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ON RELATIVE DEFICIENCIES OF COMMON ROOTS ON THE BASIS OF INTEGRATED MODULI OF LOGARITHMIC DERIVATIVE OF ENTIRE AND MEROMORPHIC FUNCTION

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Abstract. The main target of our paper is to derive some bounds related to the relative deficiencies of common roots from the view point of integrated moduli of logarithmic derivative of entire and meromorphic functions. Some examples are also provided to validate the results obtained.

1. Introduction

Let f_1,f_2... and f_n be any n (>1) non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. Let $n_0(r,a)$ and $\bar{n_0}(r,a)$ respectively denote the number of common roots and the number of distinct common roots in the disk $|z| \leq r$ of n equations $f_1 = a$,$f_2 = a$, ... and $f_n = a$ where $'a'$ is any complex number. For a meromorphic function f in $\mathbb{C}$, Milloux [10] introduced the concepts of absolute defect of $'a'$ with respect to the derivative f' where $a \in \mathbb{C} \cup \{\infty\}$.

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Later Xiong [17] extended the definition to the k-th derivative $f^{(k)}$ where $k > 1$. Singh [12] introduced the term relative defect for distinct zeros poles and established various relations among it, relative defect and usual defect.

The concept of finding out relative defects corresponding to common roots of meromorphic functions was initiated by Singh [13]. In this paper we deduce some more results on relative defects of common roots of meromorphic functions in the direction of Singh [13]. We do not explain the standard definitions and notations of the value distribution and the Nevanlinna theory of entire and meromorphic functions as those are available in [16] and [9].
To start our paper we require the following:

Let

$$N_0(r,a) = \int_0^r \frac{n_0(t,a) - n_0(0,a)}{t} dt + n_0(0,a)\log r$$

and

$$N_{1,2,\ldots,n}(r,a) = N\left(r, \frac{1}{f_1 - a}\right) + N\left(r, \frac{1}{f_2 - a}\right) + \\ +\ldots + N\left(r, \frac{1}{f_n - a}\right) - nN_0(r,a).$$

Similarly we may define $\bar{N}_0(r,a)$ and $\bar{N}_{1,2,\ldots,n}(r,a)$ as follows

$$\bar{N}_0(r,a) = \int_0^r \frac{\bar{n}_0(t,a) - \bar{n}_0(0,a)}{t} dt + \bar{n}_0(0,a)\log r$$

and

$$\bar{N}_{1,2,\ldots,n}(r,a) = \bar{N}\left(r, \frac{1}{f_1 - a}\right) + \bar{N}\left(r, \frac{1}{f_2 - a}\right) + \\ + \cdots + \bar{N}\left(r, \frac{1}{f_n - a}\right) - n\bar{N}_0(r,a).$$

Also let $\bar{n}_0^{(k)}(r,a)$, $\bar{N}_{1,2,\cdots,n}^{(k)}(r,a)$ etc. denote the corresponding quantities with respect to $f_1^{(k)}$, $f_2^{(k)}$,... and $f_n^{(k)}$ where k is any non-negative integer.

The following definition is well known.

Definition 1.1. *The order ρ_f of a meromorphic function f is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}$$

If $\rho_f < \infty$ then f is of finite order.

If f ise a meromorphic function in the complex plane then the integrated moduli of the logarithmic derivative $I(r,f)$ is defined by

$$I(r,f) = \frac{r}{2\pi}\int_0^{2\pi} \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta \text{ for } 0 < r < +\infty.$$

Also let $\bar{n}_0^{(k)}(r,a)$, $\bar{N}_{1,2,\cdots,n}^{(k)}(r,a)$ etc. denote the corresponding quantities with respect to $f_1^{(k)}$, $f_2^{(k)}$, ... and $f_n^{(k)}$ where k is any non-negative integer.

We now define the following terms by using the concept of $I(r,f)$

$${}_I\delta_{1,2,\ldots,n}(a) = 1 - \limsup_{r\to\infty} \frac{N_{1,2,\ldots,n}(r,a)}{I(r,f_1)+I(r,f_2)+\ldots+I(r,f_n)}$$

$${}_I\delta_{1,2,\ldots,n}^{(k)}(a) = 1 - \limsup_{r\to\infty} \frac{N_{1,2,\ldots,n}^{(k)}(r,a)}{I(r,f_1)+I(r,f_2)+\ldots+I(r,f_n)}$$

$${}_I\delta_0(a) = 1 - \limsup_{r\to\infty} \frac{N_0(r,a)}{I(r,f_1)+I(r,f_2)+\ldots+I(r,f_n)}$$

$${}_I\Theta_{1,2,\ldots,n}(a) = 1 - \limsup_{r\to\infty} \frac{\bar{N}_{1,2,\ldots,n}(r,a)}{I(r,f_1)+I(r,f_2)+\ldots+I(r,f_n)}$$

$${}_I\Theta_{1,2,\ldots,n}^{(k)}(a) = 1 - \limsup_{r\to\infty} \frac{\bar{N}_{1,2,\ldots,n}^{(k)}(r,a)}{I(r,f_1)+I(r,f_2)+\ldots+I(r,f_n)}$$

$${}_I\Theta_0(a) = 1 - \limsup_{r\to\infty} \frac{\bar{N}_0(r,a)}{I(r,f_1)+I(r,f_2)+\ldots+I(r,f_n)}.$$

In this paper we establish some theorems on the relative defects corresponding to the common roots of $f_1 = a$, $f_2 = a$,... and $f_n = a$ in the direction of [2]. The term $S(r,f)$ denotes any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise.

For a transcendental meromorphic function f in $|z| < \infty$ we denote by $S(f)$ the set of all meromorphic functions a in $|z| < \infty$ which satisfy $T(r,a) = o\{T(r,f)\}$ as $r \to \infty$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [9] *Let k be any positive integer and $\Psi = \sum_{i=0}^{k} a_i f^{(i)}$ where a_i are meromorphic functions such that $T(r,a_i) = S(r,f)$. Then*

$$m(r, \frac{\Psi}{f}) = S(r,f).$$

Lemma 2.2. [9] *Let a_1, a_2 and a_3 be three distinct elements in $S(f)$. Then*

$$\{1 + o(1)\} T(r,f) < \sum_{j=1}^{3} \overline{N}\left(r, \frac{1}{f - a_j}\right) + S(r,f) \ \ \textit{as } r \to \infty.$$

Lemma 2.3. $(p.41, [9])$*Let f be meromorphic and non-constant in $|z| \le R_0$. Then*

$$\frac{S(r,f)}{T(r,f)} \to 0 \qquad\qquad (*)$$

as $r \to R_0$ with the following provisions :

(a) $(*)$ holds without restrictions if $R_0 = +\infty$ and f is of finite order in the plane.

(b) If f has infinite order in the plane, $(*)$ still holds as $r \to \infty$ outside a certain exceptional set E of finite length. Here E depends only on f.

(c) If $R_0 < +\infty$ and

$$\limsup_{r\to\infty} \frac{T(r,f)}{\log\{\frac{1}{R_0 - r}\}} = +\infty \ ,$$

then $(*)$ holds as $r \to R_0$ through a suitable sequence r_n, which depends on f only.

Lemma 2.4. *Let $f_1, f_2, ...$ and f_n be n (> 1) non-constant meromorphic functions of finite order in $\mathbb{C}$. Then*

$$\lim_{r\to\infty} \frac{S(r, f_1) + S(r, f_2) + ... + S(r, f_n)}{T(r, f_1) + T(r, f_2) + ... + T(r, f_n)} = 0.$$

Proof. In view of Lemma 2.3, we get that

$$\lim_{r\to\infty} \frac{S(r, f_i)}{T(r, f_i)} = 0 \quad \text{for each } i = 1, 2, ..., n.$$

Then

$$\lim_{r\to\infty} \frac{S(r, f_1)}{T(r, f_1)} = \lim_{r\to\infty} \frac{S(r, f_2)}{T(r, f_2)} = ... = \lim_{r\to\infty} \frac{S(r, f_n)}{T(r, f_n)} = 0.$$

Hence by ratio proportion formula we obtain that

$$\lim_{r\to\infty} \frac{S(r, f_1) + S(r, f_2) + ... + S(r, f_n)}{T(r, f_1) + T(r, f_2) + ... + T(r, f_n)} = \frac{0 + 0 + ... + 0}{1 + 1 + ... + 1} = 0.$$

This proves the lemma. □

Lemma 2.5. *Let $f_1, f_2, ...$ and f_n be n (> 1) non-constant meromorphic functions of finite order in $\mathbb{C}$. Then*

$$\lim_{r\to\infty} \frac{S(r, f_1) + S(r, f_2) + ... + S(r, f_n)}{I(r, f_1) + I(r, f_2) + ... + I(r, f_n)} = 0.$$

Proof. In view of Lemma 2.4, we obtain that

$$\lim_{r\to\infty} \frac{S(r, f_1) + S(r, f_2) + ... + S(r, f_n)}{T(r, f_1) + T(r, f_2) + ... + T(r, f_n)} = 0.$$

Now

$$\begin{aligned} &= \lim_{r\to\infty} \frac{S(r, f_1) + S(r, f_2) + ... + S(r, f_n)}{T(r, f_1) + T(r, f_2) + ... + T(r, f_n)} \\ &\quad \cdot \lim_{r\to\infty} \frac{T(r, f_1) + T(r, f_2) + ... + T(r, f_n)}{I(r, f_1) + I(r, f_2) + ... + I(r, f_n)} \\ &= 0. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.6. [15] *Let f be an entire function of finite order $'\rho'$ with no zeros in $\mathbb{C}$. Then*

$$\lim_{r\to\infty} \frac{I(r, f)}{T(r, f)} = \pi\rho.$$

Lemma 2.7. *Let $f_1, f_2, f_3, ..., f_n$ be n (> 1) entire functions of finite order $'\rho_i\,'$ with each f_i for $i = 1, 2, ..., n$ having no zeros in $\mathbb{C}$. Then*

$$\lim_{r\to\infty} \frac{I(r, f_1) + I(r, f_2) + ... + I(r, f_n)}{T(r, f_1) + T(r, f_2) + ... + T(r, f_n)} = \frac{\pi\,(\rho_1 + \rho_2 + ... + \rho_n)}{n}.$$

Proof. In view of Lemma 2.6, we get that

$$\lim_{r\to\infty} \frac{I(r, f_i)}{T(r, f_i)} = \pi\rho_i \quad \text{for each } i = 1, 2, ..., n.$$

Then

$$\lim_{r\to\infty} \frac{I(r, f_1)}{T(r, f_1)} = \pi\rho_1, \;\; \lim_{r\to\infty} \frac{I(r, f_2)}{T(r, f_2)} = \pi\rho_2, ...$$

and

$$\lim_{r\to\infty} \frac{I(r, f_l)}{T(r, f_l)} = \pi\rho_n.$$

Hence by ratio proportion formula

$$\begin{aligned} \lim_{r\to\infty} \frac{I(r, f_1) + I(r, f_2) + ... + I(r, f_n)}{T(r, f_1) + T(r, f_2) + ... + T(r, f_n)} &= \frac{\pi\rho_1 + \pi\rho_2 + ... + \pi\rho_n}{1 + 1 + ... + 1} \\ &= \frac{\pi\,(\rho_1 + \rho_2 + ... + \rho_n)}{n}. \end{aligned}$$

This proves the lemma. □

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let $f_1,\ f_2,\ ...,\ f_n$ be n (> 1) entire functions of finite order $'\rho_i\,'$ with each f_i for $i = 1, 2, ..., n$ having no zeros in $\mathbb{C}$. Then*

$$\delta_{1,2,...,n}(a) + \pi\rho \geq 1 + \pi\rho \cdot \;{}_I\delta_{1,2,...,n}(a).$$

Proof. From the definition of Datta et. al. [2], [3], [4] and [5] it follows that

$$\begin{aligned}
\delta_{1,2,...,n}(a) &= 1 - \limsup_{r\to\infty} \frac{N_{1,2,...,n}(a)}{T(r,f_1)+T(r,f_2)+...+T(r,f_n)} \\
&= 1 - \limsup_{r\to\infty} \frac{N_{1,2,...,n}(a)}{I(r,f_1)+I(r,f_2)+...+I(r,f_n)} \\
&\cdot \frac{I(r,f_1)+I(r,f_2)+...+I(r,f_n)}{T(r,f_1)+T(r,f_2)+...+T(r,f_n)} \\
&\geq 1 - \limsup_{r\to\infty} \frac{N_{1,2,...,n}(a)}{I(r,f_1)+I(r,f_2)+...+I(r,f_n)} \cdot \pi\rho \\
&= \pi\rho - \limsup_{r\to\infty} \frac{N_{1,2,...,n}(a)}{I(r,f_1)+I(r,f_2)+...+I(r,f_n)} \cdot \pi\rho + (1-\pi\rho) \\
&= \pi\rho\left(1 - \limsup_{r\to\infty} \frac{N_{1,2,...,n}(a)}{I(r,f_1)+I(r,f_2)+...+I(r,f_n)}\right) + (1-\pi\rho) \\
&= \pi\rho \cdot_I \delta_{1,2,...,n}(a) + (1-\pi\rho).
\end{aligned}$$

Therefore

$$\delta_{1,2,...,n}(a) + \pi\rho \geq 1 + \pi\rho \cdot_I \delta_{1,2,...,n}(a).$$

Hence the theorem is proved. □

Remark 3.1. *In the same way we may prove*

$$\delta^{(k)}_{1,2,...,n}(a) + \pi\rho \geq 1 + \pi\rho \cdot_I \delta^{(k)}_{1,2,...,n}(a),$$

$$\delta_0(a) + \pi\rho \geq 1 + \pi\rho \cdot_I \delta_0(a),$$

$$\Theta_{1,2,...,n}(a) + \pi\rho \geq 1 + \pi\rho \cdot_I \Theta_{1,2,...,n}(a),$$

$$\Theta^{(k)}_{1,2,...,n}(a) + \pi\rho \geq 1 + \pi\rho \cdot_I \Theta^{(k)}_{1,2,...,n}(a)$$

and

$$\Theta_0(a) + \pi\rho \geq 1 + \pi\rho \cdot_I \Theta_0(a),$$

where k is any non-negative integer.

