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ON A WEAKER FORM OF MINIMAL OPEN SETS AND A STRONGER FORM OF MEAN OPEN SETS

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Abstract. In this paper, we introduce the notions of locally minimal open (resp. locally minimal closed) and s-mean open (resp. s-mean closed) sets at a certain point in a topological space and investigate some properties of such sets. We see that a minimal open (resp. minimal closed) set is locally minimal open (resp. locally minimal closed) at each of its points and the notion of s-mean open (resp. s-mean closed) sets is stronger than the notion of mean open (resp. mean closed) sets.

1. INTRODUCTION

Unless otherwise mentioned, X stands for the topological space $(X, \mathbb{T})$. By a proper open set (resp. closed set) of a topological space X, we mean an open set $G \neq \emptyset, X$ (resp. a closed set $E \neq \emptyset, X$). By $A \subseteq B$ we mean A is a subset of B and by $A \subsetneq B$ we mean A is a proper subset of B. We write $\mathbb{R}$ to denote the set of all real numbers.

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Nakaoka and Oda [5, 6, 7] introduced and studied the notion minimal open set: a proper open set U (resp. proper closed set E) of a topological space X is called a minimal open set if U and $\emptyset$ are only open sets contained in U. Now it is obvious that if U is a minimal open set and $x \in U$ then there does not exist any proper open set V satisfying $x \in V \subsetneq U$. Thus in general, if U is a proper open set containing a point x of a topological space X then there may not exists a proper open set V such that $x \in V \subsetneq U$. This observation leads us to introduce the notion of locally minimal open sets (Definition 7). We investigate the same for the case of closed sets and introduce the notion of locally minimal closed sets (Definition 8).

In the contrary of minimal open and maximal open sets, Mukharjee et al. [4] introduced the notion of mean open set: a proper open set U of a topological space X is called a mean open set if there exist proper open sets G and H such that $G \subsetneq U \subsetneq H$. Now if $x \in U$ then $x \in H$ but it may happen that $x \notin G$. This observation insists us to study the notion of s-mean open sets (Definition 9).

2. BASIC DEFINITIONS AND RESULTS

Firstly, we recall the followings.

Definition 1. (Nakaoka and Oda [5, 7]) A proper open set U of X is said to be a minimal open set if G is an open set of X contained in U, then $G = \emptyset$ or $G = U$.

Definition 2. (Nakaoka and Oda [7] A proper closed set E of X is said to be a minimal closed set if F is a closed set of X contained in E, then $E = \emptyset$ or $E = F$.

Definition 3. (Nakaoka and Oda [5, 6] A proper open set U of X is said to be a maximal open set if U is contained in an open set G of X, then $G = U$ or $G = X$.

Definition 4. (Nakaoka and Oda [7]) A proper closed set E of X is said to be a maximal closed set if E is contained in a closed set F of X, then $F = E$ or $F = X$.

Definition 5. (Mukharjee and Bagchi [4]) An open set M of X is said to be a mean open set X if there exist two proper open sets U and V of X satisfying $U \subsetneq M \subsetneq V$.

Definition 6. (Mukherjee and Bagchi [4]) A closed set E of X is said to be a mean closed set X if there exist two proper closed sets D and F of X satisfying $D \subsetneq E \subsetneq F$.

Theorem 1. *(Nakaoka and Oda [6]) If U is a maximal open set and W is an open set in X, then either $U \cup W = X$ or $W \subseteq U$. If W is also a maximal open set distinct from U, then $U \cup W = X$.*

Theorem 2. *(Nakaoka and Oda [7]) If E is a maximal closed set and F is a closed set in X, then either $E \cup F = X$ or $F \subseteq E$. If F is also a maximal closed set distinct from E, then $E \cup F = X$.*

Theorem 3. *(Nakaoka and Oda [5]) If U is a minimal open set and W is an open set in X, then either $U \cap W = \emptyset$ or $U \subseteq W$. If W is also a minimal open set distinct from U, then $U \cap W = \emptyset$.*

Theorem 4. *(Nakaoka and Oda [7]) If E is a minimal closed set and F is a closed set in X, then either $E \cap F = \emptyset$ or $E \subseteq F$. If F is also a minimal closed set distinct from E, then $E \cap F = \emptyset$.*

Lemma 1. (Bagchi and Mukharjee [1]) Each nonempty open set G of a T_1 connected topological space X is infinite and is not minimal open in X.

Theorem 5. *(Bagchi and Mukharjee [1]) Let $(X, \mathbb{T})$ be a T_1 connected topological space and $\mathbb{T}_{mo}$ denotes the family of all mean open sets in X. Then $\mathbb{B} = \{\emptyset\} \cup \mathbb{T}_{mo}$ forms a basis of the topology $\mathbb{T}$ on X.*

3. LOCALLY MINIMAL OPEN AND CLOSED SETS

Definition 7. Let X be a topological space and $x \in X$. An open set $U(\neq X)$ containing x is said to be a locally minimal open set at x if V is an open set satisfying $x \in V \subseteq U$ implies $U = V$.

