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$\mathcal{S}_{gf}$-CLOSED SETS IN A FUZZY TOPOLOGICAL SPACE ENDOWED WITH A FUZZY STACK

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Abstract. The main purpose of this article is to introduce and study a concept of a generalized type of fuzzy closed (open) sets in a fuzzy topological space that is endowed with a fuzzy stack. Also several properties of these sets have been studied. Finally by using the aforesaid kind of generalized fuzzy closed sets, we have defined and studied two classes of functions between such spaces, which turn out to be respectively smaller and larger than that of fuzzy continuous functions.

1. Introduction and preliminaries

In 1947, Choquet ([6]) introduced the concept of grill on a point set topological space. In 1981, Azad ([2]) initiated the ideas of fuzzy grills and fuzzy stacks on fuzzy topological spaces and since then these have been powerful tools besides fuzzy nets, fuzzy filters and fuzzy ultrafilters etc. for obtaining useful applications to the study of some fuzzy topological concepts such as fuzzy basic proximity, the theory of convergence, fuzzy compactness, fuzzy almost compactness and other extension problems of different kinds(see [9], [17]). Many other mathematicians (e.g. see ([7], [8], [14])) have enriched different areas of fuzzy topology by use of fuzzy grill.

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In 1970, Levine ([10]) was the first author to introduce the concept of generalized closed sets in topological spaces. In fuzzy setting, Balasubramanian and Sundaram ([3]) studied generalized fuzzy closed (gf-closed, for short) sets, generalizations of fuzzy continuous functions, fuzzy connectedness and fuzzy compactness.

In recent years a huge number of papers is devoted to the ramifications of different generalized approaches. In this paper, we have introduced and investigated a kind of generalized fuzzy closed sets, termed $\mathcal{S}_{gf}$-closed sets, the definition being formulated in terms of a fuzzy stack $\mathcal{S}$ on the underlying space. We investigate some basic properties of such $\mathcal{S}_{gf}$-closed sets and show that the class of $\mathcal{S}_{gf}$-closed sets strictly contains that of gf-closed sets. As applications of the newly introduced class of sets, we introduce a few variant forms of fuzzy continuous functions. Interrelations between these classes of functions and fuzzy continuity are also investigated.

Throughout the paper, (X, τ) and (Y, σ) (or simply by X and Y) shall denote fuzzy topological spaces (fts's) as defined by Chang ([5]), I^X denotes the set of all fuzzy sets on X and I stands for $[0, 1]$. For a fuzzy set U in X, the set $\{x \in X : U(x) > 0\}$ is called the support of U and is denoted by $supp\, U$. By 0_X and 1_X we mean, as usual, the constant fuzzy sets taking on the constant values 0 and 1 on X respectively ([19]).

For two fuzzy sets A and B, we shall write AqB to mean that A is quasi-coincident (q-coincident) with B, i.e., there exists $x \in X$ such that $A(x)+B(x) > 1$ ([12]). The negation of this statement is denoted by $A\bar{q}B$.

A fuzzy singleton or a fuzzy point ([12]) with support x and value α $(0 < \alpha \leq 1)$ is denoted by x_α. We shall write $x_\alpha \leq A$ iff $\alpha \leq A(x)$, and $A \leq B$ iff $A(x) \leq B(x)$, for all x of X.

A fuzzy set U is non-null if $supp\, U \neq \phi$, i.e., $U \neq 0_X$. A fuzzy set A is called a q-neighborhood (shortly, *q-nbd*) of a fuzzy set B ([12]) if there exists a fuzzy open set G in X such that BqG and $G \leq A$; if in addition, A itself is fuzzy open then it is called an open *q-nbd* of B. The collection of all open *q-nbds* of any fuzzy point x_α is denoted by $Q(x_\alpha)$. For a fuzzy set A in an fts X, $cl(A), int(A), 1_X - A$ (or sometimes $1-A$) respectively denote closure, interior and complement of A in X.

Definition 1.1. *([11]) For any two fuzzy sets U and V in an fts X, the Lukasiewicz conjunction $U*V$ is the fuzzy set defined as follows:*

$$(U*V)(x)=\begin{cases} U(x)+V(x)-1, & if \quad U(x)+V(x)>1 \\ 0, & otherwise \end{cases}$$

where $x \in X$.

Definition 1.2. *([2]) A non-empty collection $\mathcal{S}$ of fuzzy sets in an fts (X,τ) is called a fuzzy stack on X if*

(i) $0_X \notin \mathcal{S}$

(ii) for any fuzzy sets A, B in X, $A \in \mathcal{S}$ and $A \leq B \Rightarrow B \in \mathcal{S}$.