Theorem 3.2. *Let $f_1, f_2, ..., f_n$ be $n\ (>1)$ entire functions of non-zero finite order $'\rho_i\ '$ with each f_i having no zeros in $\mathbb{C}$ such that $m(r,f_i) = S(r,f_i)$ for $i = 1,2,...,n$. Also let $a_i(i = 1,2,...,p)$ and*

$b_j (j = 1, 2, ..., q)$ *be two sets of distinct finite non-zero complex numbers. Then for any positive integer* k,

$$_I\Theta_{1,2,...,n}(0) + q \cdot {}_I\delta_{1,2,...,n}(0) + \sum_{i=1}^{p} {}_I\Theta_{1,2,...,l}(a_i) + \sum_{j=1}^{q} {}_I\Theta^{(k)}_{1,2,\cdots,n}(b_j)$$

$$+n \cdot {}_I\Theta_0(0) + qn \cdot {}_I\delta_0(0) + n\sum_{i=1}^{p} {}_I\Theta_0(a_i) + n\sum_{j=1}^{q} {}_I\Theta^{(k)}_{1,2,...,n}(b_j)$$

$$+(p+q) \cdot \frac{n}{\pi(\rho_1 + \rho_2 + ... + \rho_n)} \le (1+n)(1+p+2q).$$

Proof. From the relation $T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$, we get that

$$T(r, f) = N\left(r, \frac{1}{f}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{1}{f^{(k)}}\right) + O(1)$$

$$i.e., T(r, f) = N\left(r, \frac{1}{f}\right) + T\left(r, \frac{1}{f^{(k)}}\right) - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f)$$

$$i.e., T(r, f) = N\left(r, \frac{1}{f}\right) + T\left(r, f^{(k)}\right) - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \tag{1}$$

Again by Nevanlinna's second fundamental theorem we obtain from Equality (1) that

$$\begin{aligned} qT\left(r, f^{(k)}\right) &\le \bar{N}\left(r, f^{(k)}\right) + \sum_{j=1}^{q} \bar{N}\left(r, \frac{1}{f^{(k)} - b_j}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \end{aligned} \tag{2}$$

Therefore from Inequality (1) and Inequality (2) we get that

$$\begin{aligned} qT(r, f) &\le \bar{N}\left(r, f^{(k)}\right) + \bar{N}\left(r, \frac{1}{f^{(k)}}\right) - qN\left(r, \frac{1}{f^{(k)}}\right) \\ &\quad + \sum_{j=1}^{q} \bar{N}\left(r, \frac{1}{f^{(k)} - b_j}\right) + qN\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned} \tag{3}$$

Since $\bar{N}\left(r, f^{(k)}\right) = \bar{N}(r, f)$ and $\bar{N}\left(r, \frac{1}{f^{(k)}}\right) - qN\left(r, \frac{1}{f^{(k)}}\right) \leq 0$, it follows from Inequality (3) that

$$(4)\quad qT(r, f) \leq \sum_{j=1}^{q} \bar{N}\left(r, \frac{1}{f^{(k)} - b_j}\right) + \bar{N}(r, f) + qN\left(r, \frac{1}{f}\right) + S(r, f).$$

In view of Nevanlinna's second fundamental theorem we get from Inequality (4) that

$$(5)\quad pT(r, f) \leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) + \sum_{i=1}^{p} \bar{N}\left(r, \frac{1}{f - a_i}\right) + S(r, f).$$

Adding Inequality (4) and Inequality (5) it follows that

$$(p+q)T(r, f) \leq \bar{N}\left(r, \frac{1}{f}\right) + qN\left(r, \frac{1}{f}\right) + 2\bar{N}(r, f)$$

$$(6)\qquad + \sum_{i=1}^{p} \bar{N}\left(r, \frac{1}{f - a_i}\right) + \sum_{j=1}^{q} \bar{N}\left(r, \frac{1}{f^{(k)} - b_j}\right) + S(r, f).$$

Applying Inequality (6) for $f_1, f_2, f_3, ..., f_n$ and using $\bar{N}(r, f_1) = S(r, f_1), \bar{N}(r, f_2) = S(r, f_2), \bar{N}(r, f_3) = S(r, f_3), ..., \bar{N}(r, f_n) = S(r, f_n)$ we get that

$$(p+q)\{T(r, f_1) + T(r, f_2) + ... + T(r, f_n)\}$$
$$\leq \left[\bar{N}\left(r, \frac{1}{f_1}\right) + \bar{N}\left(r, \frac{1}{f_2}\right) + ... + \bar{N}\left(r, \frac{1}{f_n}\right)\right]$$
$$+q\left[N\left(r, \frac{1}{f_1}\right) + N\left(r, \frac{1}{f_2}\right) + ... + N\left(r, \frac{1}{f_n}\right)\right]$$
$$+\sum_{i=1}^{p}\left[\bar{N}\left(r, \frac{1}{f_1 - a_i}\right) + \bar{N}\left(r, \frac{1}{f_2 - a_i}\right) + ... + \bar{N}\left(r, \frac{1}{f_n - a_i}\right)\right]$$
$$+\sum_{j=1}^{q}\left[\bar{N}\left(r, \frac{1}{f_1^{(k)} - b_j}\right) + \bar{N}\left(r, \frac{1}{f_1^{(k)} - b_j}\right) + ... + \bar{N}\left(r, \frac{1}{f_n^{(k)} - b_j}\right)\right]$$
$$+S(r, f_1) + S(r, f_2) + ... + S(r, f_n)$$

$i.e., (p+q)\{T(r, f_1) + T(r, f_2) + ... + T(r, f_n)\} \leq \left[\bar{N}_{1,2,...,n}(r, 0) + n\bar{N}_0(r, 0)\right]$

$$+q\left[N_{1,2,...,n}(r, 0) + nN_0(r, 0)\right] + \sum_{i=1}^{p}\left[\bar{N}_{1,2,...,n}(r, a_i) + n\bar{N}_0(r, a_i)\right]$$

$$+\sum_{j=1}^{q}\left[\bar{N}^{(k)}_{1,2,\ldots,n}(r,b_j)+n\bar{N}^{(k)}_{0}(r,b_j)\right]$$

$$+\left[S(r,f_1)+S(r,f_2)+...+S(r,f_n)\right]. \tag{7}$$

Now dividing both sides of Inequality (7) by $\{I(r,f_1)+I(r,f_2)+...+I(r,f_n)\}$ and taking limit superior we get in view of Lemma 2.5 and Lemma 2.7 that

$$(p+q)\cdot\frac{T(r,f_1)+T(r,f_2)+...+T(r,f_n)}{I(r,f_1)+I(r,f_2)+...+I(r,f_n)}$$
$$\leq\{1-{}_I\Theta_{1,2,\ldots,n}(0)\}+n\{1-{}_I\Theta_0(0)\}+q\{1-{}_I\delta_{1,2,\ldots,n}(0)\}$$

$$+qn\{1-{}_I\delta_0(0)\}+\sum_{i=1}^{p}\{1-{}_I\Theta_{1,2,\ldots,n}(a_i)\}+n\sum_{i=1}^{p}\{1-{}_I\Theta_0(a_i)\}$$

$$+\sum_{j=1}^{q}\{1-{}_I\Theta_{1,2,\ldots,n}(b_j)\}+n\sum_{j=1}^{q}\left\{1-{}_I\Theta_0^{(k)}(b_j)\right\}$$

$$i.e., (p+q)\cdot\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)}\leq(1+n+q+qn+p+pn+q+qn)$$

$$-{}_I\Theta_{1,2,\ldots,n}(0)-n\cdot{}_I\Theta_0(0)-q\cdot{}_I\delta_{1,2,\cdots,n}(0)-qn\cdot{}_I\delta_0(0)-\sum_{i=1}^{p}{}_I\Theta_{1,2,\ldots,n}(a_i)$$

$$-n\sum_{i=1}^{p}{}_I\Theta_0(a_i)-\sum_{j=1}^{q}{}_I\Theta^{(k)}_{1,2,\ldots,l}(b_j)-n\sum_{j=1}^{q}{}_I\Theta_0^{(k)}(b_j)$$

$$i.e.,{}_I\Theta_{1,2,\ldots,n}(0)+n\cdot{}_I\Theta_0(0)+q\cdot{}_I\delta_{1,2,\cdots,n}(0)+qn\cdot{}_I\delta_0(0)+$$
$$+\textstyle\sum_{i=1}^{p}{}_I\Theta_{1,2,\ldots,n}(a_i)+n\sum_{i=1}^{p}{}_I\Theta_0(a_i)+$$
$$+\textstyle\sum_{j=1}^{q}{}_I\Theta^{(k)}_{1,2,\ldots,n}(b_j)+n\sum_{j=1}^{q}{}_I\Theta_0^{(k)}(b_j)$$

$$+(p+q)\cdot\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)}\leq(1+n)(1+p+2q).$$

Hence the theorem is proved. □

Remark 3.2. *The condition $'a_i$ and b_j be two sets of distinct finite non-zero complex numbers$'$ in Theorem 3.2 is necessary as we see by considering $n=2$, $f_1=\exp z$, $f_1=\exp(-z)$, $p=q=1$, $a_1=0,\infty$ and $b_1=0,\infty$. Then we get that $N_{1,2}(r,0)=0$, $\overline{N}_{1,2}(r,0)=0$,*

$\overline{N}^{(k)}_{1,2}(r,\infty) = 0,\ N_0(r,0) = 0,\ \overline{N}_0(r,0) = 0,\ \overline{N}^{(k)}_{1,2}(r,\infty) = 0$ *and* $\rho_1 = \rho_2 = 1$. *So,* $I(r,f_1) = r^2 \neq 0$ *and* $I(r,f_2) = r^2 \neq 0$. *Now,*

$$ {}_I\Theta_{1,2,\ldots,n}(0) = {}_I\Theta_0(0) = {}_I\delta_{1,2,\ldots,n}(0) = {}_I\delta_0(0) = 1, $$

$$ \sum_{i=1}^{p} {}_I\Theta_{1,2,\ldots,n}(a_i) = {}_I\Theta(a_1)_{1,2} = {}_I\Theta_{1,2}(0) = {}_I\Theta_{1,2}(\infty) = 1, $$

$$ \sum_{i=1}^{p} {}_I\Theta_0(a_i) = {}_I\Theta_0(a_1) = {}_I\Theta_0(0) = {}_I\Theta_0(\infty) = 1, $$

$$ \sum_{j=1}^{q} {}_I\Theta^{(k)}_{1,2,\ldots,n}(b_j) = {}_I\Theta^{(k)}_{1,2}(b_1) = {}_I\Theta^{(k)}_{1,2}(0) = {}_I\Theta^{(k)}_{1,2}(\infty) = 1 $$

and

$$ \sum_{j=1}^{q} {}_I\Theta^{(k)}_0(b_j) = {}_I\Theta^{(k)}_0(b_1) = {}_I\Theta^{(k)}_0(\infty) = {}_I\Theta^{(k)}_0(\infty) = 1. $$

Hence,

$$ \begin{gathered} {}_I\Theta_{1,2,\ldots,n}(0) + n \cdot_I \Theta_0(0) + q \cdot_I \delta_{1,2,\ldots,n}(0) + \\ +qn \cdot_I \delta_0(0) + \textstyle\sum_{i=1}^{p} {}_I\Theta_{1,2,\ldots,n}(a_i) + n \sum_{i=1}^{p} {}_I\Theta_0(a_i) + \\ + \textstyle\sum_{j=1}^{q} {}_I\Theta^{(k)}_{1,2,\ldots,n}(b_j) + n \sum_{j=1}^{q} {}_I\Theta^{(k)}_0(b_j) + \\ +(p+q) \cdot \dfrac{n}{\pi(\rho_1 + \rho_2 + \ldots + \rho_n)} \\ = 1+2+1+2+1+2+1+2+\dfrac{4}{2\pi} = 12 + \dfrac{2}{\pi} \end{gathered} $$

and

$$ (n+1)(1+p+2q) = 12, $$

which is contrary to Theorem 3.2.

Theorem 3.3. *Let* $f_1,\ f_2,\ \ldots,\ f_n$ *be* $n\ (>1)$ *entire functions of non-zero finite order* $'\rho_i\,'$ *with each* f_i *for* $i = 1,2,\ldots,n$ *having no zeros in* $\mathbb{C}$. *Then*

$$ \begin{gathered} {}_I\Theta^{(k)}_{1,2,\ldots,n}(0) + q\textstyle\sum_{i=1}^{p} {}_I\delta_{1,2,\ldots,n}(a_i) + \sum_{j=1}^{q} {}_I\Theta^{(k)}_{1,2,\ldots,n}(b_j) + n \cdot_I \Theta^{(k)}_0(0) \\ +qn\textstyle\sum_{i=1}^{p} {}_I\delta_0(a_i) + n\sum_{j=1}^{q} {}_I\Theta^{(k)}_0(b_j) + \dfrac{pqn}{\pi(\rho_1 + \rho_2 + \ldots + \rho_n)} \\ \leq (1+n)(1+pq+q), \end{gathered} $$

where $'a_i'$ $(i = 1,2,\ldots,p)$ *and* $'b_j'$ $(j = 1,2,\ldots,q)$ *be any two sets of distinct finite non-zero complex numbers and* k *is any positive integer.*

Proof. Let us consider

$$F=\sum_{i=1}^{p}\frac{1}{f-a_i}.$$

$$i.e., \sum_{i=1}^{p} m\left(r,\frac{1}{f-a_i}\right)\leq m\left(r,F\right)+O\left(1\right)$$

$$i.e., \sum_{i=1}^{p} m\left(r,\frac{1}{f-a_i}\right)\leq m\left(r,Ff^{(k)}\right)+m\left(r,\frac{1}{f^{(k)}}\right)+O\left(1\right)$$

$$i.e., \sum_{i=1}^{p} m\left(r,\frac{1}{f-a_i}\right)\leq m\left(r,\frac{f^{(k)}}{f-a_i}\right)+m\left(r,\frac{1}{f^{(k)}}\right)+O\left(1\right)$$

$$i.e., \sum_{i=1}^{p} m\left(r,\frac{1}{f-a_i}\right)\leq m\left(r,\frac{1}{f^{(k)}}\right)+S\left(r,f\right). \tag{8}$$

