2n+1}, z_{2n+2}) = \max\{d_1(x_{2n+1}, x_{2n+2}), d_2(y_{2n+2}, y_{2n+3}), d_3(z_{2n+2}, z_{2n+3})\}.$$

But $f_2(y_{2n+1}, y_{2n+2}, z_{2n+1}, z_{2n+2}) = d_2(y_{2n}, y_{2n+1})d_2(y_{2n+1}, y_{2n+2}) = 0$. If $d_2(y_{2n+2}, y_{2n+3}) > 0$, then using inequality (2) we obtain $d_2(y_{2n+2}, y_{2n+3}) = d_2(BECy_{2n+1}, AFDy_{2n+2}) \leq 0$, a contradiction. Then $y_{2n+3} = y_{2n+2}$, i.e. $Au = v$.

This completes the proof of the Existence part.

Uniqueness of fixed points

To prove uniqueness, suppose that ECA and FDB have a second common fixed point $u' \neq u$ in X. Then, applying inequality (1), we have

$$\begin{aligned} d_1(u, u') &= d_1(ECAu, FDBu') \\ &\leq c\frac{(f_1(u, u', v, v))}{(g_1(u, u', v, v)} \\ &= c\frac{[d_1(u, u')]^2}{\max\{d_1(u, u'), d_2(Au, Bu'), d_3(CAu, DBu')\}} \\ &\leq cd_1(u, u'). \end{aligned}$$

We obtain a contradiction since $c < 1$. The fixed point u must therefore be unique.

We can prove similarly that v is the unique common fixed point of AEC and BFD and w is the unique common fixed point of CAE and DBF, using inequalities (2) and (3), respectively.

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