"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 29 (2019), No. 2, 121-138

CONSTRUCTIVE ORDERED ALGEBRAIC STRUCTURES

MARIAN ALEXANDRU BARONI

Abstract. Ordered algebraic structures are examined within the framework of Bishop-style constructive mathematics. In the constructive approach, the partial order is replaced by the classically equivalent, but constructively stronger, notion of co-order. While one could define an ordered algebraic structure by requiring certain properties of monotonicity of the algebraic operations, the constructive counterpart of strong monotonicity could be more appropriate for a constructive examination.

1. Introduction

Although linearly ordered algebraic structures have been thoroughly investigated in constructive mathematics ([7] or [9] for example), this is far from being the case when the linear order is replaced by a partial order relation. However, one can find a natural constructive definition of ordered vector spaces (first presented in [1]), as well as a more detailed examination of these structures [2]. On the other hand, noticeable efforts towards a constructive theory of ordered semigroups have been done by Romano and several co-authors [6, 13].

Keywords and phrases: constructive mathematics, strongly monotone function, co-ordered semigroup, co-ordered ring.
(2010) Mathematics Subject Classification: 03F60, 06F05, 06F25

Classically, an algebraic ordered structure is an algebraic structure which is endowed with a partial order relation which is compatible in a certain way with the algebraic operations. This compatibility can be expressed by the monotonocity of the operations. For example, an ordered vector space is a vector space with a partial order which is invariant under addition and multiplication by positive scalars. In other words, for each vector v and each scalar $\alpha \geq 0$, the addition to v and the multiplication by α are increasing operations. Our aim is to define the constructive counterpart of an ordered algebraic structure by using the notion of strong monotonicity of the operations with respect to a co-order relation, the natural constructive replacement of partial order.

Our investigation of order is done in a constructive manner based on the strict interpretation of existence as computability. In constructive mathematics, the impossibility of nonexistence does not imply existence. More precisely, we work in Bishop-style constructive mathematics (**BISH**) which, roughly speaking, is nothing else than mathematics carried out with intuitionstic logic. Every theorem of **BISH** is valid in classical mathematics and also in other varieties of constructive mathematics. The basic ideas of doing mathematics in this way are presented in Bishop's seminal book [4] and in the revised edition [5]. Further information on the varieties of constructive mathematics can be found in [8, 14]. For a constructive development of algebra, one might consult [11].

When working in **BISH**, the logical connectives and quantifiers should be interpreted in a different manner. For example, to prove $P \vee Q$ we need either a proof of P or a proof of Q. The strict use of intuitionistic logic entails serious restrictions which aim to protect our proofs from non-constructive logical principles such as the **law of excluded middle** $P \vee \neg P$. Moreover, we cannot apply the **weak law of the excluded middle** $\neg P \vee \neg\neg P$ or the **double negation** principle $\neg\neg P \Rightarrow P$. We can show that a certain proposition P is non-constructive by using a **Brouwerian example**, that is, we prove that P implies some non-constructive principle. The next three sources of non-constructivism, the so-called omniscience principles, are frequently used to produce Brouwerian examples.

- **The limited principle of omniscience (LPO)**: for every binary sequence (a_n) either $a_n = 0$ for all n, or else there exists n such that $a_n = 1$.

- **The weak limited principle of omniscience (WLPO)**: for every binary sequence (a_n) either $a_n = 0$ for all n, or it is contradictory that $a_n = 0$ for all n.
- **The lesser limited principle of omniscience (LLPO)**: if (a_n) is a binary sequence containing at most one term equal to 1, then either $a_{2n} = 0$ for all n, or else $a_{2n+1} = 0$ for all n.

We also did not accept in **BISH** the following logical principle.

- **Markov's principle (MP)**: if (a_n) is a binary sequence and $\neg\forall n\ (a_n = 0)$, then there exists n such that $a_n = 1$.

More details on non-constructive principles can be found in [10].

One of the main advantages of the constructive approach is the possibility of revealing distinctions between classically equivalent propositions. Thus, a major task is to give appropriate definitions which should be classically equivalent to the classical ones. By using appropriate constructive definitions we could prove one or more constructive counterparts of a classical theorem.

2. Basic definitions and notations

When working constructively, besides the peculiar interpretation of the logical operations, we should clarify the notions of set and function. Thus, to define a set we need to describe how we construct its elements and to determine when two elements are equal. Therefore each set comes has an **equality** $=$, which is nothing else than an equivalence relation. However, it would be more appropriate to have a method to prove that two elements are distinct. Consequently, we will consider each set S equipped with an **apartness relation** $\neq$, that is, an **irreflexive**, **symmetric**, and **cotransitive relation**:

$$\forall x, y \in S\ (x \neq y \Rightarrow \neg(x = y));$$

$$\forall x, y \in S\ (x \neq y \Rightarrow y \neq x);$$

$$\forall x, y, z \in S\ (x \neq y \Rightarrow (x \neq z \vee y \neq z)).$$

In addition, we will consider the apartness of each set as **tight**, that is,

$$\forall x, y \in S\ (\neg(x \neq y) \Rightarrow x = y).$$

Contrary to the classical case, the converse implication does not hold in **BISH**; if it were valid in general, then **MP** would hold.

To construct a function f from $(A, =_1, \neq_1)$ to $(B =_2, \neq_2)$, we need an algorithm which, applied to an element x of the set A, produces an element $f(x) \in B$. Each function is **extensional**, that is,

$$\forall x_1, x_2 \in A \ (x_1 =_1 x_2 \Rightarrow f(x_1) =_2 f(x_2)).$$

However, we have more computational information when we deal with strongly extensional functions. Let us consider the sets with apartness $(A, =_1, \neq_1)$ and $(B, =_2, \neq_2)$. A function $f : A \to B$ is called **strongly extensional** if

$$\forall x_1, x_2 \in A \ (f(x_1) \neq_2 f(x_2) \Rightarrow x_1 \neq_1 x_2).$$

Although strong extensionality is classically equivalent to extensionality, we cannot expect to prove that each extensional function is strongly extensional. As pointed out in [14], for a constructive approach to algebraic structures, it would be suitable to consider strongly extensional algebraic operations. By defining a co-ordered semigroup as a set with an associative strongly increasing operation, we obtain the strong extensionality of the semigroup operation. We will deal in a similar way with other algebraic structures.