Adding $\sum\limits_{i=1}^{p} N\left(r,\frac{1}{f-a_i}\right)$ to both sides of Inequality (8) we can obtain that

$$\sum_{i=1}^{p} N\left(r,\frac{1}{f-a_i}\right)+\sum_{i=1}^{p} m\left(r,\frac{1}{f-a_i}\right)\leq \sum_{i=1}^{p} N\left(r,\frac{1}{f-a_i}\right)+m\left(r,\frac{1}{f^{(k)}}\right)+S\left(r,f\right)$$

$$i.e., \sum_{i=1}^{p} T\left(r,\frac{1}{f-a_i}\right)\leq T\left(r,\frac{1}{f^{(k)}}\right)+\sum_{i=1}^{p} N\left(r,\frac{1}{f-a_i}\right)+S\left(r,f\right)$$

i.e.,

$$pqT\left(r,f\right)\leq qT\left(r,f^{(k)}\right)+q\sum_{i=1}^{p} N\left(r,\frac{1}{f-a_i}\right)+S\left(r,f\right). \tag{9}$$

Again by Nevanlinna's second fundamental theorem and in view of $\bar{N}\left(r,f^{(k)}\right)=\bar{N}\left(r,f\right)$, we get from Inequality (9) that

$$\begin{aligned} qT\left(r,f^{(k)}\right) &\leq \bar{N}\left(r,f\right)+\bar{N}\left(r,\frac{1}{f^{(k)}}\right) \\ &\quad +\sum_{j=1}^{q}\bar{N}\left(r,\frac{1}{f^{(k)}-b_j}\right)+S\left(r,f\right). \end{aligned} \tag{10}$$

Now from Inequality (9) and (10) we obtain from Inequality (10) that

$$pqT\left(r,f\right) \leq \bar{N}\left(r,f\right)+\bar{N}\left(r,\frac{1}{f^{(k)}}\right)+q\sum_{i=1}^{p}N\left(r,\frac{1}{f-a_i}\right)$$

$$+\sum_{j=1}^{q}\bar{N}\left(r,\frac{1}{f^{(k)}-b_j}\right)+S\left(r,f\right). \tag{11}$$

Now applying Inequality (11) for $f_1, f_2, f_3, ..., f_n$ we get that

$$pq\{T(r,f_1)+T(r,f_2)+...+T(r,f_n)\}$$

$$\leq \left[\bar{N}\left(r,\frac{1}{f_1^{(k)}}\right)+\bar{N}\left(r,\frac{1}{f_2^{(k)}}\right)+...+\bar{N}\left(r,\frac{1}{f_n^{(k)}}\right)\right]$$

$$+q\sum_{i=1}^{p}\left[N\left(r,\frac{1}{f_1-a_i}\right)+N\left(r,\frac{1}{f_2-a_i}\right)+...+N\left(r,\frac{1}{f_n-a_i}\right)\right]+$$

$$\sum_{j=1}^{q}\left\{\bar{N}\left(r,\frac{1}{f_1^{(k)}-b_j}\right)+\bar{N}\left(r,\frac{1}{f_2^{(k)}-b_j}\right)+...+\bar{N}\left(r,\frac{1}{f_n^{(k)}-b_j}\right)\right\}$$

$$+S\left(r,f_1\right)+S\left(r,f_2\right)+...+S\left(r,f_n\right)$$

$$=\left\{\bar{N}_{1,2,...,n}^{(k)}\left(r,0\right)+n\cdot\bar{N}_{o}^{(k)}\left(r,0\right)\right\}+q\sum_{i=1}^{p}\left\{N_{1,2,..,n}\left(r,a_i\right)+n\cdot N_0\left(r,a_i\right)\right\}$$

$$+\sum_{j=1}^{q}\left\{\bar{N}_{1,2,...,n}^{(k)}\left(r,b_j\right)+n\cdot\bar{N}_{0}^{(k)}\left(r,b_j\right)\right\}$$

$$+\left\{S\left(r,f_1\right)+S\left(r,f_2\right)+...+S\left(r,f_n\right)\right\}. \tag{12}$$

Now dividing both sides of Inequality (12) by $\{I(r,f_1)+I(r,f_2)+...+I(r,f_n)\}$ and taking limit superior we get in view of Lemma 2.5 and Lemma 2.7 that

$$\frac{pqn}{\pi\left(\rho_1+\rho_2+...+\rho_n\right)} \leq \left\{1-_I\Theta_{1,2,...,n}^{(k)}\left(0\right)\right\}+n\left\{1-_I\Theta_{0}^{(k)}\left(0\right)\right\}$$
$$+q\sum_{i=1}^{p}\left\{1-_I\delta_{1,2,...,n}\left(a_i\right)\right\}+qn\sum_{i=1}^{p}\left\{1-_I\delta_0\left(a_i\right)\right\}$$
$$+\sum_{j=1}^{q}\left\{1-_I\Theta_{1,2,...,n}^{(k)}\left(b_j\right)\right\}+n\sum_{j=1}^{q}\left\{1-_I\Theta_{0}^{(k)}\left(b_j\right)\right\}$$

$$i.e., {}_I\Theta^{(k)}_{1,2,...,n}(0) + q\sum_{i=1}^{p} {}_I\delta_{1,2,...,n}(a_i) + \sum_{j=1}^{q} {}_I\Theta^{(k)}_{1,2,...,n}(b_j) + n \cdot {}_I\Theta^{(k)}_0(0)$$
$$+ qn\sum_{i=1}^{p} {}_I\delta_0(a_i) + n\sum_{j=1}^{q} {}_I\Theta^{(k)}_0(b_j) + \frac{pqn}{\pi(\rho_1+\rho_2+...+\rho_n)}$$
$$\leq 1+n+pq+pqn+q+qn = (1+n)(1+q+pq).$$

Hence the theorem is established. □

Remark 3.3. *The condition 'a_i and b_j be any two sets of distinct finite non-zero complex numbers' in Theorem 3.3 is essential which is evident by taking* $n = 2$, $f_1 = \exp(2z)$, $f_1 = \exp(-2z)$, $p = q = 1$, $a_1 = 0, \infty$ *and* $b_1 = 0, \infty$. *Then we see that* $N_{1,2}(r,0) = 0$, $\overline{N}_{1,2}(r,0) = 0$, $\overline{N}^{(k)}_{1,2}(r,0) = 0$, $N_0(r,0) = 0$, $\overline{N}^{(k)}_0(r,0) = 0$, $\overline{N}^{(k)}_{1,2}(r,\infty) = 0$, $\rho_1 = \rho_2 = 1$, $I(r,f_1) = I(r,\exp(2z)) = 2r^2 \neq 0$ *and* $I(r,f_2) = I(r,\exp(-2z)) = 2r^2 \neq 0$. *Therefore*

$${}_I\Theta^{(k)}_{1,2,...,n}(0) = {}_I\Theta^{(k)}_0(0) = 1,$$
$$\sum_{i=1}^{p} {}_I\delta_{1,2,...,n}(a_i) = {}_I\delta_{1,2}(0) = {}_I\delta_{1,2}(\infty) = 1,$$
$$\sum_{j=1}^{q} {}_I\Theta^{(k)}_{1,2,...,n}(b_j) = {}_I\Theta^{(k)}_{1,2}(0) = {}_I\Theta^{(k)}_{1,2}(\infty) = 1,$$
$$\sum_{i=1}^{p} {}_I\delta_0(a_i) = {}_I\delta_0(0) = {}_I\delta_0(\infty) = 1$$

and

$$\sum_{j=1}^{q} {}_I\Theta^{(k)}_0(b_j) = {}_I\Theta^{(k)}_0(0) = {}_I\Theta^{(k)}_0(\infty) = 1.$$

Thus

$${}_I\Theta^{(k)}_{1,2,...,n}(0) + q\sum_{i=1}^{p} {}_I\delta_{1,2,...,n}(a_i) + \sum_{j=1}^{q} {}_I\Theta^{(k)}_{1,2,...,n}(b_j) + n \cdot {}_I\Theta^{(k)}_0(0)$$
$$+ qn\sum_{i=1}^{p} {}_I\delta_0(a_i) + n\sum_{j=1}^{q} {}_I\Theta^{(k)}_0(b_j) + \frac{pqn}{\pi(\rho_1+\rho_2+...+\rho_n)}$$
$$= 1+1+1+2+2+2+\frac{2}{2\pi} = 9+\frac{1}{\pi}$$

and

$$(1+n)(1+q+pq)=9,$$

which contradicts Theorem 3.3.

Theorem 3.4. *Let f_1, f_2, ..., f_n be n (>1) entire functions of non-zero finite order 'ρ_i ' with each f_i for $i=1,2,...,n$ having no zeros in $\mathbb{C}$. Also let a_i $(i=1,2,...,p)$ and b_j $(j=1,2,...,q)$ be the finite non-zero distinct complex numbers. Then*

$$
\begin{aligned}
&{}_I\Theta^{(k)}_{1,2,...,n}(0)+{}_I\Theta_{1,2,...,n}(0)+q\cdot{}_I\delta_{1,2,...,n}(0)+(1+q)\sum_{i=1}^{p}{}_I\delta_{1,2,...,n}(a_i)\\
&+2\sum_{j=1}^{q}{}_I\Theta^{(k)}_{1,2,...,n}(b_j)+n\cdot{}_I\Theta^{(k)}_0(0)+n\cdot{}_I\Theta_0(0)+qn\cdot{}_I\delta_0(0)+n(1+q)\sum_{i=1}^{p}{}_I\delta_0(a_i)\\
&+2n\sum_{j=1}^{q}{}_I\Theta_0(b_j)+\frac{(p+q+pq)\,n}{\pi(\rho_1+\rho_2+...+\rho_n)}\le(1+n)(2+p+3q+pq).
\end{aligned}
$$

Proof. Adding Inequality (3) and Inequality (11) we obtain that

$$
\begin{aligned}
&(p+q+pq)\,T(r,f)\le 3\bar{N}(r,f)+\bar{N}\left(r,\frac{1}{f^{(k)}}\right)+\bar{N}\left(r,\frac{1}{f}\right)+qN\left(r,\frac{1}{f}\right)\\
(13)\quad&+(1+q)\sum_{i=1}^{p}N\left(r,\frac{1}{f-a_i}\right)+2\sum_{j=1}^{q}\bar{N}\left(r,\frac{1}{f^{(k)}-b_j}\right)+S(r,f).
\end{aligned}
$$

Now applying Inequality (13) for $f_1, f_2, f_3, ..., f_n$ we get that

$$
\begin{aligned}
&(p+q+pq)\{T(r,f_1)+T(r,f_2)+...+T(r,f_n)\}\\
&\le\left\{\bar{N}\left(r,\frac{1}{f_1^{(k)}}\right)+\bar{N}\left(r,\frac{1}{f_2^{(k)}}\right)+...+\bar{N}\left(r,\frac{1}{f_n^{(k)}}\right)\right\}\\
&+\left\{\bar{N}\left(r,\frac{1}{f_1}\right)+...+\bar{N}\left(r,\frac{1}{f_n}\right)\right\}+q\left\{N\left(r,\frac{1}{f_1}\right)+...+N\left(r,\frac{1}{f_n}\right)\right\}\\
&+(1+q)\sum_{i=1}^{p}\left\{N\left(r,\frac{1}{f_1-a_i}\right)+N\left(r,\frac{1}{f_2-a_i}\right)+...+N\left(r,\frac{1}{f_n-a_i}\right)\right\}\\
&+2\sum_{j=1}^{q}\left\{\bar{N}\left(r,\frac{1}{f_1^{(k)}-b_j}\right)+\bar{N}\left(r,\frac{1}{f_2^{(k)}-b_j}\right)+...+\bar{N}\left(r,\frac{1}{f_n^{(k)}-b_j}\right)\right\}\\
(14)\qquad&+\{S(r,f_1)+S(r,f_2)+...+S(r,f_n)\}.
\end{aligned}
$$

Now dividing both sides of Inequality (14) by $\{I(r,f_1)+I(r,f_2)+...+I(r,f_n)\}$ and taking limit superior we get in view of Lemma 2.5 and Lemma 2.7 that

$$\frac{(p+q+pq)\,n}{\pi\left(\rho_1+\rho_2+...+\rho_n\right)} \leq \left\{1-\ {}_I\Theta^{(k)}_{1,2,\cdots,n}(0)\right\}+n\left\{1-\ {}_I\Theta^{(k)}_0(0)\right\}$$
$$+\left\{1-\ {}_I\Theta_{1,2,...,n}(0)\right\}+n\left\{1-\ {}_I\Theta_0(0)\right\}$$
$$+q\left\{1-\ {}_I\delta_{1,2,...,n}(0)\right\}+qn\left\{1-\ {}_I\delta_0(0)\right\}$$
$$+(1+q)\textstyle\sum_{i=1}^{p}\left\{1-\ {}_I\delta_{1,2,\cdots,n}(a_i)\right\}+n(1+q)\textstyle\sum_{i=1}^{p}\left\{1-\ {}_I\delta_0(a_i)\right\}$$
$$+2\textstyle\sum_{j=1}^{q}\left\{1-\ {}_I\Theta^{(k)}_{1,2,...,n}(b_j)\right\}+2n\textstyle\sum_{j=1}^{q}\left\{1-\ {}_I\Theta^{(k)}_0(b_j)\right\}.$$
$$i.e.,{}_I\,\Theta^{(k)}_{1,2,...,n}(0)+\ {}_I\Theta^{(k)}_0(0)+q\cdot\ {}_I\delta_{1,2,\cdots,n}(0)$$
$$+(1+q)\textstyle\sum_{i=1}^{p}\ {}_I\delta_{1,2,...,n}(a_i)+2\ \textstyle\sum_{j=1}^{q}\ {}_I\Theta^{(k)}_{1,2,\cdots,n}(b_j)$$
$$+n\cdot\ {}_I\Theta^{(k)}_0(0)+n\cdot\ {}_I\Theta_0(0)+qn\cdot\ {}_I\delta_0(0)+n(1+q)\textstyle\sum_{i=1}^{p}\ {}_I\delta_0(a_i)$$
$$+2n\textstyle\sum_{j=1}^{q}\ {}_I\Theta_0(b_j)+\frac{(p+q+pq)\,n}{\pi\left(\rho_1+\rho_2+...+\rho_n\right)}$$
$$\leq 1+n+1+n+q+qn+(1+q)p+n(1+q)p+2q+2nq$$
$$=(1+n)(2+p+3q+pq).$$