If, in addition, the following condition (iii) is satisfied, then $\mathcal{S}$ is called a fuzzy grill on X, where

(iii) $A, B \in I^X$ and $A \vee B \in \mathcal{S} \Rightarrow A \in \mathcal{S}$ or $B \in \mathcal{S}$.

An fts (X,τ) endowed with a fuzzy stack $\mathcal{S}$ will be called an sfts or a fuzzy $\mathcal{S}$-space, to be denoted by $(X,\tau,\mathcal{S})$.

Definition 1.3. *([15]) For an sfts $(X,\tau,\mathcal{S})$, an operator $\Phi_{\mathcal{S}}: I^X \to I^X$ is defined by $\Phi_{\mathcal{S}}(A)=\vee\{x_\alpha : A*U \in \mathcal{S}, \ \forall\, U \in Q(x_\alpha)\}$.*

Theorem 1.4. *([15]) In an sfts $(X,\tau,\mathcal{S})$, the following properties hold:*

(i) $\Phi_{\mathcal{S}}(0_X)=0_X$, $\Phi_{\mathcal{S}}(1_X)=1_X$.

(ii) For any $U,V \in I^X$, $U \leq V \Rightarrow \Phi_{\mathcal{S}}(U) \leq \Phi_{\mathcal{S}}(V)$.

(iii) If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two fuzzy stacks on an fts (X,τ) with $\mathcal{S}_1 \leq \mathcal{S}_2$, then $\Phi_{\mathcal{S}_1}(U) \leq \Phi_{\mathcal{S}_2}(U)$, for any $U \in I^X$.

(iv) $\Phi_{\mathcal{S}}(U) \leq cl(U)$, for any $U \in I^X$.

(v) $\Phi_{\mathcal{S}}(U)$ is closed, for any $U \in I^X$.

(vi) $\Phi_{\mathcal{S}}(\Phi_{\mathcal{S}}(U)) \leq \Phi_{\mathcal{S}}(U)$ for any $U \in I^X$.

(vii) For any $U \in I^X$, $U \notin \mathcal{S} \Rightarrow \Phi_{\mathcal{S}}(U)=0_X$.

Theorem 1.5. *([15]) For any fuzzy set A in an sfts $(X,\tau,\mathcal{S})$, $\Phi_{\mathcal{S}}(A \vee \Phi_{\mathcal{S}}(A))=\Phi_{\mathcal{S}}(A)$.*

Definition 1.6. *([15]) For an sfts $(X,\tau,\mathcal{S})$, an operator $\Psi_{\mathcal{S}}: I^X \to I^X$ is defined by $\Psi_{\mathcal{S}}(A)=A \vee \Phi_{\mathcal{S}}(A)$ for $A \in I^X$.*

Definition 1.7. *([15]) Let $\mathcal{S}$ be a fuzzy stack on an fts X. We define a collection $\tau_{\mathcal{S}}$ of fuzzy sets given by $\tau_{\mathcal{S}}=\{U \in I^X : \Psi_{\mathcal{S}}(1_X-U)=1_X-U\}$.*

Definition 1.8. *([16]) Let X be a non empty set and τ be a collection of fuzzy sets on X (i.e. $\tau \leq I^X$). τ is called a generalized fuzzy topology on X, if the following conditions are satisfied*

(i) $0_X \in \tau$,

(ii) *If* $U_\alpha \in \tau$ *for* $\alpha \in J$, *then* $\bigvee_{\alpha \in J} U_\alpha \in \tau$.

If, in addition, $1_X \in \tau$, then τ is called a strong generalized topology on X.

Theorem 1.9. *([15]) In a fuzzy $\mathcal{S}$-space $(X, \tau, \mathcal{S})$, $\tau_{\mathcal{S}}$ forms a strong generalized topology on X.*

Definition 1.10. *([15]) For any fuzzy set A in X, $\tau_{\mathcal{S}}$-$int(A) = \vee\{U \leq I^X : U \leq A, U \in \tau_{\mathcal{S}}\}$, and*

$$\tau_{\mathcal{S}}\text{-}cl(A) = \wedge\{F \leq I^X : A \leq F, 1_X - F \in \tau_{\mathcal{S}}\}.$$

Theorem 1.11. *([15]) Let $\mathcal{S}$ be a fuzzy stack on an fts X. Then the following hold:*

(i) $\tau_{\mathcal{S}}$-$cl(A) = \Psi_{\mathcal{S}}(A)$, *for* $A \in I^X$.

(ii) *If* $A \notin \mathcal{S}$, *then* $1_X - A \in \tau_{\mathcal{S}}$, *for* $A \in I^X$.