The **Cartesian product** of the sets $(A, =_1, \neq_1)$ and $(B, =_2, \neq_2)$ is the set $(A \times B, =, \neq)$ with $A \times B = \{(a, b) : a \in A, \ b \in B\}$,

$$(a_1, b_1) = (a_2, b_2) \Leftrightarrow (a_1 =_1 b_1 \wedge a_2 =_2 b_2);$$

$$(a_1, b_1) \neq (a_2, b_2) \Leftrightarrow (a_1 \neq_1 b_1 \wedge a_2 \neq_2 b_2).$$

A **binary relation** on S is a subset of $S \times S$.

When dealing with order, the use of partial order relations will be replaced by a constructively stronger notion of co-order, based on von Plato's approach [12].

Definition 1. *Let $(S, =, \neq)$ be a set endowed with a binary relation $\nleq$. The relation $\nleq$ is called a* **partial co-order** *relation, or simply a* **co-order**, *if it is cotransitive and*

$$\forall x, y \in S \ (x \neq y \Leftrightarrow (x \nleq y) \vee (y \nleq x).$$

In this case, we say that $(S, =, \neq, \nleq)$ is a **co-ordered set**.

Following [12], we say that x **exceeds** y whenever $x \nleq y$. Clearly, a co-order is necessarily **strongly irreflexive**, $x \nleq y \Rightarrow x \neq y$, hence irreflexive. One can easily verify (as in [12]) that we obtain a **partial order** $\leq$ and a **strict partial order** $<$ as follows:

$$x \leq y \Rightarrow \neg(x \nleq y);$$

$$x < y \Leftrightarrow (x \leq y \wedge x \neq y).$$

We might write as usual, $x \geq y$ instead of $y \leq x$ and $x > y$ if $y < x$. A co-order relation $\nleq$ on S is a **linear order** if it is **asymmetric**: for all $x, y \in S$, $x \nleq y \Rightarrow \neg(y \nleq x)$ which is equivalent to $x \nleq y \Rightarrow x > y$.

For an arbitrary co-ordered set $(S, =, \neq, \nleq)$, we cannot prove constructively any of the next four properties. It suffices to consider the linear ordering of the real number set **R** to observe that each condition entails the non-constructive principle on the right-hand side. One might consider the real numbers as constructed in [4] or presented axiomatically in [7].

- $\forall x, y \in S \ (x \leq y \vee x \nleq y)$; (**LPO**)
- $\forall x, y \in S \ (x \leq y \vee \neg(x \nleq y))$; (**WLPO**)
- $\forall x, y \in S \ (x \leq y \vee y \leq x \vee (x \nleq y \wedge y \nleq x))$; (**LLPO**)
- $\forall x, y \in S \ (\neg(x \leq y) \Rightarrow x \nleq y)$. (**MP**)

3. Strongly increasing algebraic operations

In classical mathematics, an ordered semigroup is a semigroup with a partial order relation which is preserved by the semigroup operation. In other words, a semigroup $(S, \cdot)$ with a partial order $\leq$ is an ordered semigroup if for each element $x \in S$ the right multiplication and the left multiplication by x are increasing functions from S to S. Since the partial order is too weak a notion for a constructive examination of order, the partial order $\leq$ will be replaced by a co-order relation $\nleq$. Correspondingly, the classical monotonicity of the algebraic operations should be replaced by a stronger notion of monotonicity defined in [2].

Definition 2. *Let* $(S_1, =_1, \neq_1, \nleq_1)$ *and* $(S_2, =_2, \neq_2, \nleq_2)$ *be co-ordered sets and* f *a function from* S_1 *to* S_2*. The function* f *is said to be:*

- **increasing** *if* $\forall x, y \in S_1 \ (x \leq_1 y \Rightarrow f(x) \leq_2 f(y))$;
- **strongly increasing** *if* $\forall x, y \in S_1 \ (f(y) \nleq_2 f(x) \Rightarrow y \nleq_1 x)$.

The notion of **strongly decreasing** function is defined correspondingly:

$$\forall x, y \in S_1 \ (f(y) \nleq_2 f(x) \Rightarrow x \nleq_1 y).$$

As in the classical case, where a function is monotone whenever it is increasing or decreasing, we say that f is **strongly monotone** if it is strogly increasing or strongly decreasing. One can easily observe that every strongly increasing function is increasing and strongly extensional. However we cannot expect to prove constructively that every increasing function is strongly increasing. We can prove that, provided

S_1 is a lattice, each strongly extensional increasing function is necessarily strongly increasing. The proof and more details on monotonicity can be found in Chapter 5 of [2].

We can use this notion of strong monotonicity to define co-ordered algebraic structures; for the classical ordered structures we will use the term 'weakly ordered'. For example, let us consider a semigroup with apartness $(S, =, \neq, \cdot)$, endowed with a co-order relation $\nleq$. Then $(S, =, \neq, \cdot, \nleq)$ is a **co-ordered semigroup** if for all $z \in S$ the right multiplication R_z and the left multiplication L_z defined by $R_z(y) = y \cdot z$ and $L_z(y) = z \cdot y$ are strongly increasing functions from S to S. Thus we obtain the natural constructive definition of a co-ordered semigroup:

$$\forall x, y, z \in S\ ((x \cdot z \nleq y \cdot z \vee z \cdot x \nleq z \cdot y) \Rightarrow x \nleq y),$$

as given, for example, in [6]. The semigroup $(S, =, \cdot)$ endowed with a partial order relation $\leq$ is a **weakly ordered semigroup** if for all $z \in S$ the right multiplication R_z and the left multiplication L_z are increasing functions from S to S. In other words, $(S, =, \cdot, \leq)$ satisfies the classical definition:

$$\forall x, y, z \in S\ (x \leq y \Rightarrow (x \cdot z \leq y \cdot z \vee z \cdot x \leq z \cdot y)).$$

Since every strongly increasing function is increasing and strongly extensional, if S is a co-ordered semigroup then S is a weakly ordered semigroup and the right and left multiplications are strongly extensional. We will prove, as a consequence of the following theorem, that the semigroup operation is strongly increasing, and therefore strongly extensional, as a function from $S \times S$ to S.