This proves the theorem. □

Remark 3.4. *The necessity of the condition 'a_i and b_j be any two sets of distinct finite non-zero complex numbers' in Theorem 3.4 can be verified by considering* $n=2,\ f_1=\exp\left(z^2\right),\ f_2=\exp\left(-z^2\right),\ p=q=1,\ a_1=0,\infty$ *and* $b_1=0,\infty$. *Then we get that* $N_{1,2}(r,0)=0,\ \overline{N}_{1,2}(r,0)=0,\ \overline{N}^{(k)}_{1,2}(r,\infty)=0,\ N_0(r,0)=0,\ \overline{N}_0(r,0)=0$ *and* $\overline{N}^{(k)}_{1,2}(r,\infty)=0$. *So,* $I(r,f_1)=\frac{2r^3}{\pi}\cdot I_0(1)\neq 0$, $I(r,f_2)=\frac{2r^3}{\pi}\cdot I_0(1)\neq 0$, *where* $I_n(z)$ *is the Modified Bessel Function of the first kind such that* $I_n(z)=\frac{1}{\pi}\int_0^{\pi}e^{z\cos\theta}\cdot\cos n\theta d\theta$ *and* $\rho_1=\rho_2=2$. *Now*

$${}_I\Theta^{(k)}_{1,2,...,n}(0)=\ {}_I\Theta^{(k)}_0(0)=\ {}_I\delta_{1,2,\cdots,n}(0)=\ {}_I\Theta^{(k)}_0(0)=\ {}_I\Theta_0(0)=\ {}_I\delta_0(0)=1,$$

$$\sum_{i=1}^{p}\ {}_I\delta_{1,2,...,n}(a_i)=\ {}_I\delta_{1,2}(0)=\ {}_I\delta_{1,2}(\infty)=1,$$

$$\sum_{j=1}^{q}\ {}_I\Theta^{(k)}_{1,2,\cdots,n}(b_j)=\ {}_I\Theta^{(k)}_{1,2}(0)=\ {}_I\Theta^{(k)}_{1,2}(\infty)=1,$$

$$\sum_{i=1}^{p}\ {}_I\delta_0(a_i)=\ {}_I\delta_0(0)=\ {}_I\delta_0(\infty)=1$$

and

$$\sum_{j=1}^{q} {}_I\Theta_0\left(b_j\right) = {}_I\Theta_0\left(0\right) = {}_I\Theta_0\left(\infty\right) = 1.$$

Hence,

$$\begin{aligned} &{}_I\Theta^{(k)}_{1,2,\ldots,n}\left(0\right) + {}_I\Theta^{(k)}_0\left(0\right) + q\cdot{}_I\,\delta_{1,2,\cdots,n}\left(0\right) + \left(1+q\right)\sum_{i=1}^{p} {}_I\delta_{1,2,\ldots,n}\left(a_i\right) \\ &+2\sum_{j=1}^{q} {}_I\Theta^{(k)}_{1,2,\cdots,n}\left(b_j\right) + n\cdot{}_I\,\Theta^{(k)}_0\left(0\right) + n\cdot{}_I\,\Theta_0\left(0\right) + qn\cdot{}_I\,\delta_0\left(0\right) \\ &+n\left(1+q\right)\sum_{i=1}^{p} {}_I\delta_0\left(a_i\right) + 2n\sum_{j=1}^{q} {}_I\Theta_0\left(b_j\right) + \frac{\left(p+q+pq\right)n}{\pi\left(\rho_1+\rho_2+\ldots+\rho_n\right)} \\ &= 1+1+1+2+2+2+2+2+4+4 = 21 + \frac{6}{2\pi} \end{aligned}$$

and

$$(1+n)(2+p+3q+pq) = 21.$$

So, we arrive at a contradiction.

Theorem 3.5. *Let f_1, f_2, ..., f_n be n (>1) entire functions of non-zero finite order $'\rho_i\,'$ with each f_i having no zeros in $\mathbb{C}$ such that $N\left(r, \frac{1}{f_i}\right) = S\left(r, f_i\right)$ for $i = 1, 2, ..., n$ then for any two finite non-zero distinct complex numbers a, b,*

$$\begin{aligned} &{}_I\Theta^{(k)}_{1,2,\ldots,n}\left(a\right) + {}_I\Theta^{(k)}_{1,2,\ldots,n}\left(b\right) + n\cdot {}_I\Theta^{(k)}_0\left(a\right) + n\cdot {}_I\Theta^{(k)}_0\left(b\right) \\ &+\frac{n}{\pi\left(\rho_1+\rho_2+\ldots+\rho_n\right)} \le 2\left(1+n\right). \end{aligned}$$

Proof. In view of Lemma 2.1, we obtain for any entire function f that

$$\begin{aligned} m\left(r, \frac{1}{f}\right) &\le m\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{1}{f^{(k)}}\right) \\ &= m\left(r, \frac{1}{f^{(k)}}\right) + S\left(r, f\right) \\ &= T\left(r, \frac{1}{f^{(k)}}\right) - N\left(r, \frac{1}{f^{(k)}}\right) + S\left(r, f\right). \end{aligned} \tag{15}$$

Now using the relation $T(r,f) = T\left(r,\frac{1}{f}\right) + O(1)$ and Nevanlinna's second fundamental theorem, we get from Inequality (15) that

$$m\left(r,\frac{1}{f}\right) \leq \bar{N}\left(r,\frac{1}{f^{(k)}}\right) + \bar{N}\left(r,\frac{1}{f^{(k)}-a}\right) + \bar{N}\left(r,\frac{1}{f^{(k)}-b}\right)$$

$$-N\left(r,\frac{1}{f^{(k)}}\right) + S(r,f) \leq \bar{N}\left(r,\frac{1}{f^{(k)}-a}\right) + \bar{N}\left(r,\frac{1}{f^{(k)}-b}\right) + S(r,f)$$

$$\begin{aligned} i.e., T(r,f) \ \leq \ & \bar{N}\left(r,\frac{1}{f^{(k)}-a}\right) + \bar{N}\left(r,\frac{1}{f^{(k)}-b}\right) \\ & + N\left(r,\frac{1}{f}\right) + S(r,f). \end{aligned} \tag{16}$$

Now applying Inequality (16) for $f_1, f_2, f_3, ..., f_n$ we get that

$$\{T(r,f_1) + T(r,f_2) + ... + T(r,f_n)\}$$

$$\leq \left\{\bar{N}\left(r,\frac{1}{f_1^{(k)}-a}\right) + \bar{N}\left(r,\frac{1}{f_2^{(k)}-a}\right) + ... + \bar{N}\left(r,\frac{1}{f_n^{(k)}-a}\right)\right\}$$

$$+ \left\{\bar{N}\left(r,\frac{1}{f_1^{(k)}-b}\right) + \bar{N}\left(r,\frac{1}{f_2^{(k)}-b}\right) + ... + \bar{N}\left(r,\frac{1}{f_n^{(k)}-b}\right)\right\}$$
$$+ \left\{N\left(r,\frac{1}{f_1}\right) + N\left(r,\frac{1}{f_2}\right) + ... + N\left(r,\frac{1}{f_n}\right)\right\}$$

$$+ \{S(r,f_1) + S(r,f_2) + ... + S(r,f_n)\}. \tag{17}$$

Now $N\left(r,\frac{1}{f_i}\right) = S(r,f_i)$ for $i = 1,2,...,n$, we obtain from Inequality (17) that

$$\{T(r,f_1) + T(r,f_2) + ... + T(r,f_n)\} \leq \left\{\bar{N}_{1,2,...,n}^{(k)}(a) + n \cdot \bar{N}_0^{(k)}(a)\right\}$$

$$+ \left\{\bar{N}_{1,2,...,n}^{(k)}(b) + n \cdot \bar{N}_0^{(k)}(b)\right\} + \{S(r,f_1) + S(r,f_2) + ... + S(r,f_n)\}.$$

Now dividing both sides of Inequality (17) by $\{I(r,f_1) + I(r,f_2) + ... + I(r,f_n)\}$ and taking limit superior we get in view of Lemma 2.5 and Lemma 2.7 that

$$\begin{aligned}\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)} &\le \left\{1-{}_I\Theta^{(k)}_{1,2,...,n}(a)\right\}+n\cdot\left\{1-{}_I\Theta^{(k)}_0(a)\right\}\\ &\quad+\left\{1-{}_I\Theta^{(k)}_{1,2,...,n}(b)\right\}+n\cdot\left\{1-{}_I\Theta^{(k)}_0(b)\right\}\end{aligned}$$

$$\begin{aligned}i.e., {}_I\Theta^{(k)}_{1,2,...,n}(a)+{}_I\Theta^{(k)}_{1,2,...,n}(b)+n\cdot{}_I\Theta^{(k)}_0(a)+n\cdot{}_I\Theta^{(k)}_0(b)\\ +\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)}\le 2(1+n).\end{aligned}$$

Hence the theorem is established. □

Remark 3.5. *The condition 'a and b are two finite non-zero distinct complex numbers' in Theorem 3.5 is necessary as we see by taking* $n=2$, $f_1=\exp z$, $f_2=\exp(-z)$, $a=0,\infty$ *and* $b=0,\infty$. *Then we get that* $I(r,f_1)=I(r,\exp z)=r^2\ne 0$, $I(r,f_2)=I(r,\exp(-z))=r^2\ne 0$ *and* $\rho_1=\rho_2=1$. *Thus,*

$$\begin{aligned}{}_I\Theta^{(k)}_{1,2,...,n}(a) &= {}_I\Theta^{(k)}_{1,2}(0)={}_I\Theta^{(k)}_{1,2}(\infty)=1,\\ {}_I\Theta^{(k)}_{1,2,...,n}(b) &= {}_I\Theta^{(k)}_{1,2}(0)={}_I\Theta^{(k)}_{1,2}(\infty)=1,\end{aligned}$$

$$\begin{aligned}{}_I\Theta^{(k)}_0(a) &= {}_I\Theta^{(k)}_0(0)={}_I\Theta^{(k)}_0(\infty)=1 \text{ and}\\ {}_I\Theta^{(k)}_0(b) &= {}_I\Theta^{(k)}_0(0)={}_I\Theta^{(k)}_0(\infty)=1.\end{aligned}$$

Hence,

$$\begin{aligned}&{}_I\Theta^{(k)}_{1,2,...,n}(a)+{}_I\Theta^{(k)}_{1,2,...,n}(b)+n\cdot{}_I\Theta^{(k)}_0(a)\\ &+n\cdot{}_I\Theta^{(k)}_0(b)+\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)}\\ =\;&1+1+2+2+\frac{1}{\pi}=6+\frac{1}{\pi}\end{aligned}$$

and

$$2(1+n)=6,$$

which is contrary to Theorem 3.5.

Theorem 3.6. *Let* f_1, f_2, ..., f_n *be* n (>1) *entire functions of non-zero finite order* 'ρ_i' *with each* f_i *for* $i=1,2,...,n$ *having no zeros in* $\mathbb{C}$. *Also let* $a_1,a_2,...,a_7$ *be distinct elements of* $S(f_n)$ *for* $n\,(\ge 2)$. *Then*

$$\sum_{j=1}^{7}{}_I\Theta_{1,2,...,n}(a_j)+n\sum_{j=1}^{7}{}_I\Theta_0(a_j)+\frac{7n}{3\pi(\rho_1+\rho_2+...+\rho_n)}\le 7(1+n).$$

Proof. For any three distinct integers s, t and u such that $1 \leq s, t, u \leq 7$, we obtain in view of Lemma 2.2 that

$$\begin{aligned}&\{1 + o\,(1)\}\, T\,(r, f)\\&< \bar{N}\left(r, \frac{1}{f - a_s}\right) + \bar{N}\left(r, \frac{1}{f - a_t}\right) + \bar{N}\left(r, \frac{1}{f - a_u}\right) + S(r, f).\end{aligned}$$

When we choose three elements from $a_1, a_2, ...$ and a_7 then there are ${}^7C_3$ different combinations. Therefore we obtain that

$$ {}^7C_3\,(1 + o(1))\, T\,(r, f) < {}^6C_2 \sum_{j=1}^{7} \bar{N}\left(r, \frac{1}{f - a_j}\right) + S(r, f)$$

$$ i.e., 35\,(1 + o(1))\, T\,(r, f) < 15 \sum_{j=1}^{7} \bar{N}\left(r, \frac{1}{f - a_j}\right) + S(r, f)$$

$$ i.e., \frac{7}{3}\,(1 + o(1))\, T\,(r, f) < \sum_{j=1}^{7} \bar{N}\left(r, \frac{1}{f - a_j}\right) + S(r, f). \tag{18}$$

Applying Inequality (18) for $f_1, f_2, f_3, ..., f_n$ we get from Inequality (18) that

$$\begin{aligned}&\left(\frac{7}{3} + o(1)\right)\{T(r, f_1) + T(r, f_2) + ... + T(r, f_n)\}\\&< \sum_{j=1}^{7}\left\{\bar{N}\left(r, \frac{1}{f_1 - a_j}\right) + \bar{N}\left(r, \frac{1}{f_2 - a_j}\right) + ... + \bar{N}\left(r, \frac{1}{f_7 - a_j}\right)\right\}\\&+ [S(r, f_1) + S(r, f_2) + ... + S(r, f_n)]\end{aligned}$$

$$ i.e., \left(\frac{7}{3} + o(1)\right)\{T(r, f_1) + T(r, f_2) + ... + T(r, f_n)\}$$

$$ < \sum_{j=1}^{7}\left\{\bar{N}_{1,2,...,n}\,(r, a_j) + n\bar{N}_0\,(r, a_j)\right\}$$

$$ + [S(r, f_1) + S(r, f_2) + ... + S(r, f_n)]\,. \tag{19}$$

On dividing both sides of Inequality (19) by $\{I(r, f_1) + I(r, f_2) + ... + I(r, f_n)\}$ and taking limit superior we get in view of Lemma 2.5 and Lemma 2.7 that

$$\frac{7}{3}\cdot\frac{n}{\pi\left(\rho_1+\rho_2+...+\rho_n\right)}\leq\sum_{j=1}^{7}\left\{1-_I\Theta_{1,2,\cdots,n}\left(a_j\right)\right\}+n\sum_{j=1}^{7}\left\{1-_I\Theta_0\left(a_j\right)\right\}$$

$$i.e.,\frac{7}{3}\cdot\frac{n}{\pi\left(\rho_1+\rho_2+...+\rho_n\right)}\leq 7-\sum_{j=1I}^{7}{}_I\Theta_{1,2,\cdots,n}\left(a_j\right)+7n-n\sum_{j=1}^{7}{}_I\Theta_0\left(a_j\right)$$

$$i.e.,\sum_{j=1}^{7}{}_I\Theta_{1,2,\cdots,n}\left(a_j\right)+n\sum_{j=1}^{7}{}_I\Theta_0\left(a_j\right)+\frac{7n}{3\pi\left(\rho_1+\rho_2+...+\rho_n\right)}\leq 7\left(1+n\right).$$