(iii) $\Phi_{\mathcal{S}}(A)$ *is* $\tau_{\mathcal{S}}$-*closed, for any* $A \in I^X$.

(iv) $\tau_{\mathcal{S}}$-$cl(1_X - A) = 1_X - \tau_{\mathcal{S}}$-$int(A)$, *for any* $A \in I^X$.

Theorem 1.12. *([15]) Let $\mathcal{S}$ be a fuzzy stack on an fts (X, τ). Then $\tau \leq \tau_{\mathcal{S}}$.*

2. $\mathcal{S}_{gf}$-closed and $\mathcal{S}_{gf}$-open sets in fuzzy $\mathcal{S}$-spaces

As already proposed, we introduce in this section, the concepts of $\mathcal{S}_{gf}$-closed and $\mathcal{S}_{gf}$-open sets in a fuzzy $\mathcal{S}$-space and study some of their basic properties. Our definition is patterned after the following one, recalled from ([3]).

Definition 2.1. *A fuzzy set U in an fts (X, τ) is called generalized fuzzy closed (in short, gf-closed) if $cl(U) \leq G$ whenever $U \leq G$ and G is a fuzzy open set.*

A fuzzy set U is called generalized fuzzy open (in short, gf-open) if its complement $(1_X - U)$ is gf-closed.

We now propose the definition of generalized fuzzy closed sets in an sfts as follows.

Definition 2.2. *Let $(X, \tau, \mathcal{S})$ be an sfts. Then a fuzzy set $U \in I^X$ is called a generalized fuzzy closed set with respect to a fuzzy stack $\mathcal{S}$ (in short, $\mathcal{S}_{gf}$-closed) if $\Phi_{\mathcal{S}}(U) \leq G$ whenever $U \leq G$ and G is a fuzzy open set with respect to τ.*

U is called a generalized fuzzy open set (in short, $\mathcal{S}_{gf}$-open) if its complement $(1_X - U)$ is $\mathcal{S}_{gf}$-closed.

Remark 2.3. *In ([3]) it is observed that every fuzzy closed (fuzzy open) set is a gf-closed (gf-open) set. Now, in a fuzzy $\mathcal{S}$-space $(X, \tau, \mathcal{S})$, we see that*

(i) if $A \notin \mathcal{S}$, then $\Phi_{\mathcal{S}}(A) = 0_X$ [by Theorem 1.4.(vii)], and hence every non-member of $\mathcal{S}$ is an $\mathcal{S}_{gf}$-closed set in $(X, \tau, \mathcal{S})$.

(ii) for any fuzzy set $U \in I^X$, $\Phi_{\mathcal{S}}(\Phi_{\mathcal{S}}(U)) \leq \Phi_{\mathcal{S}}(U)$ [by Theorem 1.4.(vi)], so that $\Phi_{\mathcal{S}}(U)$ is an $\mathcal{S}_{gf}$-closed set.

(iii) in view of Theorem 1.10. a fuzzy set A is $\mathcal{S}_{gf}$-closed if and only if ($A \leq G \in \tau \Rightarrow \tau_{\mathcal{S}}$-$cl(A) = \Psi_{\mathcal{S}}(A) = A \vee \Phi_{\mathcal{S}}(A) \leq G$).

(iv) every fuzzy closed (resp. fuzzy open) as well as every gf-closed [resp. gf-open] set is $\mathcal{S}_{gf}$-closed (resp. $\mathcal{S}_{gf}$-open) but converses are not be true in general. In fact, in [3] it is shown that a gf-closed set need not be fuzzy closed. We now show below that an $\mathcal{S}_{gf}$-closed set may be neither gf-closed nor fuzzy closed.

Example 2.4. *Let $(X, \tau, \mathcal{S})$ be an sfts, where X is an infinite set, $\tau = \{U \in I^X : 0 \leq U(x) \leq 1/3,\ x \in X\}$ together with 1_X, and $\mathcal{S} = \{W \in I^X : 0.6 \leq W(x) \leq 1,\ x \in X\}$. Let x_0 be a chosen point of X and A be the fuzzy set in X given by $A(x_0) = 0.3$ and $A(x) = 0$ for $x \in X \setminus \{x_0\}$. Then A is not gf- closed, since for the fuzzy open set U where $U(x) = 1/3, \forall x \in X$, we have $A \leq U$, $(clA)(x) = 2/3, \forall x \in X$ and hence $cl(A) \not\leq U$. We check that A is $\mathcal{S}_{gf}$-closed. To avoid triviality, consider the case when $A \leq U(\neq 1_X) \in \tau$. Here for any fuzzy point x_α in X, 1_X is the only open q-nbd of x_α and we have $A * 1_X \notin \mathcal{S}$. Thus $\Phi_{\mathcal{S}}(A) = 0_X \leq U$. Therefore A is $\mathcal{S}_{gf}$-closed.*