Theorem 1. *Let us consider the co-ordered sets* $(S_1, =_1, \neq_1, \nleq_1)$, $(S_2, =_2, \neq_2, \nleq_2)$ *and* $(S_3, =_3, \neq_3, \nleq_3)$ *and the function* $f : S_1 \times S_2 \to S_3$. *For each* $z \in S_1$ *and* $w \in S_2$, *let us consider the functions* $R_z : S_1 \to S_3,\ R_z(x) = f(x, z)$ *and* $L_w : S_2 \to S_3,\ L_w(y) = f(w, y)$. *Then the following conditions are equivalent:*

(1) *For all* $z \in S_1$ *and* $w \in S_2$, *the functions* R_z *and* L_w *are strongly increasing.*

(2) *The function* f *is strongly increasing.*

Proof. Denote by $\nleq$ the product co-order on $S_1 \times S_2$, that is,

$$\forall (x_1, y_1), (x_2, y_2) \in S_1 \times S_2 ((x_1, y_1) \nleq (x_2, y_2) \Leftrightarrow (x_1 \nleq_1 x_2 \vee y_1 \nleq_2 y_2)).$$

(See Section 6.)

We first prove (1) $\Rightarrow$ (2). Assuming (1), we have to prove that

$$\forall (x_1, y_1), (x_2, y_2) \in S_1 \times S_2 \ (f(x_1, y_1) \nleq_3 f(x_2, y_2) \Rightarrow (x_1, y_1) \nleq (x_2, y_2)).$$

To this end, let us consider $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$ such that $f(x_1, y_1) \nleq_3 f(x_2, y_2)$. Then, it follows from the cotransitivity of $\nleq_3$ that $f(x_1, y_1) \nleq_3 f(x_1, y_2)$ or $f(x_1, y_2) \nleq_3 f(x_2, y_2)$. In the former case, since L_{x_1} is strongly increasing, we obtain $y_1 \nleq_2 y_2$. In the latter one, since R_{y_2} is strongly increasing, $x_1 \nleq_1 x_2$. Consequently, $x_1 \nleq_1 x_2$ or $y_1 \nleq_2 y_2$, that is, $(x_1, y_1) \nleq (x_2, y_2)$.

To prove (2) $\Rightarrow$ (1), let z and w be arbitrary elements of S_1, respectively S_2; we have to prove that the functions R_z and L_w are strongly increasing. Let x_1, x_2 be elements of S_1 such that $R_z(x_1) \nleq_3 R_z(x_2)$, that is, $f(x_1, z) \nleq_3 f(x_2, z)$. Since f is strongly increasing, it follows that $(x_1, z) \nleq (x_2, z)$ and, consequently, $x_1 \nleq_1 x_2$ or $z \nleq_2 z$. The latter is contradictory, so that $x_1 \nleq_1 x_2$ which, in turn, entails that R_z is strongly increasing. Similarly, we can prove that for all y_1, y_2 in S_2, $L_w(y_1) \nleq_3 L_w(y_2)$ implies $y_1 \nleq_2 y_2$ hence L_w is a strongly increasing mapping. □

Corollary 2. *Let $(S, =, \neq, \cdot, \nleq)$ be a semigroup endowed with a co-order $\nleq$. Then S is a co-ordered semigroup if and only if the semigroup operation is a strongly increasing function from $S \times S$ to S, that is,*

$$\forall (x_1, x_2), (y_1, y_2) \in S \times S \ (x_1 \cdot y_1 \nleq x_2 \cdot y_2 \Rightarrow (x_1 \nleq y_1 \vee x_2 \nleq y_2)).$$

Proof. This is a direct consequence of Theorem 1. □

Lemma 3. *If S is a co-ordered semigroup, then for all $x, y, z, w \in S$*

$$(z \cdot x \cdot w \nleq z \cdot y \cdot w \Rightarrow x \nleq y).$$

Proof. If $z \cdot x \cdot w \nleq z \cdot y \cdot w$, then $x \cdot w \nleq y \cdot w$, which entails $x \nleq y$. □

For a commutative semigroup we have simpler conditions, using only one-sided (right or left) multiplications.

Corollary 4. *Let $(S, =, \neq, \cdot, \nleq)$ a commutative semigroup. Then the following conditions are equivalent.*

(1) *The semigroup S is a co-ordered semigroup*

(2) *For each $z \in S$, the right mutiplication by z is strongly increasing, that is,*

$$\forall x, y, z \in S \ (x \cdot z \nleq y \cdot z \Rightarrow x \nleq y).$$

(3) *For each $z \in S$, the left mutiplication by z is strongly increasing, that is,*

$$\forall x, y, z \in S \ (z \cdot x \nleq z \cdot y \Rightarrow x \nleq y).$$

Proof. The proof is straightforward. □

Since a monoid is a semigroup with an identity element, a **co-ordered monoid** is a monoid $(M, =, \neq, \cdot, \nleq)$ which is a co-ordered semigroup with respect to the co-order.

Proposition 5. *Let $(M, =, \neq, \cdot, \nleq)$ a monoid with a co-order relation. Then M is a co-ordered monoid if and only if*

$$\forall x, y, z, w \in M\ (z \cdot x \cdot w \nleq z \cdot y \cdot w \Rightarrow x \nleq y).$$

Proof. It follows from Lemma 3 that each co-ordered monoid satisfies the above condition. Conversely, let x, y, z be elements of M such that $x \cdot z \nleq y \cdot z$ or $z \cdot x \nleq z \cdot y$. If 1 is the identity element, then $1 \cdot x \cdot z \nleq 1 \cdot y \cdot z$ or $z \cdot x \cdot 1 \nleq z \cdot y \cdot 1$ and, as a consequence, $x \nleq y$. □

To illustrate these notions, let us consider the sets S and M of all the continuous functions from $\mathbf{R}$ to the set $\{x \in \mathbf{R} : x > 1\}$, respectively to $\{x \in \mathbf{R} : x \geq 1\}$, with the standard equality of functions and the natural apartness $f \neq g \Leftrightarrow \exists x\ (f(x) \neq g(x))$. Then M is a co-ordered monoid and S is a co-ordered semigroup with respect to the standard multiplication of functions and the co-order defined by $f \nleq g \Leftrightarrow \exists x\ (f(x) > g(x))$.