This proves the theorem. □

Theorem 3.7. *If f_1, f_2, ..., f_n be n (>1) entire functions of non-zero finite order $'\rho_i\,'$ with each f_i for $i=1,2,...,n$ having no zeros in $\mathbb{C}$ then for any complex number $a\neq 0,\infty$ and for any non-negative integer k,*

$$_I\delta^{(k)}_{1,2,..,n}\left(\infty\right)+{}_I\Theta_{1,2,...,n}\left(0\right)+{}_I\Theta_{1,2,...,n}\left(a\right)+n\cdot{}_I\delta^{(k)}_0\left(\infty\right)+n\cdot{}_I\Theta_0\left(0\right)$$
$$+n\cdot{}_I\Theta_0\left(a\right)+\frac{n}{\pi\left(\rho_1+\rho_2+...+\rho_n\right)}\leq 3\left(1+n\right).$$

Proof. For any entire function f, the following inequality is well known ($cf.$ $p.$43, [9])

$$T\left(r,f\right)\leq\bar{N}\left(r,f\right)+\bar{N}\left(r,\frac{1}{f}\right)+\bar{N}\left(r,\frac{1}{f-a}\right)+S\left(r,f\right).$$

For all non-negative integers k we get from the above inequality

$$T\left(r,f\right)\leq k\bar{N}\left(r,f\right)+N\left(r,f\right)+\bar{N}\left(r,\frac{1}{f}\right)+\bar{N}\left(r,\frac{1}{f-a}\right)+S\left(r,f\right)$$

$$(20)\quad i.e.,T\left(r,f\right)\leq N\left(r,f^{(k)}\right)+\bar{N}\left(r,\frac{1}{f}\right)+\bar{N}\left(r,\frac{1}{f-a}\right)+S\left(r,f\right).$$

Now applying Inequality (20) for $f_1, f_2, f_3, ..., f_n$ it follows that

$$\left\{T(r,f_1)+T(r,f_2)+...+T(r,f_n)\right\}$$
$$\leq\left\{N\left(r,f_1^{(k)}\right)+N\left(r,f_2^{(k)}\right)+...+N\left(r,f_n^{(k)}\right)\right\}$$
$$+\left\{\bar{N}\left(r,\frac{1}{f_1}\right)+\bar{N}\left(r,\frac{1}{f_2}\right)+...+\bar{N}\left(r,\frac{1}{f_2}\right)\right\}$$

$$+\left\{\bar{N}\left(r,\frac{1}{f_1-a}\right)+\bar{N}\left(r,\frac{1}{f_2-a}\right)+...+\bar{N}\left(r,\frac{1}{f_n-a}\right)+\right\}$$
$$+\left\{S(r,f_1)+S(r,f_2)+...+S(r,f_n)\right\}$$

$$i.e., \{T(r,f_1)+T(r,f_2)+...+T(r,f_n)\} \leq \left\{N^{(k)}_{1,2,...,n}(\infty)+n\cdot N^{(k)}_0(\infty)\right\}$$
$$+\left\{\bar{N}_{1,2,...,n}(0)+n\cdot \bar{N}_0(0)\right\}+\left\{\bar{N}_{1,2,...,n}(a)+n\cdot \bar{N}_0(a)\right\}$$
$$+\left\{S(r,f_1)+S(r,f_2)+...+S(r,f_n)\right\}. \tag{21}$$

On dividing both sides of Inequality (21) by $\{I(r,f_1)+I(r,f_2)+...+I(r,f_n)\}$ and taking limit superior we get in view of Lemma 2.5 and Lemma 2.7 that

$$\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)} \leq \left\{1-{}_I\,\delta^{(k)}_{1,2,...,n}(\infty)\right\}+\left\{1-{}_I\,\delta^{(k)}_0(\infty)\right\}$$
$$+\{1-{}_I\,\Theta_{1,2,...,n}(0)\}+n\{1-{}_I\,\Theta_0(0)\}$$
$$+\{1-{}_I\,\Theta_{1,2,...,n}(a)\}+\{1-{}_I\,\Theta_0(a)\}$$

$$i.e.,{}_I\,\delta^{(k)}_{1,2,..,n}(\infty)+{}_I\,\Theta_{1,2,...,n}(0)+{}_I\,\Theta_{1,2,...,n}(a)+n\cdot{}_I\,\delta^{(k)}_0(\infty)+n\cdot{}_I\,\Theta_0(0)$$
$$+n\cdot{}_I\,\Theta_0(a)+\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)} \leq 3+3n$$

$$i.e.,{}_I\,\delta^{(k)}_{1,2,..,n}(\infty)+{}_I\Theta_{1,2,...,n}(0)+{}_I\Theta_{1,2,...,n}(a)+n\cdot{}_I\delta^{(k)}_0(\infty)+n\cdot{}_I\Theta_0(0)$$
$$+n\cdot{}_I\Theta_0(a)+\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)} \leq 3(1+n).$$

Thus the theorem is proved. □

Remark 3.6. *The condition '$a \neq 0,\infty$ in Theorem 3.7' is essential which is evident by considering $n=2$, $f_1=\exp(2z)$, $f_2=\exp(-2z)$ and $a=0,\infty$. Then we see that $I(r,f_1)=I(r,\exp(2z))=2r^2\neq 0$, $I(r,f_2)=I(r,\exp(-2z))=2r^2\neq 0$ and $\rho_1=\rho_2=1$. Hence*

$${}_I\delta^{(k)}_{1,2,..,n}(\infty)={}_I\delta^{(k)}_{1,2}(\infty)=1,\ {}_I\Theta_{1,2,..,n}(0)={}_I\Theta_{1,2}(0)=1,$$
$${}_I\delta^{(k)}_0(\infty)=1,\ {}_I\Theta_{1,2,..,n}(a)={}_I\Theta_{1,2}(0)={}_I\Theta_{1,2}(\infty)=1$$
$$\text{and } {}_I\Theta_0(a)={}_I\Theta_0(0)={}_I\Theta_0(\infty)=1.$$

Therefore,

$$i.e.,{}_I\,\delta^{(k)}_{1,2,..,n}(\infty)+{}_I\Theta_{1,2,...,n}(0)+{}_I\Theta_{1,2,...,n}(a)+n\cdot{}_I\delta^{(k)}_0(\infty)+n\cdot{}_I\Theta_0(0)$$
$$+n\cdot{}_I\Theta_0(a)+\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)}=1+1+1+2+2+2=9+\frac{1}{\pi}.$$

and

$$3(1+n)=9,$$

which is contrary to Theorem 3.7.

Theorem 3.8. *Let f_1, f_2, ..., f_n be n (>1) entire functions of non-zero finite order $'\rho_i'$ with each f_i for $i=1,2,...,n$ having no zeros in $\mathbb{C}$. Also let $\bar{N}(r,f_1)=S(r,f_1), \bar{N}(r,f_2)=S(r,f_2),...$ and $\bar{N}(r,f_n)=S(r,f_n)$ then for every positive integer k,*

$$\begin{aligned} & {}_I\delta^{(k)}_{1,2,...,n}(a)+ {}_I\delta_{1,2,...,n}(a)+n\cdot {}_I\delta^{(k)}_0(a) \\ & +n\cdot {}_I\delta_0(a)+\frac{n}{\pi\left(\rho_1+\rho_2+...+\rho_n\right)}\le 2\left(1+n\right), \end{aligned}$$

where $'a'$ is any finite non-zero complex number.

Proof. From the identity

$$\frac{1}{f-a}=\frac{1}{a}\left\{\frac{f^{(k)}}{f-a}-\frac{f^{(k)}-a}{f^{(l)}}\cdot\frac{f^{(l)}}{f-a}\right\},$$

where $0\le k<l$ and by Milloux's theorem {p. 55, [9]}, we get that

$$m\left(r,\frac{1}{f-a}\right)\le m\left(r,\frac{f^{(k)}-a}{f^{(l)}}\right)+S\left(r,f\right). \tag{23}$$

Now by Nevanlinna's first fundamental theorem it follows from Inequality (23) that

$$m\left(r,\frac{1}{f-a}\right)\le T\left(r,\frac{f^{(k)}-a}{f^{(l)}}\right)-N\left(r,\frac{f^{(k)}-a}{f^{(l)}}\right)+S\left(r,f\right)$$

$$\text{i.e., } m\left(r,\frac{1}{f-a}\right)\le T\left(r,\frac{f^{(l)}}{f^{(k)}-a}\right)-N\left(r,\frac{f^{(k)}-a}{f^{(l)}}\right)+S\left(r,f\right)$$

$$\begin{aligned} \text{i.e., } m\left(r,\frac{1}{f-a}\right) \le\ & N\left(r,\frac{f^{(l)}}{f^{(k)}-a}\right) \\ & -N\left(r,\frac{f^{(k)}-a}{f^{(l)}}\right)+S\left(r,f\right). \end{aligned} \tag{24}$$

Now in view of {p.34, [9]} and as $N\left(r,\frac{1}{f^{(n)}}\right)\ge 0$, we obtain from Inequality (24) that

$$\begin{aligned} & m\left(r,\frac{1}{f-a}\right) \\ & \le N\left(r,f^{(l)}\right)+N\left(r,\frac{1}{f^{(k)}-a}\right)-N\left(r,f^{(k)}-a\right)-N\left(r,\frac{1}{f^{(l)}}\right)+S\left(r,f\right) \end{aligned}$$

$$\text{i.e., } T\left(r,\frac{1}{f-a}\right)$$
$$\leq \left\{N\left(r,f\right)+n\cdot\bar{N}\left(r,f\right)\right\}+N\left(r,\frac{1}{f^{(k)}-a}\right)+N\left(r,\frac{1}{f-a}\right)$$
$$-\left\{N\left(r,f\right)+k\cdot\bar{N}\left(r,f\right)\right\}+S\left(r,f\right)$$
$$\text{i.e., } T\left(r,f\right)\leq\left(n-k\right)\bar{N}\left(r,f\right)+N\left(r,\frac{1}{f^{(k)}-a}\right)$$
$$+N\left(r,\frac{1}{f-a}\right)+S\left(r,f\right). \tag{25}$$

Applying Inequality (25) for $f_1, f_2, ..., f_n$ it follows from Inequality (25) that

$$T(r,f_1)+T(r,f_2)+...+T(r,f_n)$$
$$\leq\left(n-k\right)\left\{\bar{N}\left(r,f_1\right)+\bar{N}\left(r,f_2\right)+...+\bar{N}\left(r,f_n\right)\right\}$$
$$+\left\{N\left(r,\frac{1}{f_1^{(k)}-a}\right)+N\left(r,\frac{1}{f_2^{(k)}-a}\right)+...+N\left(r,\frac{1}{f_n^{(k)}-a}\right)\right\}$$
$$+\left\{N\left(r,\frac{1}{f_1-a}\right)+N\left(r,\frac{1}{f_2-a}\right)+...+N\left(r,\frac{1}{f_n-a}\right)\right\}$$
$$+\{S(r,f_1)+S(r,f_2)+...+S(r,f_n)\}$$

$$\text{i.e., } T(r,f_1)+T(r,f_2)+...+T(r,f_n)$$
$$\leq\left(n-k\right)\left\{\bar{N}\left(r,f_1\right)+\bar{N}\left(r,f_2\right)+...+\bar{N}\left(r,f_n\right)\right\}$$
$$+\left\{N_{1,2,...,n}^{(k)}\left(r,a\right)+n\cdot N_0^{(k)}\left(r,a\right)\right\}+\left\{N_{1,2,\cdots,n}\left(r,a\right)+n\cdot N_0\left(r,a\right)\right\}$$
$$+\{S(r,f_1)+S(r,f_2)+...+S(r,f_n)\}. \tag{26}$$

As $\bar{N}(r,f_1) = S(r,f_1)$, $\bar{N}(r,f_2) = S(r,f_2)$, ... and $\bar{N}(r,f_n) = S(r,f_n)$, dividing both sides of Inequality (26) by $\{I(r,f_1)+I(r,f_2)+\cdots+I(r,f_n)\}$ and taking limit superior we get in view of Lemma 2.5 and Lemma 2.7 that

$$\frac{n}{\pi\left(\rho_1+\rho_2+...+\rho_n\right)} \leq \left\{1-_I\delta_{1,2,...,n}^{(k)}(a)\right\}+n\cdot\left\{1-_I\delta_0^{(k)}(a)\right\}$$
$$+\{1-_I\delta_{1,2,...,n}(a)\}+n\cdot\{1-_I\delta_0(a)\}$$

$$i.e., \ {}_I\delta^{(k)}_{1,2,...,n}(a) +_I \delta_{1,2,...,n}(a) + n \cdot_I \delta^{(k)}_0(a) + n \cdot_I \delta_0(a)$$
$$+\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)} \leq 2+2n$$
$$i.e., \ {}_I\delta^{(k)}_{1,2,...,n}(a) + \ {}_I\delta_{1,2,...,n}(a) + n \cdot \ {}_I\delta^{(k)}_0(a) + n \cdot \ {}_I\delta_0(a)$$
$$+\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)} \leq 2(1+n).$$

This proves the theorem. □

Remark 3.7. *The condition 'a be a finite non-zero complex number' in Theorem 3.8 is necessary as we see by taking* $n = 2$, $f_1 = \exp(z^2)$, $f_2 = \exp(-z^2)$ *and* $a = 0, \infty$. *Then we see that* $I(r, f_1) = \frac{2r^3}{\pi} \cdot I_0(1) \neq 0$, $I(r, f_2) = \frac{2r^3}{\pi} \cdot I_0(1) \neq 0$, *where* $I_n(z)$ *is the Modified Bessel Function of the first kind such that* $I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta} \cdot \cos n\theta d\theta$ *and* $\rho_1 = \rho_2 = 2$. *Hence,*

$$\begin{aligned} {}_I\delta^{(k)}_{1,2}(0) &= \ {}_I\delta^{(k)}_{1,2}(\infty) = \ {}_I\delta_{1,2}(0) = \ {}_I\delta_{1,2}(\infty) \ \textit{and} \\ {}_I\delta^{(k)}_0(0) &= \ {}_I\delta^{(k)}_0(\infty) = \ {}_I\delta_0(0) = \ {}_I\delta_0(\infty) = 1. \end{aligned}$$

Thus,

$${}_I\delta^{(k)}_{1,2,...,n}(a) + \ {}_I\delta_{1,2,...,n}(a) + n \cdot \ {}_I\delta^{(k)}_0(a) + n \cdot \ {}_I\delta_0(a) + \frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)}$$
$$= 1+1+2+2 = 6 + \frac{1}{2\pi}$$

and

$$2(1+n) = 6.$$

So, we arrive at a contradiction.