Theorem 2.5. *Let $(X, \tau, \mathcal{S})$ be a fuzzy $\mathcal{S}$-space. A fuzzy set P in X is $\mathcal{S}_{gf}$-open if and only if $W \leq \tau_{\mathcal{S}}$-$int(P)$, whenever W is fuzzy closed and $W \leq P$.*

Proof. Let P be an $\mathcal{S}_{gf}$-open set and W be a fuzzy closed set such that $W \leq P$. Then $1_X - W$ is fuzzy open and $1_X - P \leq 1_X - W$. Since

$(1_X - P)$ is $\mathcal{S}_{gf}$-closed, $\Phi_{\mathcal{S}}(1_X - P) \leq 1_X - W$, i.e., $\tau_{\mathcal{S}}$-$cl(1_X - P) = 1_X - \tau_{\mathcal{S}}$-$int(P) \leq 1_X - W$. Thus $W \leq \tau_{\mathcal{S}}$-$int(P)$.

Conversely, suppose that P is a fuzzy set such that $W \leq \tau_{\mathcal{S}}$-$int(P)$, whenever W is fuzzy closed and $W \leq P$. We have to prove that $(1_X - P)$ is an $\mathcal{S}_{gf}$-closed set. So let $1_X - P \leq V$, where V is fuzzy open. Now since $1_X - P \leq V$, $1_X - V \leq P$. Hence by assumption we have $1_X - V \leq \tau_{\mathcal{S}}$-$int(P)$, i.e., $1_X - \tau_{\mathcal{S}}$-$int(P) \leq V$. Hence $\Phi_{\mathcal{S}}(1_X - P) \leq \tau_{\mathcal{S}}$-$cl(1_X - P) \leq V$, which implies that $1_X - P$ is $\mathcal{S}_{gf}$-closed. Thus P is $\mathcal{S}_{gf}$-open. □

Theorem 2.6. *Let $(X, \tau, \mathcal{S})$ be a fuzzy $\mathcal{S}$-space. Then the following are equivalent:*

(i) every fuzzy set in X is $\mathcal{S}_{gf}$-closed.
(ii) every fuzzy open set is $\tau_{\mathcal{S}}$-closed.

Proof. $(i) \Rightarrow (ii)$: Let U be an open set in $(X, \tau, \mathcal{S})$. Then by hypothesis, U is $\mathcal{S}_{gf}$-closed so that $\Phi_{\mathcal{S}}(U) \leq U$ and hence U is $\tau_{\mathcal{S}}$-closed.

$(ii) \Rightarrow (i)$: Let A be any fuzzy set in $(X, \tau, \mathcal{S})$ with $A \leq U$, where U is a fuzzy open set. Then by (ii), $\Phi_{\mathcal{S}}(A) \leq \Phi_{\mathcal{S}}(U) \leq U$ [as U is $\tau_{\mathcal{S}}$-closed]. Thus A is $\mathcal{S}_{gf}$-closed. □

Theorem 2.7. *Let $(X, \tau, \mathcal{S})$ be a fuzzy $\mathcal{S}$-space, where $\mathcal{S}$ is so taken that $\tau \wedge \mathcal{S} = \{1_X\}$. Then each $B(\neq 1_X) \in \tau$ is an $\mathcal{S}_{gf}$-closed set.*

Proof. Given that $B(\neq 1_X)$ is fuzzy open in $(X, \tau, \mathcal{S})$. Then by hypothesis, $B \notin \mathcal{S} \Rightarrow B$ is $\tau_{\mathcal{S}}$-closed [by Theorem 1.11.] $\Rightarrow \Phi_{\mathcal{S}}(B) \leq B$. Then for any fuzzy open set U with $B \leq U$ we have $\Phi_{\mathcal{S}}(B) \leq U$. Thus B is $\mathcal{S}_{gf}$-closed. □

Remark 2.8. *The union of two $\mathcal{S}_{gf}$-closed sets may not be $\mathcal{S}_{gf}$-closed, as is shown by the example given below.*