4. Co-ordered groups

We can define a weakly ordered group, respectively a co-ordered group, as a group which is also a weakly ordered semigroup, respectively a co-ordered semigroup, with respect to the group operation.

Definition 3. (i) *Let $(G, =, +)$ be a group and $\leq$ a partial order on G. Then $(G, =, +, \leq)$ is a* **weakly ordered group** *if*

$$\forall x, y, z \in G\ (x \leq y \Rightarrow ((z + x \leq z + y) \wedge (x + z \leq y + z)).$$

(ii) *Let $(G, =, \neq, +)$ be a group with apartness endowed with a co-order relation $\nleq$. Then $(G, =, \neq, +, \nleq)$ is a* **co-ordered group** *if*

$$\forall x, y, z \in G\ ((z + x \nleq z + y \vee x + z \nleq y + z) \Rightarrow x \nleq y).$$

Clearly, each co-ordered group is a weakly ordered group. Since every co-ordered group is a co-ordered semigroup, the one-sided additions as well as the binary operation of addition are strongly increasing, hence strongly extensional. Each co-ordered group is a co-ordered monoid too, so we can apply Proposition 5 to give another equivalent condition for a co-ordered group, respectively a co-ordered commutative group.

Corollary 6. *Let $(G, =, \neq, \cdot, \nleq)$ a group with a co-order relation. Then G is a co-ordered group if and only if*

$$\forall x, y, z, w \in G\ z + x + w \nleq z + y + w \Rightarrow x \nleq y.$$

Proof. It easily follows from Proposition 5. □

When dealing with a group, the compatibility between the co-order relation and the group operation can be expressed not only by using the strong monotonocity, as in the case of semigroups and monoids, but also by requiring the group operation to preserve the co-order. Actually, this means that for each z the left and right addition with $-z$ is strongly increasing which is another way of asserting that G is a co-ordered group.

Proposition 7. *Let $(G, =, \neq, +, \nleq)$ be a group.*

(i) *The group G is a co-ordered group if and only if*

$$\forall x, y, z \in G\ (x \nleq y \Rightarrow (x + z \nleq y + z \wedge z + x \nleq z + y)).$$

(ii) *If G is commutative, then G is a co-ordered group if and only if*

$$\forall x, y, z \in G\ (x \nleq y \Rightarrow x + z \nleq y + z)$$

and, in addition, if and only if

$$\forall x, y, z \in G\ (x \nleq y \Rightarrow z + x \nleq z + y).$$

Proof. (i) Assume that G is a co-ordered group and consider $x, y \in G$ with $x \nleq y$. Then $x+z-z \nleq y+z-z$ and it follows from the definition that $x+z \nleq y+z$. Similarly we can prove that $z+x \nleq z+y$. Conversely, if $x + z \nleq y + z$ then $x + z - z \nleq y + z - z$. Similarly, we can prove that $x \nleq y$ whenever $z + x \nleq z + y$.

(ii) It easily follows from (i). □

We denote the identity element of the group G by 0.

Corollary 8. *Let G be a co-ordered group. Then for all $x, y \in G$*

(i) $x \nleq y \Leftrightarrow -y \nleq -x$;

(ii) $x \neq y \Leftrightarrow x - y \neq 0$.

Proof. (i) According to Proposition 7, the condition $x \nleq y$ is equivalent to $0 \nleq y - x$ which, in turn, is equivalent to $-y \nleq -x$.

(ii) From the definition of co-order, $x \neq y \Leftrightarrow x \nleq y \vee y \nleq x$. In the former case $x - y \nleq 0$ and in the latter $0 \nleq x - y$. □

Corollary 9. *Let x, y be elements of the co-ordered group G. Then the function*

$$f : G \to G,\ f(x) = -x$$

is strongly decreasing.

Proof. It is an immediate consequence of Corollary 8. □

Let x be an element of the co-ordered group G. Then x is said to be **positive** if $x \geq 0$, that is, $0 \leq x$. A **negative** element x is defined correspodingly by the condition $x \leq 0$. Clearly, x is positive if and only if $-x$ is negative. We denote, as usual by G^+ and G^- the set of the positive, respectively negative, elements of G.

Example 1. *Let $(G, =, \neq, +)$ a group with a strongly extensional addition. Then G is a co-ordered group with respect to the co-order $\nleq$ defined by $x \nleq y \Leftrightarrow x \neq y$, $x \leq y \Leftrightarrow x = y$, and there is no element of G with $x < 0 \vee x > 0$. Moreover, the functions*

$$f_k : G \to G,\ f(x) = k \cdot x;\ k \in \mathbf{Z}^*$$

are both strongly increasing and strongly decreasing.

5. Co-ordered rings

We now examine algebraic structures endowed with more than one operation. For example, if we consider a ring, we should require an increasing addition, therefore the ring should be a weakly ordered group with respect to addition. Moreover, both right and left multiplications by positive elements should be increasing. We will define the corresponding constructive counterpart, the co-ordered ring, to get a strongly increasing addition, strongly increasing multiplications by positive elements and, as expected, a strongly extensional binary operation of multiplication. As usual, we will denote by 0 the additive identity element of the ring.