Theorem 3.9. *Let* $f_1, f_2, ..., f_n$ *be* n (>1) *entire functions of non-zero finite order '*ρ_i*' with each* f_i *for* $i = 1, 2, ..., n$ *having no zeros in* $\mathbb{C}$ *such that* $\sum\limits_{\alpha\neq\infty}\delta(\alpha; f_1) = \delta(\infty; f_1) = 1$, $\sum\limits_{\alpha\neq\infty}\delta(\alpha; f_2) = \delta(\infty; f_2) = 1, ...$ *and* $\sum\limits_{\alpha\neq\infty}\delta(\alpha; f_n) = \delta(\infty; f_n) = 1$. *Also let* a *be a finite complex number and* b, c *be two distinct non zero complex numbers. Then*

$${}_I\delta_{1,2,...,n}(a) + \ {}_I\Theta^{(k)}_{1,2,\cdots,n}(b) + \ {}_I\Theta^{(k)}_{1,2,\cdots,n}(c) + n \cdot \ {}_I\delta_0(a) + n \cdot \ {}_I\Theta^{(k)}_0(b)$$
$$+n \cdot \ {}_I\Theta^{(k)}_0(c) + \frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)} \leq 3(n+1).$$

Proof. Since

$$\frac{1}{f-a}=\frac{f^{(k)}}{f-a}\cdot\frac{1}{f^{(k)}},$$

by Milloux's theorem {p.55, [9]} , we obtain that

$$m\left(r,\frac{1}{f-a}\right)\le m\left(r,\frac{1}{f^{(k)}}\right)+S\left(r,f\right). \tag{27}$$

Applying Nevanlinna's first fundamental theorem we get from Inequality (27) that

$$m\left(r,\frac{1}{f-a}\right)\le T\left(r,f^{(k)}\right)-N\left(r,\frac{1}{f^{(k)}}\right)+S\left(r,f\right). \tag{28}$$

Now by Nevanlinna's second fundamental theorem and Lemma 2.1, it follows from Inequality (28) that

$$\begin{aligned} & m\left(r,\frac{1}{f-a}\right) \\ & \le \bar{N}\left(r,\frac{1}{f^{(k)}}\right)+\bar{N}\left(r,\frac{1}{f^{(k)}-b}\right)+\bar{N}\left(r,\frac{1}{f^{(k)}-c}\right) \\ & -N\left(r,\frac{1}{f^{(k)}}\right)+S\left(r,f\right). \end{aligned} \tag{29}$$

Since $\bar{N}\left(r,\frac{1}{f^{(k)}}\right)-N\left(r,\frac{1}{f^{(k)}}\right)\le 0$, we obtain from Inequality (29) that

$$m\left(r,\frac{1}{f-a}\right)\le\bar{N}\left(r,\frac{1}{f^{(k)}-b}\right)+\bar{N}\left(r,\frac{1}{f^{(k)}-c}\right)+S\left(r,f\right)$$

$$\begin{aligned} \text{i.e., } T\left(r,f\right) & \le N\left(r,\frac{1}{f-a}\right)+\bar{N}\left(r,\frac{1}{f^{(k)}-b}\right) \\ & +\bar{N}\left(r,\frac{1}{f^{(k)}-c}\right)+S\left(r,f\right). \end{aligned} \tag{30}$$

Now applying Inequality (30) for $f_1, f_2, ..., f_n$ it follows from Inequality (30) that

$$\begin{aligned}
&T(r,f_1)+T(r,f_2)+...+T(r,f_n)\\
&\leq\left\{N\left(r,\frac{1}{f_1-a}\right)+N\left(r,\frac{1}{f_2-a}\right)+...+N\left(r,\frac{1}{f_n-a}\right)\right\}\\
&+\left\{\bar{N}\left(r,\frac{1}{f_1^{(k)}-b}\right)+\bar{N}\left(r,\frac{1}{f_2^{(k)}-b}\right)+...+\bar{N}\left(r,\frac{1}{f_n^{(k)}-b}\right)\right\}\\
&+\left\{\bar{N}\left(r,\frac{1}{f_1^{(k)}-c}\right)+\bar{N}\left(r,\frac{1}{f_2^{(k)}-c}\right)+...+\bar{N}\left(r,\frac{1}{f_n^{(k)}-c}\right)\right\}\\
&+\{S(r,f_1)+S(r,f_2)+...+S(r,f_n)\}
\end{aligned}$$

$$\begin{aligned}
&\text{i.e., } T(r,f_1)+T(r,f_2)+...+T(r,f_n)\\
&\leq\{N_{1,2,...,n}(r,a)+n\cdot N_0(r,a)\}+\left\{\bar{N}^{(k)}_{1,2,\cdots,n}(r,b)+n\cdot\bar{N}^{(k)}_0(r,b)\right\}\\
&+\left\{\bar{N}^{(k)}_{1,2,...,n}(r,c)+n\cdot\bar{N}^{(k)}_0(r,c)\right\}
\end{aligned}$$

$$+\{S(r,f_1)+S(r,f_2)+...+S(r,f_n)\}. \tag{31}$$

Dividing both sides of Inequality (31) by $\{I(r,f_1)+I(r,f_2)+...+I(r,f_n)\}$ and taking limit superior we get in view of Lemma 2.5 and Lemma 2.7 that

$$\begin{aligned}
\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)}&\leq\{1-{}_I\delta_{1,2,...,n}(a)\}+n\{1-{}_I\delta_0(a)\}\\
&+\left\{1-{}_I\Theta^{(k)}_{1,2,\cdots,n}(b)\right\}+n\left\{1-{}_I\Theta^{(k)}_0(b)\right\}\\
&+\left\{1-{}_I\Theta^{(k)}_{1,2,...,n}(c)\right\}+n\left\{1-{}_I\Theta^{(k)}_0(c)\right\}
\end{aligned}$$

$$\textit{i.e., } {}_I\delta_{1,2,...,n}(a)+{}_I\Theta^{(k)}_{1,2,...,n}(b)+{}_I\Theta^{(k)}_{1,2,\cdots,n}(c)+n\cdot{}_I\delta_0(a)+n\cdot{}_I\Theta^{(k)}_0(b)$$

$$+n\cdot{}_I\Theta^{(k)}_0(c)+\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)}\leq 3n+3$$

$$\textit{i.e., } {}_I\delta_{1,2,...,n}(a)+{}_I\Theta^{(k)}_{1,2,...,n}(b)+{}_I\Theta^{(k)}_{1,2,\cdots,n}(c)+n\cdot{}_I\delta_0(a)$$

$$+n\cdot{}_I\Theta^{(k)}_0(b)+n\cdot{}_I\Theta^{(k)}_0(c)+\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)}\leq 3(n+1).$$

Thus the theorem is proved. □

Theorem 3.10. *Let f_1, f_2, ..., f_n be n (>1) entire functions of non-zero finite order $'\rho_i\,'$ with each f_i having no zeros in $\mathbb{C}$ such that $\bar{N}(r, f_i) = S(r, f_i)$ and $T\left(r, f_i^{(k)}\right) \sim n' T\,(r, f_i)$ for $i = 1, 2, ..., n$. Then for any positive integer k and any finite complex number $'a'$*

$$_I\delta_{1,2,...,l}(0) + n\ _I\delta_0(0) + n'\frac{n}{\pi\,(\rho_1 + \rho_2 + ... + \rho_n)} \leq n + 1,$$

where $n \geq 2$ and $n' \geq 1$.

Proof. Let $b \neq a$ be a finite complex number. Since

$$\frac{a-b}{f^{(k)} - a} = \frac{f}{f^{(k)} - a}\left\{\frac{f^{(k)} - b}{f} - \frac{f^{(k)} - a}{f}\right\},$$

we obtain in view of Milloux's theorem (p. 55, [9]) and Nevanlinna's first fundamental theorem that

$$m\left(r, \frac{a-b}{f^{(k)} - a}\right) \leq m\left(r, \frac{f}{f^{(k)} - a}\right) + S\,(r, f)$$

$$\text{i.e., } m\left(r, \frac{1}{f^{(k)} - a}\right) \leq T\left(r, \frac{f}{f^{(k)} - a}\right) - N\left(r, \frac{f}{f^{(k)} - a}\right) + S\,(r, f)$$

$$\text{i.e., } m\left(r, \frac{1}{f^{(k)} - a}\right) \leq N\left(r, \frac{f^{(k)} - a}{f}\right) - N\left(r, \frac{f}{f^{(k)} - a}\right) + S\,(r, f). \tag{32}$$

In view of (p. 34, [9]), it follows from Inequality (32) that

$$m\left(r, \frac{1}{f^{(k)} - a}\right) \leq N\left(r, f^{(k)} - a\right) + N\left(r, \frac{1}{f}\right) - N\,(r, f) - N\left(r, \frac{1}{f^{(k)} - a}\right) + S\,(r, f)$$

$$\text{i.e., } T\left(r, \frac{1}{f^{(k)} - a}\right) \leq \left\{N\,(r, f) + k\bar{N}\,(r, f)\right\} + N\left(r, \frac{1}{f}\right) - N\,(r, f) + S\,(r, f)$$

$$\text{i.e., } T\left(r, f^{(k)}\right) \leq k\bar{N}\,(r, f) + N\left(r, \frac{1}{f}\right) + S\,(r, f). \tag{33}$$

Since $\bar{N}(r,f_i) = S(r,f_i)$ and $T\left(r,f_i^{(k)}\right) \sim l'T\left(r,f_i\right)$ for $i = 1,2,...,n$, applying Inequality (33) for $f_1, f_2, ..., f_n$ it follows that

$$n'\{T(r,f_1)+T(r,f_2)+...+T(r,f_n)\}$$
$$\leq \left\{N\left(r,\frac{1}{f_1}\right)+N\left(r,\frac{1}{f_2}\right)+...+N\left(r,\frac{1}{f_n}\right)\right\}$$
$$+\{S(r,f_1)+S(r,f_2)+...+S(r,f_n)\}$$

$$\text{i.e., } n'\{T(r,f_1)+T(r,f_2)+\cdots+T(r,f_n)\}$$
$$\leq \{N_{1,2,...,n}(r,0)+nN_0(r,0)\}$$
$$+\{S(r,f_1)+S(r,f_2)+...+S(r,f_n)\}. \tag{34}$$

Now dividing both sides of Inequality (34) by $\{I(r,f_1)+I(r,f_2)+...+I(r,f_n)\}$ and taking limit superior it follows in view of Lemma 2.5 and Lemma 2.7 that

$$n'\frac{n}{\pi\left(\rho_1+\rho_2+...+\rho_n\right)} \leq \{1- {}_I\delta_{1,2,...,n}(0)\}+n\{1- {}_I\delta_0(0)\}$$

$$i.e.,{}_I\,\delta_{1,2,...,l}(0)+n_I\delta_0(0)+n'\frac{n}{\pi\left(\rho_1+\rho_2+...+\rho_n\right)} \leq n+1.$$

This completes the proof of the theorem. □

Theorem 3.11. *Let f_1, f_2, ..., f_n be n (>1) entire functions of non-zero finite order $'\rho_i\,'$ with each f_i for $i=1,2,...,n$ having no zeros in $\mathbb{C}$ such that $\bar{N}(r,f_1)=S(r,f_1)$, $\bar{N}(r,f_2)=S(r,f_2)$, ... , $\bar{N}(r,f_n)=S(r,f_n)$ and $'\alpha'$ be a non zero finite complex number. Then*

$$\begin{gathered}{}_I\delta^{(k)}_{1,2,...,n}(\alpha)+ {}_I\delta_{1,2,...,n}(0)+n\cdot {}_I\delta^{(k)}_0(\alpha)\\ +n\cdot {}_I\delta_0(0)+\frac{n}{\pi\left(\rho_1+\rho_2+...+\rho_n\right)}\\ \leq 2\left(n+1\right).\end{gathered}$$

Proof. Considering the identity

$$\frac{\alpha}{f}=\frac{f^{(k)}}{f}-\frac{f^{(k)}-\alpha}{f^{(k+1)}}\cdot\frac{f^{(k+1)}}{f},$$

we get in view of Milloux's theorem {p. 55, [9]} and Nevanlinna's first fundamental theorem that

$$m\left(r,\frac{1}{f}\right)\leq m\left(r,\frac{f^{(k)}-\alpha}{f^{(k+1)}}\right)+S\left(r,f\right)$$

$$\text{i.e., } m\left(r,\frac{1}{f}\right)\leq T\left(r,\frac{f^{(k)}-\alpha}{f^{(k+1)}}\right)-N\left(r,\frac{f^{(k)}-\alpha}{f^{(k+1)}}\right)+S\left(r,f\right)$$

$$\text{i.e., } m\left(r,\frac{1}{f}\right)\leq T\left(r,\frac{f^{(k+1)}}{f^{(k)}-\alpha}\right)-N\left(r,\frac{f^{(k)}-\alpha}{f^{(k+1)}}\right)+S\left(r,f\right)$$

$$\text{(35) i.e., } m\left(r,\frac{1}{f}\right)\leq N\left(r,\frac{f^{(k+1)}}{f^{(k)}-\alpha}\right)-N\left(r,\frac{f^{(k)}-\alpha}{f^{(k+1)}}\right)+S\left(r,f\right).$$