Example 1. Let us consider the topological space $(X, \mathbb{T})$, where $X = \mathbb{R}$ and $\mathbb{T} = \{\emptyset, X, (a, \infty), [a, \infty)\}$. Here we see that $[a, \infty)$ is the only locally minimal open set at a. But $[a, \infty)$ is not a minimal open set at all, in fact (a, ∞) is an open set in X contained in $[a, \infty)$. Also we see that $[a, \infty)$ is not a locally minimal open set at any point in it other than a. But (a, ∞) is a locally minimal open set at each of its points.

Theorem 6. *Let U be a minimal open set in a topological space X. Then U is a locally minimal open set at each of its points.*

Proof. If possible, let there be a $u \in U$ such that U be not a locally minimal open set at u. Then there is an open set V in X satisfying $u \in V \subsetneq U$. But This contradicts the minimality of U. ■

Theorem 7. *Let X be a topological space and $x \in X$. If U and V are locally minimal open sets at x, then $U = V$.*

Proof. If possible, let $U \neq V$. By the nature of U and V, neither $U \subsetneq V$ nor $V \subsetneq U$. Then $x \in U \cap V \subsetneq U$ as well as $x \in U \cap V \subsetneq V$ and these contradicts the nature of U and V. Thus $U = V$. ■

Lemma 2. Let X be a topological space and $x \in X$. If U is a locally minimal open set at x and V is an open set containing x, then $U \subseteq V$

Proof. Clearly $U \cap V$ is an open set satisfying $x \in U \cap V \subseteq U$. By the locally minimality of U, we have $U \cap V = U$ and hence $U \subseteq V$. ■

Theorem 8. *Let X be a topological space and $x \in X$. If U is locally minimal open at x, then there is no nonempty closed set E such that $E \subseteq U - \{x\}$.*

Proof. Let E be a nonempty closed set satisfying $E \subseteq U - \{x\}$. Then $x \notin E$ and $E \subsetneq U$. So $X - E$ is a proper open set containing x. Then by the Lemma 2, $U \subseteq X - E$, i. e., $E \subseteq X - U$. As $U \cap (X - U) = \emptyset$, it follows that $E \subsetneq U$ and $E \subseteq X - U$ can not hold simultaneously. ■

Corollary 1. *Let X be a T_1 connected topological space and $x \in X$. Then there is no locally minimal open set at x.*

Proof. Let U be a locally minimal open set at x. By the Lemma 1, $U - \{x\}$ is an infinite open set and so we can find a point y in $U - \{x\}$. As X satisfies T_1 axiom, it follows that $\{y\}$ is a nonempty closed set satisfying $\{y\} \subseteq U - \{x\}$ and this contradicts the previous theorem. ■

Theorem 9. *Let X be a topological space and $x \in X$. If U is an open set which is both locally open at x and maximal open, then U is the only proper open set in X containing x.*

Proof. Let G be a proper open set in X containing x. By locally minimality of U at x we have $U \subseteq G$ (by the Lemma 2). Again by the maximality of U we have either $G \subseteq U$ or $G \cup U = X$ (by the Theorem 1). But $G \cup U = X$ contradicts $U \subseteq G$. Hence $G \subseteq U$ and $U \subseteq G$ hold simultaneously and hence $U = G$. ■

Definition 8. Let X be a topological space and $x \in X$. A closed set $F(\neq X)$ containing x is said to be a locally minimal closed set at x if E is a closed set satisfying $x \in E \subseteq F$ implies $E = F$.

If a topological space X satisfies T_1 axiom then for each $x \in X$, $\{x\}$ is the locally minimal closed set at x. Thus each minimal closed set is the locally minimal closed set at each x.

Theorem 10. *Let E be a minimal closed set in a topological space X. Then E is a locally minimal closed set at each of its points.*

Proof. Proof is similar to proof of the Theorem 6. ■

Theorem 11. *Let X be a topological space and $x \in X$. If E and F are locally minimal closed sets at x, then $E = F$.*

Proof. Proof is similar to proof of the Theorem 7. ■

Lemma 3. Let X be a topological space and $x \in X$. If E is a locally closed open set at x and F is a closed set containing x, then $E \subseteq F$

Proof. Proof is similar to proof of the Lemma 2. ■

Theorem 12. *Let X be a topological space and $x \in X$. If E is locally minimal closed at x, then there is no nonempty open set U such that $U \subseteq E - \{x\}$.*

Proof. Proof is similar to proof of the Theorem 8. ■

Theorem 13. *Let X be a topological space and $x \in X$. If E is a closed set which is both locally closed at x and maximal closed, then E is the only proper closed set in X containing x.*