Definition 4. (i) *The ring $(R, =, +, \cdot, \leq)$ is a* **weakly ordered ring** *if R is a weakly ordered group with respect to the addition $+$ and*

$$\forall x, y, z \in R\ ((x \leq y \wedge 0 \leq z) \Rightarrow (x \cdot z \leq y \cdot z \wedge z \cdot x \leq z \cdot y)).$$

(ii) *The ring $(R, =, \neq, +, \cdot, \nleq)$ is a* **co-ordered ring** *if R is a co-ordered group with respect to the addition and*

$$\forall x, y \in R\ (x \cdot y \neq 0 \Rightarrow (x \neq 0 \wedge y \neq 0));$$

$$\forall x, y, z \in R\ ((z \cdot x \nleq z \cdot y \vee x \cdot z \nleq y \cdot z) \Rightarrow (0 \nleq z \vee x \nleq y)).$$

We should notice that although $x \neq 0$ and $y \neq 0$ whenever $x \cdot y \neq 0$, this is not the case for

$$\forall x, y \in R\ (x \cdot y = 0 \Rightarrow (x = 0 \vee y = 0)).$$

To prove this, it suffices to consider the ring of the real numbers with the usual addition and multiplication. If this property were true, then

LLPO would hold. Nevertheless the property of extensionality from the definition of a co-ordered ring ensures the strong extensionality of the multiplication.

Corollary 10. *If R is a co-ordered ring, then the binary operations are strongly extensional functions from $R \times R$ to R.*

Proof. Since R is a co-ordered group, the addition is strongly extensional. To prove that the multiplication satisfies the same property, let (x, y) and (z, w) be elements of $R \times R$ such that $x \cdot y \neq z \cdot w$. Then, either $x \cdot y \neq x \cdot w$ or $x \cdot w \neq z \cdot w$. In the former case, we obtain, $x \cdot y - x \cdot w \nleq 0$ hence $x \cdot (y - w) \nleq 0)$. It follows from the definition of the co-ordered ring that $y - w \neq 0$ and, in turn, by applying Corollary 8, $y \neq w$. In the latter case, we obtain in a similar way $x \neq z$. Consequently, $(x, y) \neq (z, w)$. □

Proposition 11. *Let $(R, =, \neq, +, \cdot)$ be a ring equipped with a co-order $\nleq$. Then the following conditions are equivalent.*

(1) *The ring R is a co-ordered ring.*

(2) *For all elements x, y of R, $x \cdot y \neq 0 \Rightarrow (x \neq 0 \wedge y \neq 0)$ and $0 \nleq x \cdot y \Rightarrow (0 \nleq x \vee 0 \nleq y)$.*

Proof. Assuming (1), we have to prove that $0 \nleq x \cdot y$ entails $0 \nleq x$ or $0 \nleq y$. If $0 \nleq x \cdot y$, then $x \cdot 0 \nleq x \cdot y$. It follows that $0 \nleq x$ or $0 \nleq y$. Therefore (1) $\Rightarrow$ (2).

Assume now (2) and let $x, y, z \in R$ with $z \cdot x \nleq z \cdot y$. Then, according to Corollary 8, $0 \nleq z \cdot (y - x)$ hence $0 \nleq z$ or $0 \nleq y - x$, the latter being equivalent to $x \nleq y$. Similarly, from $x \cdot z \nleq y \cdot z$, we obtain $0 \nleq z$ or $x \nleq y$. Therefore (2) implies (1). □

For each element z of a ring, we will denote by L_z andR_z the left, respectively the right, multiplication by z:

$$L_z : R \to R, \;\; L_z(x) = z \cdot x; \;\; R_z : R \to R, \;\; R_z(x) = x \cdot z.$$

Lemma 12. *Let $(R, =, \neq, +, \cdot)$ be a ring with a coorder and assume that $(R, =, \neq, +, \cdot)$ is a co-ordered group. Let us consider the following properties.*

(1) *For each $z \geq 0$, L_z is strongly increasing.*

(2) *For each $z \leq 0$, L_z is strongly decreasing.*

(3) *For each $z \geq 0$, R_z is strongly increasing.*

(4) *For each $z \leq 0$, R_z is strongly decreasing.*

(5) *The function $f : R^+ \times R^+ \to R$, $f(x, y) = x \cdot y$ is strongly increasing.*

(6) *The function* $g : R^{-} \times R^{-} \to R$, $g(x,y) = x \cdot y$ *is strongly decreasing.*

Then (1) $\Leftrightarrow$ (2), (3) $\Leftrightarrow$ (4), *and* (5) $\Leftrightarrow$ (6).

Proof. To prove (1) $\Rightarrow$ (2) let x, y, z be elements of R such that $z \leq 0$ and $L_z(x) \nleq L_z(y)$, that is, $z \cdot x \nleq z \cdot y$. Then $-z \geq 0$ and $(-z) \cdot (-x) \nleq (-z) \cdot (-y)$ or, equivalently, $L_{-z}(-x) \nleq L_{-z}(-y)$. Acording to (1), $-x \nleq -y$, and, from Corollary 8, $y \nleq x$. Thus, L_z is strongly decreasing. Similarly, we can prove the converse implication. The equivalence of (3) and (4) can be proved in a similar manner.

Let us now prove (5) $\Rightarrow$ (6). To this end, let x, y, z, w be elements of R^{-} such that $g(x,y) \nleq g(z,w)$., that is, $x \cdot y \nleq z \cdot w$ or, equivalently, $(-x) \cdot (-y) \nleq (-z) \cdot (-w)$ which, in turn, is equivalent to $f(-x,-y) \nleq f(-z,-w)$. Since f is strongly increasing, we obtain $(-x,-y) \nleq (-z,-w)$ which is equivalent to $-x \nleq -z$ or $-y \nleq -w$. Therefore $z \nleq x$ or $w \nleq y$, that is, $(z,w) \nleq (x,y)$ and, as a consequence, g is strongly decreasing. In a similar way, we prove the converse implication. □

Proposition 13. *Let* $(R, =, \neq, +, \cdot)$ *be a ring endowed with a co-order relation* $\nleq$ *such that* $(R, =, \neq, +)$ *is a co-ordered group. Let us consider the following conditions:*