Now in view of [9, p. 34] and as $N\left(r,\frac{1}{f^{(k+1)}}\right)\geq 0$, it follows from Inequality (35) that

$$m\left(r,\frac{1}{f}\right)\leq N\left(r,f^{(k+1)}\right)+N\left(r,\frac{1}{f^{(k)}-\alpha}\right)-N\left(r,f^{(k)}-\alpha\right)$$
$$-N\left(r,\frac{1}{f^{(k+1)}}\right)+S\left(r,f\right)$$

$$\text{i.e., } m\left(r,\frac{1}{f}\right)\leq N\left(r,\frac{1}{f^{(k)}-\alpha}\right)+N\left(r,f^{(k+1)}\right)-N\left(r,f^{(k)}\right)+S\left(r,f\right)$$

$$\text{i.e., } m\left(r,\frac{1}{f}\right)\leq N\left(r,\frac{1}{f^{(k)}-\alpha}\right)+\bar{N}\left(r,f\right)+S\left(r,f\right)$$

$$\text{(36) i.e., } T\left(r,f\right)\leq N\left(r,\frac{1}{f^{(k)}-\alpha}\right)+N\left(r,\frac{1}{f}\right)+\bar{N}\left(r,f\right)+S\left(r,f\right).$$

Applying Inequality (36) for $f_1, f_2, ..., f_n$ we obtain that

$$T(r,f_1)+T(r,f_2)+...+T(r,f_n)$$
$$\leq\left\{N^{(k)}_{1,2,...,n}\left(r,\alpha\right)+nN^{(k)}_{0}\left(r,\alpha\right)\right\}+\left\{N_{1,2,\cdots,n}\left(r,0\right)+nN_{0}\left(r,0\right)\right\}$$
$$+\left\{\bar{N}\left(r,f_1\right)+\bar{N}\left(r,f_2\right)+\cdots+\bar{N}\left(r,f_n\right)\right\}$$
$$\text{(37)}\qquad +\{S(r,f_1)+S(r,f_2)+...+S(r,f_n)\}.$$

As $\bar{N}(r,f_i)=S(r,f_i)$ for $i=1,2,...,n$, dividing both sides of Inequality (37) by $\{I(r,f_1)+I(r,f_2)+\cdots+I(r,f_n)\}$ and taking limit superior we get in view of Lemma 2.5 and Lemma 2.7 that

$$\frac{n}{\pi\left(\rho_1+\rho_2+...+\rho_n\right)}\leq\left\{1-{}_I\,\delta^{(k)}_{1,2,...,n}(\alpha)\right\}+n\left\{1-{}_I\delta^{(k)}_{0}(\alpha)\right\}$$
$$+\left\{1-{}_I\delta_{1,2,...,n}(0)\right\}+n\left\{1-{}_I\delta_{0}(0)\right\}$$

$$\text{i.e., } {}_I\delta^{(k)}_{1,2,...,n}(\alpha)+{}_I\delta_{1,2,...,n}(0)+n\cdot{}_I\delta^{(k)}_{0}(\alpha)+n\cdot{}_I\delta_{0}(0)$$

$$+\frac{n}{\pi\left(\rho_1+\rho_2+\ldots+\rho_n\right)} \leq 2\left(n+1\right). \tag{38}$$

Thus the theorem is established. □

Remark 3.8. *The condition* $'\alpha$ *be a non zero finite complex number*$'$ *in Theorem 3.11 is essential which is evident by taking* $n=2$, $f_1=\exp z$, $f_2=\exp(-z)$ *and* $\alpha=0,\infty$. *Then we see that* $I\left(r,f_1\right)=I\left(r,f_1\right)=r^2\neq 0$ *and* $\rho_1=\rho_2=1$. *So,*

$$\begin{aligned} {}_I\delta^{(k)}_{1,2,\ldots,n}(\alpha) &= {}_I\delta^{(k)}_{1,2}(0) ={}_I\delta^{(k)}_{1,2}(\infty)=1,\ {}_I\delta_{1,2,\ldots,n}(0)= {}_I\delta_{1,2}(0)=1,\\ {}_I\delta^{(k)}_0(\alpha) &= {}_I\delta^{(k)}_0(0)= {}_I\delta^{(k)}_0(\infty)=1 \text{ and } {}_I\delta_0(0)=1. \end{aligned}$$

Hence

$$\begin{gathered} {}_I\delta^{(k)}_{1,2,\ldots,n}(\alpha)+ {}_I\delta_{1,2,\ldots,n}(0)+n\cdot {}_I\delta^{(k)}_0(\alpha) \\ +n\cdot {}_I\delta_0(0)+\frac{n}{\pi\left(\rho_1+\rho_2+\ldots+\rho_n\right)} \\ =1+1+2+2+\frac{1}{\pi}=6+\frac{1}{\pi}. \end{gathered}$$

and

$$2\left(n+1\right)=6,$$

which contradicts Theorem 3.11.

Theorem 3.12. *Let* $f_1,\ f_2,\ \ldots,\ f_n$ *be* $n\ (>1)$ *entire functions of non-zero finite order* $'\rho_i\ '$ *with each* f_i *for* $i=1,2,\ldots,n$ *having no zeros in* $\mathbb{C}$ *such that* $a_1,a_2,\ldots,a_5$ *be distinct elements of* $S\left(f_n\right)$ *for* $n\geq 2$. *Then*

$$\begin{gathered} \sum_{j=1}^{3}\ {}_I\Theta_{1,2,\cdots,n}(a_j)+3\ {}_I\Theta_{1,2,\ldots,n}(a_4)+3\ {}_I\Theta_{1,2,\ldots,n}(a_5) \\ +l\sum_{j=1}^{3}\ {}_I\Theta_0(a_j)+3n\ {}_I\Theta_0(a_4)+3n\ {}_I\Theta_0(a_5) \\ +\frac{3n}{\pi\left(\rho_1+\rho_2+\ldots+\rho_n\right)}\leq 9\left(n+1\right). \end{gathered}$$

Proof. In view of Lemma 2.2, we get for $j=1,2,3$ that

$$\begin{aligned} &\left(1+o\left(1\right)\right)T\left(r,f\right) \\ &<\bar{N}\left(r,\frac{1}{f-a_j}\right)+\bar{N}\left(r,\frac{1}{f-a_4}\right)+\bar{N}\left(r,\frac{1}{f-a_5}\right)+S(r,f). \end{aligned}$$

Adding these inequalities for $j = 1, 2, 3$, we obtain that

$$\begin{aligned}
&(3 + o(1))\, T(r, f) \\
&< \sum_{j=1}^{3} \bar{N}\left(r, \frac{1}{f - a_j}\right) + 3\bar{N}\left(r, \frac{1}{f - a_4}\right) \\
&+ 3\bar{N}\left(r, \frac{1}{f - a_5}\right) + S(r, f). \qquad (39)
\end{aligned}$$

Applying Inequality (39) for $f_1, f_2, ..., f_n$ we get that

$$\begin{aligned}
&(3 + o(1)) \{T(r, f_1) + T(r, f_2) + \cdots + T(r, f_n)\} \\
&< \sum_{j=1}^{3} \left\{\bar{N}\left(r, \frac{1}{f_1 - a_j}\right) + \bar{N}\left(r, \frac{1}{f_2 - a_j}\right) + \cdots + \bar{N}\left(r, \frac{1}{f_n - a_j}\right)\right\} \\
&+3\left\{\bar{N}\left(r, \frac{1}{f_1 - a_4}\right) + \bar{N}\left(r, \frac{1}{f_2 - a_4}\right) + ... + \bar{N}\left(r, \frac{1}{f_n - a_4}\right)\right\} \\
&+ 3\left\{\bar{N}\left(r, \frac{1}{f_1 - a_5}\right) + \bar{N}\left(r, \frac{1}{f_2 - a_5}\right) + ... + \bar{N}\left(r, \frac{1}{f_n - a_5}\right)\right\} \\
&+ \{S(r, f_1) + S(r, f_2) + ... + S(r, f_n)\}
\end{aligned}$$

i.e., $(3 + o(1)) \{T(r, f_1) + T(r, f_2) + ... + T(r, f_n)\}$

$$\begin{aligned}
&< \sum_{j=1}^{3} \left\{\bar{N}_{1,2,\cdots,n}(r, a_j) + n\bar{N}_0(r, a_j)\right\} + 3\left\{\bar{N}_{1,2,...,n}(r, a_4) + n\bar{N}_0(r, a_4)\right\} \\
&+ 3\left\{\bar{N}_{1,2,...,n}(r, a_5) + n\bar{N}_0(r, a_5)\right\} \\
&+\{S(r, f_1) + S(r, f_2) + ... + S(r, f_n)\}. \qquad (40)
\end{aligned}$$

On dividing both sides of Inequality (40) by $\{I(r, f_1) + I(r, f_2) + ... + I(r, f_n)\}$ and taking limit superior we get in view of Lemma 2.5 and Lemma 2.7 that

$$\begin{aligned}
\frac{3n}{\pi(\rho_1 + \rho_2 + ... + \rho_n)} &\le \sum_{j=1}^{3} \{1 - {}_I\Theta_{1,2,\cdots,n}(a_j)\} + n\sum_{j=1}^{3} \{1 - {}_I\Theta_0(a_j)\} \\
&+3\{1 - {}_I\Theta_{1,2,...,n}(a_4)\} + 3n\{1 - {}_I\Theta_0(a_4)\} \\
&+3\{1 - {}_I\Theta_{1,2,...,n}(a_5)\} + 3n\{1 - {}_I\Theta_0(a_5)\}
\end{aligned}$$

$$\text{i.e., } \frac{3n}{\pi(\rho_1+\rho_2+...+\rho_n)} \leq 3-\sum_{j=1}^{3} {}_I\Theta_{1,2,\cdots,n}(a_j)+3n \; - n\sum_{j=1}^{3} {}_I\Theta_0(a_j)$$
$$+3\{1- {}_I\Theta_{1,2,...,l}(a_4)\}+3n\{1- {}_I\Theta_0(a_4)\}$$
$$+3\{1- {}_I\Theta_{1,2,...,l}(a_5)\}+3n\{1- {}_I\Theta_0(a_5)\}$$

$$\text{i.e.,} \sum_{j=1}^{3} {}_I\Theta_{1,2,\cdots,n}(a_j)+3\,{}_I\Theta_{1,2,\cdots,n}(a_4)+3\,{}_I\Theta_{1,2,...,n}(a_5)$$
$$+l\sum_{j=1}^{3} {}_I\Theta_0(a_j)+3n\,{}_I\Theta_0(a_4)+3n\,{}_I\Theta_0(a_5)$$
$$+\frac{3n}{\pi(\rho_1+\rho_2+...+\rho_n)} \leq 9(n+1).$$

This proves the theorem. □

Theorem 3.13. *Let $f_1, f_2, ..., f_n$ be $n\ (>1)$ entire functions of non-zero finite order $'\rho_i\,'$ with each f_i for $i=1,2,...,n$ having no zeros in $\mathbb{C}$ such that $\delta(0,f_t)=\delta(\infty,f_t)=1$ for $t=1,2,...,n$. Also let $a_i\,(i=1,2,...,p)$ be finite, distinct, non-zero complex numbers. Then*

$$\sum_{i=1}^{p} {}_I\Theta_{1,2,...,n}^{(k)}(a_i)+n\sum_{i=1}^{p} {}_I\Theta_0^{(k)}(a_i)+\frac{pn}{\pi(\rho_1+\rho_2+...+\rho_n)} \leq p(n+1).$$

Proof. In view of the Equality (1), Inequality (2) and $\overline{N}\left(r,f^{(k)}\right)=\overline{N}(r,f)$, we obtain by replacing q by p for all integers $p\geq 1$ that

$$pT(r,f) \leq pN\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\sum_{i=1}^{p}\overline{N}\left(r,\frac{1}{f^{(k)}-a_i}\right)$$
$$-pN\left(r,\frac{1}{f^{(k)}}\right)+S(r,f).$$

$$(41)\qquad \textit{i.e., } pT(r,f) \leq \sum_{i=1}^{p}\overline{N}\left(r,\frac{1}{f^{(k)}-a_i}\right)+S(r,f).$$

Applying Inequality (41) for $f_1, f_2, ..., f_n$ we obtain that

$$p\{T(r,f_1)+T(r,f_2)+...+T(r,f_n)\}$$
$$\leq \sum_{i=1}^{p}\left\{\overline{N}\left(r,\frac{1}{f_1^{(k)}-a_i}\right)+\overline{N}\left(r,\frac{1}{f_2^{(k)}-a_i}\right)+...+\overline{N}\left(r,\frac{1}{f_n^{(k)}-a_i}\right)\right\}$$
$$+\{S(r,f_1)+S(r,f_2)+...+S(r,f_n)\}$$

$$i.e., p\{T(r,f_1)+T(r,f_2)+...+T(r,f_n)\}$$
$$\leq \sum_{i=1}^{p}\left\{\overline{N}_{1,2,...n}^{(k)}(r,a_i)+n\overline{N}_0^{(k)}(r,a_i)\right\}$$
$$+\{S(r,f_1)+S(r,f_2)+...+S(r,f_n)\}. \tag{42}$$

On dividing both sides of Inequality (42) by $\{I(r,f_1)+I(r,f_2)+...+I(r,f_n)\}$ and taking limit superior we get in view of Lemma 2.5 and Lemma 2.7 that

$$p\cdot\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)}\leq\sum_{i=1}^{p}\left\{1-{}_I\Theta_{1,2,...,n}^{(k)}(a_i)\right\}+n\sum_{i=1}^{p}\left\{1-{}_I\Theta_0^{(k)}(a_i)\right\}$$

$$i.e., p\cdot\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)}\leq p-\sum_{i=1}^{p}{}_I\Theta_{1,2,...,n}^{(k)}(a_i)+np-n\sum_{i=1}^{p}{}_I\Theta_0^{(k)}(a_i)$$

$$i.e., \sum_{i=1}^{p}{}_I\Theta_{1,2,...,n}^{(k)}(a_i)+n\sum_{i=1}^{p}{}_I\Theta_0^{(k)}(a_i)+\frac{pn}{\pi(\rho_1+\rho_2+...+\rho_n)}\leq p(n+1).$$

This completes the proof of the theorem. □

Remark 3.9. *The condition 'a_i be finite, distinct, non-zero complex numbers' in Theorem 3.13 is essential which is evident by considering $n=2$, $f_1=\exp(2z)$, $f_2=\exp(-2z)$, $p=1$ and $a_1=0,\infty$. Then we see that $I(r,f_1)=I(r,f_2)=2r^2\neq 0$ and $\rho_1=\rho_2=1$. Thus,*

$$\sum_{i=1}^{p}{}_I\Theta_{1,2,...,n}^{(k)}(a_i)={}_I\Theta_{1,2}^{(k)}(a_1)={}_I\Theta_{1,2}^{(k)}(0)={}_I\Theta_{1,2}^{(k)}(\infty)=1$$

and

$$\sum_{i=1}^{p}{}_I\Theta_0^{(k)}(a_i)={}_I\Theta_0^{(k)}(a_1)={}_I\Theta_0^{(k)}(0)={}_I\Theta_0^{(k)}(\infty)=1.$$

So,

$$\sum_{i=1}^{p}{}_I\Theta_{1,2,...,n}^{(k)}(a_i)+n\sum_{i=1}^{p}{}_I\Theta_0^{(k)}(a_i)$$
$$+\frac{pn}{\pi(\rho_1+\rho_2+...+\rho_n)}=1+2+\frac{1}{\pi}=3+\frac{1}{\pi}$$

and

$$p(n+1)=3,$$

which contradicts Theorem 3.13.