Proof. Proof is similar to proof of the Theorem 9. ■

Theorem 14. *Let X be a topological space and $x \in X$. If U is locally minimal open at x and E is a minimal closed set not containing x, then $E \subseteq X - U$.*

Proof. As $X - U$ is closed and E is minimal closed, it follows that either $E \cap (X - U) = \emptyset$ or $E \subseteq X - U$ (by the Theorem 4). Now $E \subseteq X - U$ implies that $E \subseteq U$. Since $x \notin E$, $E \subseteq U - \{x\}$ which contradicts the previous theorem. Hence $E \subseteq X - U$. ■

Theorem 15. *Let X be a topological space and $x \in X$. If U and E are respectively locally minimal open and locally minimal closed sets at x with $U \neq E$, then following statements are true.*

(i) If $U \nsubseteq E$, then $U \cap E$ is not open.
(ii) If $E \nsubseteq U$, then $U \cap E$ is not closed.

Proof.

(i) Here $x \in U \cap E \subseteq U$. If $U \cap E$ is open, then by locally minimality of U at x implies that $U \cap E = U$ and which implies that $U \subseteq E$.
(ii) Here $x \in U \cap E \subseteq E$. If $U \cap E$ is closed, then by locally minimality of E at x implies that $U \cap E = E$ and which implies that $E \subseteq U$.

■

4. SOME RESULTS ON s-MEAN OPEN SETS

Definition 9. Let X be a topological space and $x \in X$. An open set U is said to be a s-mean open set at x if there exist open sets $V, W(\neq X)$ such that $x \in V \subsetneq U \subsetneq W$.

From the definition it is clear that if U is a s-mean open set at x then U is a mean open set containing x. On the other hand a s-mean open set V at a point x is neither a locally minimal open set at x nor a maximal open set, and conversely.

Example 2. (Steen and Seebach [8]) Let X be an infinite set and $a \in X$. We define $\mathbb{T} = \{\emptyset\} \bigcup \{G | a \in G \subseteq X\}$. In the topological space $(X, \mathbb{T})$, $\{a, b\}$ is a mean open set containing b but not a s-mean open set at x, for some $b \in X$ with $a \neq b$.

Theorem 16. *Let X be a T_1 connected topological space. Then for each x in X there is a s-mean open set at x.*

Proof. Let $x \in X$ be arbitrary. By the Theorem 5, there is a mean open set U, say, containing x. By the Lemma 1, U is infinite and so there is a $y \in U$ such that $x \neq y$. As X is T_1 , it follows that $U - \{y\}$ is a proper open set satisfying $x \in U - \{y\} \subsetneq U$. Again since U is a mean open set in X, there is a proper open set V such that $U \subsetneq V$. Thus we have proper open sets $U - \{y\}$, V satisfying $x \in U - \{y\} \subsetneq U \subsetneq V$. So U is s-mean open at x. ■

Theorem 17. *Let X be a T_1 connected topological space. Then each mean open set is s-mean open at each of its points.*

Proof. Let U be a mean open set and let $u \in U$. As U is a proper open set in X then there is a point x in $X - U$. Obviously $u \in U \subsetneq X - \{x\}$. Again by the Lemma 1, we can find a point y in U such that $u \neq y$. Since X satisfies T_1 axiom, it follows that $U - \{y\}$ is an open set containing u. Therefore we have proper open sets $X - \{x\}$ and $U - \{y\}$ such that $u \in U - \{y\} \subsetneq U \subsetneq X - \{x\}$ and so U is s-mean open at u. ■

Remark 1. If U is a locally minimal open set at a point x of a topological space X then U is unique (Theorem 7). But the same is not true for the case of s-mean open sets, i. e., at a certain point there may have multiple number of s-mean open sets. For this we consider the real number space $\mathbb{R}$ with the usual topology. $(-x, x)$ are s-mean open sets at 0, for each $x \in \mathbb{R}$ with $0 < x$.

Let X be a topological space and $x \in X$ be arbitrary. We write $\mathbb{M}_x$ to mean the family of s-mean open sets at x.

Theorem 18. *Let X be a T_1 connected topological space and $x \in X$. Then $\{\emptyset\} \cup \mathbb{M}_x$ is a local basis at x.*

Proof. Let $U(\neq X)$ be an open set containing x. By the Lemmam 1, U is not a minimal open set. If U is a mean open set then there is nothing to prove. Let U be a maximal open set containing x. Again by the Lemma 1, there must exists a y in U different from x. As X is a T_1 connected topological space, $U - \{y\}$ is an open set satisfying $x \in U - \{y\} \subsetneq U$. Then $U - \{y\}$ is not a maximal open set in X and thus by the Lemma 1, $U - \{y\}$ is a mean open set and so by the Theorem 5, $U - \{y\} \in \mathbb{M}_x$. Hence $\{\emptyset\} \cup \mathbb{M}_x$ is a local basis at x. ■

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