(1) *The ring* $(R, =, \neq, +, \cdot)$ *is a co-ordered ring.*

(2) *For all elements* x, y, z *of* R

$$(z \cdot x \nleq z \cdot y \vee x \cdot z \nleq y \cdot z) \Rightarrow (0 \nleq z \vee x \nleq y).$$

(3) *For each* $z \geq 0$, *the multiplication functions* L_z *and* R_z *are strongly increasing.*

(4) *The function* $f : R^{+} \times R^{+} \to R$, $f(x,y) = x \cdot y$ *is strongly increasing.*

(5) *The ring* R *is weakly ordered.*

Then each of the conditions (1)–(4) *implies the next one.*

Proof. Clearly, (1) implies (2). To prove (2) $\Rightarrow$ (3), let x, y, z be elements of R with $z \geq 0$ and $z \cdot x \nleq z \cdot y$. Then, according to (2), $0 \nleq z$ or $x \nleq y$. The former is contradictory to $z \geq 0$, so that the latter holds. Similarly, if $z \geq 0$ and $x \cdot z \nleq y \cdot z$, then $x \nleq y$. Therefore (2) $\Rightarrow$ (3).

The implication (3) $\Rightarrow$ (4) follows from Theorem 1. To prove (4) $\Rightarrow$ (5), let x, y, z be elements of R such that $z \geq 0$ and $x \leq y$, that is $y - x \geq 0$. Assume that $z \cdot x \nleq z \cdot y$, that is $0 \nleq (z \cdot y - x)$. Therefore $f(0,0) \nleq f(z, y - x)$. Since f is strongly increasing, $(0,0)$ exceeds

$(z, y - w)$ with respect to the product co-order on $R^+ \times R^+$ which is equivalent to $0 \nleq z$ or $0 \nleq y - x$, contradictory to the assumption $0 \leq z \wedge 0 \leq y - x$. □

As a consequence, the co-ordered rings satisfy the usual rules of signs for weakly ordered rings:

$$((x \geq 0 \wedge y \geq 0) \vee (x \leq 0 \wedge y \leq 0)) \Rightarrow (x \cdot y \geq 0);$$

$$((x \geq 0 \wedge y \leq 0) \vee (x \leq 0 \wedge y \geq 0)) \Rightarrow (x \cdot y \leq 0).$$

If the co-order is a linear order, then the conditions (1)-(4) from Proposition 13 are equivalent, as it results from the following theorem.

Theorem 14. *Let $(R, =, \neq, +, \cdot)$ be a ring that satisfies the condition $x \cdot y \neq 0 \Rightarrow (x \neq 0 \wedge y \neq 0)$ and assume that $>$ is a linear order on R. If $(R, =, \neq, +, >)$ is a linearly co-ordered group, then the following conditions are equivalent.*

(1) *The ring $(R, =, \neq, +, \cdot, >)$ is a co-ordered ring.*

(2) *For each $z \geq 0$, the multiplication functions L_z and R_z are strongly increasing.*

(3) *For each $z \leq 0$, the multiplication functions L_z and R_z are strongly decreasing.*

(4) *The function $f : R^+ \times R^+ \to R^+$ defined by $f(x, y) = x \cdot y$ is strongly increasing.*

(5) *The function $g : R^- \times R^- \to R^+$ defined by $f(x, y) = x \cdot y$ is strongly decreasing.*

Proof. On the one hand, according to Proposition 13, (1) ⇒ (2) and (2) ⇒ (4). On the other hand, (2) and (3) are equivalent, as well as (4) and (5) (Lemma 12). It suffices to prove (4) ⇒ (1). To this end, let $x, y \in R$ with $x \cdot y < 0$. We have to prove that either $x < 0$ or else $y < 0$. Since $x \cdot y \neq 0$, both $x \neq 0$ and $y \neq 0$. The co-order is linear, so $x \neq 0$ is equivalent to $x < 0 \vee x > 0$. We apply now the rules of signs; if $x > 0 \wedge y > 0$ or $x < 0 \wedge y < 0$, then $x \cdot y \geq 0$ which is contradictory to $x \cdot y < 0$. It follows that either $x < 0$ or else $y < 0$. □

We illustrate these notions by an example. Let us consider the set $(C[0, 1], =, \neq, +, \cdot, \nleq)$ the set of all continuous functions from $[0, 1]$ to $\mathbf{R}$, with the usual apartness $f \neq g \Leftrightarrow \exists x \ (f(x) \neq g(x))$, the standard addition and multiplication of functions, and the natural co-order defined by $f \nleq g \Leftrightarrow \exists x \ (f(x) > g(x))$. Then $C[0, 1]$ is a co-ordered ring.

6. Applications

If A is a co-ordered algebraic structure, then we can organize the set $A^S = \{f : S \to A\}$ as a corresponding co-ordered algebraic structure. The equality, apartness, and co-order relation on A^S are defined by

$$f = g \Leftrightarrow \forall x \in S\ (f(x) = g(x));$$

$$f \neq g \Leftrightarrow \exists x \in S\ (f(x) \neq g(x));$$

$$f \not\leq g \Leftrightarrow \exists x \in S\ (f(x) \not\leq g(x)).$$

It is straightforward to prove that we have obtained a tight apartness and a co-order on the set A^S. Further we obtain a partial order:

$$f \leq g \Leftrightarrow \neg(f \not\leq g) \Leftrightarrow \forall x \in S\ \neg(f(x) \not\leq g(x) \Leftrightarrow \forall x \in S\ (f(x) \leq g(x)).$$

If $*$ is a binary operation on A, then we can define a corresponding operation on A^S by

$$f * g : S \to A,\ f * g(x) = f(x) * g(x).$$

Clearly, if A is a semigroup, a monoid, or a group, then A^S satisfies the same property. Similarly, if A is a ring, then A^S is a ring with the corresponding operations.