Theorem 3.14. *If* f_1, f_2, ..., f_n *are* n (>1) *entire functions of non-zero finite order* $'\rho_i'$ *with each* f_i *for* $i = 1, 2, ..., n$ *has no zeros in* $\mathbb{C}$ *then for any non zero finite complex number* $'\alpha'$,

$$_I\Theta_{1,2,...,n}(\infty) +_I \Theta^{(k)}_{1,2,...,n}(a) +_I \delta_{1,2,\cdots,n}(0) + n \cdot \ _I\Theta_0(\infty)$$
$$+n \cdot \ _I\Theta^{(k)}_0(a) + n \cdot \ _I\delta_0(0) + \frac{n}{\pi(\rho_1 + \rho_2 + ... + \rho_n)} \leq 3(1+n).$$

Proof. Let us consider the following identity

$$\frac{a}{f} = \frac{f^{(k)}}{f} - \frac{f^{(k)} - a}{f^{(k+1)}} \cdot \frac{f^{(k+1)}}{f}.$$

In view of Lemma 2.1 and the relation $T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$, we get that

$$m\left(r, \frac{a}{f}\right) \leq m\left(r, \frac{f^{(k)} - a}{f^{(k+1)}}\right) + S(r, f)$$
$$i.e., m\left(r, \frac{a}{f}\right) \leq T\left(r, \frac{f^{(k)} - a}{f^{(k+1)}}\right) - N\left(r, \frac{f^{(k)} - a}{f^{(k+1)}}\right) + S(r, f)$$

(43)

$$i.e., m\left(r, \frac{a}{f}\right) \leq N\left(r, \frac{f^{(k+1)}}{f^{(k)} - a}\right) - N\left(r, \frac{f^{(k)} - a}{f^{(k+1)}}\right) + S(r, f).$$

Adding $N\left(r, \frac{a}{f}\right)$ to both sides of Inequality (43) and in view of [9, p. 34], we obtain from Inequality (43) that

$$T(r, f) \leq \left\{N\left(r, f^{(k+1)}\right) - N\left(r, f^{(k)}\right)\right\}$$
$$+\left\{N\left(r, \frac{1}{f^{(k)} - a}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right)\right\} + N\left(r, \frac{1}{f}\right) + S(r, f)$$

(44) $$i.e., T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f^{(k)} - a}\right) + N\left(r, \frac{1}{f}\right) + S(r, f).$$

Applying Inequality (44) for $f_1, f_2, ..., f_n$ we obtain that

$$\{T(r, f_1) + T(r, f_2) + ... + T(r, f_n)\}$$
$$\leq \left\{\overline{N}(r, f_1) + \overline{N}(r, f_2) + ... + \overline{N}(r, f_n)\right\}$$
$$+\left\{\overline{N}\left(r, \frac{1}{f_1^{(k)} - a}\right) + \overline{N}\left(r, \frac{1}{f_2^{(k)} - a}\right) + ... + \overline{N}\left(r, \frac{1}{f_n^{(k)} - a}\right)\right\}$$

$$\left\{N\left(r,\frac{1}{f_1}\right)+N\left(r,\frac{1}{f_2}\right)+...+N\left(r,\frac{1}{f_n}\right)\right\}$$
$$+\{S(r,f_1)+S(r,f_2)+...+S(r,f_n)\}$$

$$i.e., \{T(r,f_1)+T(r,f_2)+...+T(r,f_n)\} \leq \left\{\overline{N}_{1,2,...,n}(\infty)+n\overline{N}_0(\infty)\right\}$$
$$+\left\{\overline{N}^{(k)}_{1,2,...,n}(a)+n\overline{N}^{(k)}_0(a)\right\}+\{N_{1,2,...,n}(0)+nN_0(0)\}$$
$$+\{S(r,f_1)+S(r,f_2)+...+S(r,f_n)\}. \tag{45}$$

On dividing both sides of Inequality (45) by $\{I(r,f_1)+I(r,f_2)+...+I(r,f_n)\}$ and taking limit superior we get in view of Lemma 2.5 and Lemma 2.7 that

$$\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)} \leq \{1-\ {}_I\Theta_{1,2,...,n}(\infty)\}+n\{1-\ {}_I\Theta_0(\infty)\}$$
$$+\left\{1-\ {}_I\Theta^{(k)}_{1,2,...,n}(a)\right\}$$
$$+\left\{1-\ {}_I\Theta^{(k)}_0(a)\right\}+\{1-\ {}_I\delta_{1,2,...,n}(0)\}+n\{1-\ {}_I\delta_0(0)\}$$

$$i.e., {}_I\Theta_{1,2,...,n}(\infty)+{}_I\Theta^{(k)}_{1,2,...,n}(a)+{}_I\delta_{1,2,\cdots,n}(0)+n\cdot\ {}_I\Theta_0(\infty)$$
$$+n\cdot\ {}_I\Theta^{(k)}_0(a)+n\cdot\ {}_I\delta_0(0)+\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)} \leq 3(1+n).$$

This proves the theorem. □

Remark 3.10. *The condition 'a is any non-zero finite complex number' in Theorem 3.14 is essential which is evident by taking* $n=2$, $f_1=\exp(z^2)$, $f_2=\exp(-z^2)$ *and* $a=0,\infty$. *Then we get that* $I(r,f_1)=\frac{2r^3}{\pi}\cdot I_0(1)\neq 0$, $I(r,f_2)=\frac{2r^3}{\pi}\cdot I_0(1)\neq 0$, *where* $I_n(z)$ *is the Modified Bessel Function of the first kind such that* $I_n(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cdot\cos n\theta d\theta$ *and* $\rho=2$. *Therefore,*

$${}_I\Theta_{1,2}(\infty)=\ {}_I\Theta^{(k)}_{1,2}(0)=\ {}_I\Theta^{(k)}_{1,2}(\infty)=\ {}_I\delta_{1,2}(0)=\ {}_I\delta_{1,2}(\infty)=1$$

and

$${}_I\Theta_0(\infty)=\ {}_I\Theta^{(k)}_0(0)=\ {}_I\delta_0(0)=1.$$

Hence,

$${}_I\Theta_{1,2,...,n}(\infty)+\ {}_I\Theta^{(k)}_{1,2,...,n}(a)+\ {}_I\delta_{1,2,...,n}(0)+n\cdot\ {}_I\Theta_0(\infty)$$
$$+n\cdot\ {}_I\Theta^{(k)}_0(a)+n\cdot\ {}_I\delta_0(0)+\frac{n}{\pi(\rho_1+\rho_2+...+\rho_n)}$$

$$= 1 + 1 + 1 + 2 + 2 + 2 + \frac{1}{2\pi} = 9\frac{1}{2\pi}$$

and

$$3(1+n) = 9,$$

which is contrary to Theorem 3.14.

Theorem 3.15. *Let f_1, f_2, ..., f_n be n (> 1) entire functions of non-zero finite order $'\rho_i'$ with each f_i for $i = 1, 2, ..., n$ having no zeros in $\mathbb{C}$ with $\bar{N}(r, f_1) = S(r, f_1)$, $\bar{N}(r, f_2) = S(r, f_2)$, $\cdots$, $\bar{N}(r, f_n) = S(r, f_n)$ and $'a'$ be a non zero finite complex number. Then for any positive integer k,*

$$\begin{aligned} &{}_I\delta_{1,2,..,n}(0) + {}_I\delta_{1,2,..,n}(a) + n\ {}_I\delta_0(0) \\ &+n\ {}_I\delta_0(a) + \frac{n}{\pi(\rho_1 + \rho_2 + ... + \rho_n)} \leq 2(n+1). \end{aligned}$$

Proof. Let us consider the following identity

$$\frac{a}{f} = 1 - \frac{f-a}{f^{(k)}} \cdot \frac{f^{(k)}}{f}.$$

Since $m\left(r, \frac{1}{f}\right) = m\left(r, \frac{a}{f}\right) + O(1)$, we get from the above identity that

$$m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{f-a}{f^{(k)}}\right) + S(r,f). \tag{46}$$

Now by the relation $T(r,f) = T\left(r, \frac{1}{f}\right) + O(1)$ and Lemma 2.1, it follows from Inequality (46) that

$$m\left(r, \frac{1}{f}\right) \leq T\left(r, \frac{f-a}{f^{(k)}}\right) - N\left(r, \frac{f-a}{f^{(k)}}\right) + S(r,f)$$

$$i.e., m\left(r, \frac{1}{f}\right) \leq T\left(r, \frac{f^{(k)}}{f-a}\right) - N\left(r, \frac{f-a}{f^{(k)}}\right) + S(r,f)$$

$$i.e., m\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{f^{(k)}}{f-a}\right) - N\left(r, \frac{f-a}{f^{(k)}}\right) + S(r,f). \tag{47}$$

Now in view of {p.34, [9]} and as $N\left(r, \frac{1}{f^{(k)}}\right) \geq 0$, it follows from Inequality (47) that

$$m\left(r, \frac{1}{f}\right) \leq N\left(r, f^{(k)}\right) + N\left(r, \frac{1}{f-a}\right) - N(r, f-a) - N\left(r, \frac{1}{f^{(k)}}\right) + S(r,f)$$

$$i.e., m\left(r, \tfrac{1}{f}\right) \le N(r,f) + k\overline{N}(r,f) + N\left(r, \tfrac{1}{f-a}\right)$$
$$-N(r,f) - N\left(r, \tfrac{1}{f^{(k)}}\right) + S(r,f)$$

$$i.e., T(r,f) \le k\overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-a}\right) + S(r,f). \tag{48}$$

Applying Inequality (48) for $f_1, f_2, ..., f_n$ we obtain that

$$\{T(r,f_1) + T(r,f_2) + ... + T(r,f_n)\}$$
$$\le k\left\{\overline{N}(r,f_1) + \overline{N}(r,f_2) + ... + \overline{N}(r,f_n)\right\}$$
$$+\left\{N\left(r,\frac{1}{f_1}\right) + N\left(r,\frac{1}{f_2}\right) + ... + N\left(r,\frac{1}{f_n}\right)\right\}$$
$$+\left\{N\left(r,\frac{1}{f_1-a}\right) + N\left(r,\frac{1}{f_2-a}\right) + ... + N\left(r,\frac{1}{f_n-a}\right)\right\}$$
$$+\{S(r,f_1) + S(r,f_2) + ... + S(r,f_n)\}$$

$$i.e., \{T(r,f_1) + T(r,f_2) + ... + T(r,f_n)\}$$
$$\le k\left\{\overline{N}(r,f_1) + \overline{N}(r,f_2) + ... + \overline{N}(r,f_n)\right\}$$
$$+\{N_{1,2,...,n}(r,0) + nN_0(r,0)\} + \{N_{1,2,...,n}(r,a) + nN_0(r,a)\}$$
$$+\{S(r,f_1) + S(r,f_2) + ... + S(r,f_n)\}. \tag{49}$$

In view of $\bar{N}(r,f_i) = S(r,f_i)$ for $i = 1, 2, ..., n$, dividing both sides of Inequality (49) by $\{I(r,f_1) + I(r,f_2) + ... + I(r,f_n)\}$ and taking limit superior we get in view of Lemma 2.5 and Lemma 2.7 that

$$\frac{n}{\pi(\rho_1 + \rho_2 + ... + \rho_n)} \le \{1 - {}_I\delta_{1,2,..,n}(0)\} + n\{1 - {}_I\delta_0(0)\}$$
$$+\{1 -_I \delta_{1,2,..,n}(a)\} + n\{1 -_I \delta_0(a)\}$$

$$i.e.,_I \delta_{1,2,..,n}(0) + {}_I\delta_{1,2,..,n}(a) + n\,{}_I\delta_0(0) + n\,{}_I\delta_0(a)$$
$$+\frac{n}{\pi(\rho_1 + \rho_2 + ... + \rho_n)} \le 2(n+1).$$

Thus the theorem is established. □

Remark 3.11. *The condition 'a be a finite non-zero complex number' in Theorem 3.15 is necessary which can be shown by considering* $n = 2$, $f_1 = \exp z,\ f_2 = \exp(-z)$ *and* $a = 0, \infty$. *Then we see that* $I(r, f_1) = I(r, f_2) = r^2 \neq 0$ *and* $\rho_1 = \rho_2 = 1$. *So,*

$$\begin{aligned} {}_I\delta_{1,2,\ldots,n}(0) &= {}_I\delta_{1,2}(0) = 1,\ {}_I\delta_{1,2,\ldots,n}(a) = {}_I\delta_{1,2}(0) = {}_I\delta_{1,2}(\infty) = 1, \\ \text{and } {}_I\delta_0(a) &= {}_I\delta_0(0) = 1. \end{aligned}$$

Hence

$$\begin{aligned} &{}_I\delta_{1,2,..,n}(0) + {}_I\delta_{1,2,..,n}(a) + n\ {}_I\delta_0(0) + n\ {}_I\delta_0(a) \\ &+ \frac{n}{\pi(\rho_1 + \rho_2 + \ldots + \rho_n)} = 1 + 1 + 2 + 2 + \frac{1}{\pi} = 6 + \frac{1}{\pi} \end{aligned}$$

and

$$2(n+1) = 6,$$

which contradicts Theorem 3.15.

4. Future Prospect

In the line of the works as carried out in the paper one may think of finding out relative deficiencies of higher index in case of meromorphic functions with respect to another one on the basis of sharing of values of them and this treatment can be done under the flavour of bicomplex analysis. As a consequence, the derivation of relevant results is still open to the future workers of this branch.

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