Proposition 15. *If S is an inhabited set with apartness, then the following implications hold.*

(i) *If $(A, =, \neq, \cdot, \not\leq)$ is a co-ordered semigroup, then $(A^S, =, \neq, \cdot, \not\leq)$ is a co-ordered group.*

(ii) *If $(A, =, \neq, +, \not\leq)$ is a co-ordered group, then $(A^S, =, \neq, \cdot, \not\leq)$ is a co-ordered group.*

(iii) *If $(A, =, \neq, +, \cdot \not\leq)$ is a co-ordered ring, then $(A^S, =, \neq, \cdot, \not\leq)$ is a co-ordered ring.*

Proof. (i) If $f \cdot g \not\leq f \cdot h$, then there exists $x \in S$ such that $f(x) \cdot g(x) \not\leq f(x) \cdot h(x)$. Since A is a semigroup, it follows that $g(x) \not\leq h(x)$ hence $g \not\leq h$. Similarly, $g \cdot f \not\leq h \cdot f$ entails $g \not\leq f$. Therefore A^S is a co-ordered semigroup.

(ii) It follows from (i), taking into account that A^S is a group.

(iii) Since A is a co-ordered ring, then it is a co-ordered group with respect to addition and, from (ii), A^S is also a co-ordered group. Denote by $\mathbf{0}$ the zero function defined by $\mathbf{0}(x) = 0$ for all $x \in S$. Let f, g be functions from S to A such that $\mathbf{0} \not\leq f \cdot g$. Then there exists $x \in S$ such that $0 \not\leq f(x) \cdot g(x)$. Since R is a co-ordered ring, $0 \not\leq f(x)$ or $0 \not\leq g(x)$. Therefore $\mathbf{0} \not\leq f$ or $\mathbf{0} \not\leq g$.

We have to prove the strong extensionality of the multiplication. To this end, it suffices to prove that $f \cdot g \neq \mathbf{0}$ entails $f \neq \mathbf{0}$ and $g \neq \mathbf{0}$. If $f \cdot g \neq \mathbf{0}$, then there is $x \in S$ such that $f(x) \cdot g(x) \neq 0$ and, from the strong extensionality of the multiplication on A, $f(x) \neq 0$ and $g(x) \neq 0$ hence $f \neq \mathbf{0}$ and $g \neq \mathbf{0}$. □

We have already used the product co-order in order to examine the strong monotonicity of algebraic operations. We will show that the Cartesian product of n co-ordered semigroups is also a co-ordered semigroup with respect to the product co-order. Similar results will be proved for the other co-ordered algebraic structures studied in this paper.

To this end, let us consider the co-ordered sets $(S_i, =_i, \neq_i, \nleq_i)$, with $1 \leq i \leq n$. The standard equality and apartness on the Cartesian product $S = S_1 \times S_2 \times \cdots \times S_n$ are defined by

$$(x_1, x_2, \cdots, x_n) = (y_1, y_2, \cdots, y_n) \Leftrightarrow (\forall i \in \{1, 2, \cdots, n\}\ (x_i =_i y_i))$$

$$(x_1, x_2, \cdots, x_n) \neq (y_1, y_2, \cdots, y_n) \Leftrightarrow (\exists i \in \{1, 2, \cdots, n\}\ (x_i \neq_i y_i)).$$

Since all the apartness relations $\neq_i$ are tight, it follows that

$$\neg((x_1, x_2, \cdots, x_n) \neq (y_1, y_2, \cdots, y_n)) \Leftrightarrow (\forall i \in \{1, 2, \cdots, n\}\ (x_i =_i y_i));$$

hence the apartness on S is tight. The natural **product co-order** on S is given by

$$(x_1, x_2, \cdots, x_n) \nleq (y_1, y_2, \cdots, y_n) \Leftrightarrow (\exists i \in \{1, 2, \cdots, n\}\ (x_i \nleq_i y_i))$$

and the corresponding partial order $\leq$ is the **product order** defined by

$$(x_1, x_2, \cdots, x_n) \leq (y_1, y_2, \cdots, y_n) \Leftrightarrow (\forall i \in \{1, 2, \cdots, n\}\ (x_i \leq_i y_i)).$$

More details on product co-order can be found in [3].

Theorem 16. *Consider the sets* $(S_1, =_1, \neq_1, \nleq_1)$, $(S_2, =_2, \neq_2, \nleq_2)$, $\cdots$, $(S_n, =_n, \neq_n, \nleq_n)$ *endowed with corresponding algebraic operations. Then the following properties hold.*

(i) *If* $S_1, S_2, \cdots, S_n$ *are co-ordered semigroups, then* $S_1 \times S_2 \times \cdots \times S_n$ *is a co-ordered semigroup.*

(ii) *If* $S_1, S_2, \cdots, S_n$ *are co-ordered groups, then* $S_1 \times S_2 \times \cdots \times S_n$ *is a co-ordered group.*

(iii) *If* $S_1, S_2, \cdots, S_n$ *are co-ordered rings, then* $S_1 \times S_2 \times \cdots \times S_n$ *is a co-ordered ring.*

Proof. To avoid cumbersome notation, we will consider without loss of generality the case $n = 2$ and we will omit the indices for the algebraic operation on S_1 and S_2 and for the identity elements of the rings.

(i) Thus, we define the product $S_1 \times S_2, =, \neq, *, \nleq$ of the semigroups S_1 and S_2 with the product co-order and the usual binary operation $*$, defined by $(x_1, x_2) * (y_1, y_2) = (x_1 \cdot y_1, x_2 \cdot y_2)$. As in the classical case, the operation $*$ is associative so that $S_1 \times S_2$ is a semigroup.

We now prove that $S_1 \times S_2$ is a co-ordered semigroup. To this end, consider $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in S_1 \times S_2$ such that

$$(z_1, z_2)*(x_1, x_2) \nleq (z_1, z_2)*(y_1, y_2) \vee (x_1, x_2)*(z_1, z_2) \nleq (y_1, y_2)*(z_1, z_2),$$

that is,

$$(z_1 \cdot x_1, z_2 \cdot x_2) \nleq (z_1 \cdot y_1, z_2 \cdot y_2) \vee (x_1 \cdot z_1, x_2 \cdot z_2) \nleq (y_1 \cdot z_1, y_2 \cdot z_2).$$

In the former case, $z_1 \cdot x_1 \nleq_1 z_1 \cdot y_1 \vee z_2 \cdot x_2 \nleq_2 z_2 \cdot y_2$. Since S_1 and S_2 are co-ordered semigroups, it follows that $x_1 \nleq_1 y_1$ or $x_2 \nleq_2 y_2$, therefore $(x_1, y_1) \nleq (x_2, y_2)$. In the latter case, we obtain in a similar way the same conclusion.

(ii) The Cartesian product is a group and also a co-ordered semigroup and therefore a co-ordered group.

(iii) If S_1 and S_2 are co-ordered rings, then it follows from (ii) that $S_1 \times S_2$ is a co-ordered group. Consider now $(x_1, x_2), (y_1, y_2)$ in $S_1 \times S_2$ with $(x_1, x_2) \cdot (y_1, y_2) \neq (0, 0)$. It follows that $x_1 \cdot y_1 \neq_1 0$ or $x_2 \cdot y_2 \neq_2 0$. In the former case, $x_1 \neq_1 0$ and $y_1 \neq_1 0$; in the latter, $x_2 \neq_2 0$ and $y_2 \neq_2 0$. As a consequence, $(x_1, x_2) \neq (0, 0)$ and $(y_1, y_2) \neq (0, 0)$.

Suppose now $(0, 0) \nleq (x_1, x_2) \cdot (y_1, y_2)$, that is, $(0, 0) \nleq (x_1 \cdot y_1, x_2 \cdot y_2)$. Consequently, $0 \nleq_1 x_1 \cdot y_1$ or $0 \nleq_2 x_2 \cdot y_2$. In the former case, $0 \nleq_1 x_1$ or $0 \nleq_1 y_1$ and in the latter, $0 \nleq_2 x_2$ or $0 \nleq_2 y_2$. Thus we obtain $(0, 0) \nleq (x_1, x_2)$ or $(0, 0) \nleq (y_1, y_2)$. □

To illustrate this theorem we consider $\mathbf{R}^n$ with the standard operations:

$$(x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n) = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n);$$

$$(x_1, x_2, \cdots, x_n) * (y_1, y_2, \cdots, y_n) = (x_1 y_1, x_2 y_2, \cdots, x_n y_n);$$

Then $(\mathbf{R}^n, +)$ is a co-ordered group and $(\mathbf{R}^n, +, *)$ is a co-ordered ring.

If we consider the co-order on $\mathbf{R}$ defined in Example 1, we can define a co-order relation on $\mathbf{R} \times \mathbf{R}$ by

$$(x_1, x_2) \nleq (x_2, y_2) \Leftrightarrow (x_1 \neq y_1 \vee x_2 > y_2).$$

This co-order leads us to the partial order $\leq$:

$$(x_1, x_2) \leq (x_2, y_2) \Leftrightarrow \neg(x_1 \neq y_1 \vee x_2 > y_2) \Leftrightarrow (x_1 = y_1 \wedge x_2 \leq y_2).$$

According to Theorem 16, $\mathbf{R} \times \mathbf{R}$ is a co-ordered group with respect to this co-order too.

Other co-ordered algebraic structures can be defined in a similar manner, emphasizing the relation between the co-order and the strong monotonicity of the algebraic operations.

References

[1] M.A. Baroni, On the order dual of a Riesz space, in: *Discrete Mathematics and Theoretical Computer Science* (C.S. Calude, M.J. Dinneen, V. Vajnovski eds.), Lecture Notes in Computer Science**2731**, 109-117, Springer-Verlag, Berlin, 2003.

[2] M.A. Baroni, *The Constructive Theory of Riesz Spaces and Applications in Mathematical Economics*, PhD thesis, University of Canterbury, Christchurch, New Zealand, 2004.

[3] M.A. Baroni, On the constructive product order, *Stud. Cercet. Ştiint., Ser. Mat.*, **17**, 1-8, 2007. supplement Procedings of CNMI 2007, Bacău, Nov. 2007.

[4] E. Bishop, *Foundations of constructive analysis*, Mc-Graw Hill, New York, 1967.

[5] E. Bishop and D. Bridges, *Constructive Analysis*, Grundlehren der matematischen Wissenschaften **279**, Springer-Verlag, Berlin, 1985.

[6] D. Bogdanić, S. Crvenković, D.A. Romano, Another Isomorphism Theorem on Anti-ordered Semigroups, *Int. J. Contemp. Math. Sciences* **4** (5), 241-245, 2009.

[7] D.S. Bridges, Constructive mathematics: a foundation for computable analysis, *Theoretical Computer Science* **219**, 95-109, 1999.

[8] D.S. Bridges and F. Richman, *Varieties of Constructive Mathematics*, London Mathematical Society Lecture Notes **97**, Cambridge University Press, Cmbridge, U.K., 1987.

[9] N. Greenleaf, Linear Order in Lattices: A Constructive Study, in: *Advances in Mathematics Supplementary Studies* **1** (G-C. Rota ed.), 11-30, Academic Press, New York, 1978.

[10] H. Ishihara, Reverse mathematics in Bishop's constructive mathematics, *Philosophia Scientiæ*, Cahier spécial **6**, 43-59, 2006.

[11] R. Mines, F. Richman, and W. Ruytenburg, *A Course in Constructive Algebra*, Springer-Verlag, New York, 1988.

[12] J. von Plato, Positive lattices, in: *Reuniting the Antipodes Constructive and Nonstandard Views of the Continuum* (P. Schuster, U. Berger, H. Osswald eds.), 185-197, Kluwer Academic Publishers, Dordrecht, 2001.

[13] D.A. Romano, On co-filters of implicative semigroups with apartness, *Acta Universitatis Apulensis* **59**, 33-43, 2019.

[14] A.S. Troelstra and D. van Dalen, *Constructivism in Mathematics: An Introduction* (two volumes), North-Holland, Amsterdam, 1988.

"Dunarea de Jos" University of Galati,
Str. Domnească 111, Galati 800201,
Romania
email: marian.baroni@ugal.ro