Example 2.9. *Let $X = \{a, b\}$ and $\tau = \{0_X, 1_X, U, V\}$, where $U(a) = 0.3$ and $U(b) = V(a) = V(b) = 0.6$, be a fuzzy topology on X. Let $\mathcal{S} = \{G \in I^X : 0.3 \leq G(a) \leq 1 \text{ and } 0.2 \leq \text{G(b)} \leq 1\}$ be a fuzzy stack on X. Suppose C and D are two fuzzy sets in X such that $C(a) = 0.3, C(b) = 0$ and $D(a) = 0, D(b) = 0.4$. We can easily check that $\Phi_{\mathcal{S}}(C) = \Phi_{\mathcal{S}}(D) = 0_X$, whereas $\Phi_{\mathcal{S}}(C \vee D) = E$, where $E(a) = 0.4, E(b) = 0.4$. Thus C and D are two $\mathcal{S}_{gf}$-closed sets, whereas $C \vee D$ is not $\mathcal{S}_{gf}$-closed because $C \vee D \leq U$ but $\Phi_{\mathcal{S}}(C \vee D) \not\leq U$.*
Hence union of two $\mathcal{S}_{gf}$-closed sets may not be $\mathcal{S}_{gf}$-closed.

Remark 2.10. *Intersection of two $\mathcal{S}_{gf}$-closed sets need not be $\mathcal{S}_{gf}$-closed, as is shown by the following example.*

Example 2.11. *Let $X = \{a, b\}$ and $\tau = \{0_X, 1_X, A\}$, where $A(a) = 0.7, A(b) = 0.8$, be a fuzzy topology on X. Let $\mathcal{S} = \{G \in I^X : 0.2 \leq G(x) \leq 1,\ \forall\, x \in X\}$ be a fuzzy stack on X. Let U and V be two fuzzy sets in X defined as $U(a) = 0.8, U(b) = 0.5$ and $V(a) = 0.6, V(b) = 0.9$. Then clearly U and V are $\mathcal{S}_{gf}$-closed sets. But $U \wedge V$ is not $\mathcal{S}_{gf}$-closed because $U \wedge V \leq A$, but $\Phi_{\mathcal{S}}(U \wedge V) \not\leq A$. Indeed, $a_{0.8} \not\leq A$ but for any $W \in Q(a_{0.8})$, $(U \wedge V) * W \in \mathcal{S}$, so that $a_{0.8} \leq \Phi_{\mathcal{S}}(U \wedge V)$.*

Note 2.12. *It follows from the above examples that the union and intersection of two $\mathcal{S}_{gf}$-open sets are not necessarily $\mathcal{S}_{gf}$-open sets.*

Our next goal is to ascertain as to under what condition the union of two $\mathcal{S}_{gf}$-open sets may be an $\mathcal{S}_{gf}$-open. We answer it in the following theorem:

Theorem 2.13. *Let P and Q be two $\mathcal{S}_{gf}$-open sets in $(X, \tau, \mathcal{S})$ with $P \wedge cl(Q) = Q \wedge cl(P) = 0_X$, then $P \vee Q$ is $\mathcal{S}_{gf}$-open.*

Proof. Let F be a fuzzy closed set with $F \leq P \vee Q$. Since $Q \wedge cl(P) = 0_X$, then $F \wedge cl(P) \leq P$. Now P is $\mathcal{S}_{gf}$-open and $F \wedge cl(P)$ is a fuzzy closed set. Then by Theorem 2.5, we have $F \wedge cl(P) \leq \tau_{\mathcal{S}}\text{-}int(P)$. Similarly, we get $F \wedge cl(Q) \leq \tau_{\mathcal{S}}\text{-}int(Q)$.

Now $F = F \wedge (P \vee Q) \leq (F \wedge cl(P)) \vee (F \wedge cl(Q) \leq \tau_{\mathcal{S}}\text{-}int(P) \vee \tau_{\mathcal{S}}\text{-}int(Q) \leq \tau_{\mathcal{S}}\text{-}int(P \vee Q)$. Hence $F \leq \tau_{\mathcal{S}}\text{-}int(P \vee Q)$ and by Theorem 2.5, $P \vee Q$ is $\mathcal{S}_{gf}$-open. □

Definition 2.14. *A fuzzy set P in a fuzzy $\mathcal{S}$-space X is called $\tau_{\mathcal{S}}$-dense-in-itself if $P \leq \Phi_{\mathcal{S}}(P)$.*

Theorem 2.15. *Let $(X, \tau, \mathcal{S})$ be a fuzzy $\mathcal{S}$-space. If $B \in I^X$ is $\tau_{\mathcal{S}}$-dense-in-itself and $\mathcal{S}_{gf}$-closed, then B is gf-closed.*

Proof. Let B be $\tau_{\mathcal{S}}$-dense-in-itself and $\mathcal{S}_{gf}$-closed in $(X, \tau, \mathcal{S})$ and W be any fuzzy open set in X such that $B \leq W$. Then $B \leq \Phi_{\mathcal{S}}(B)$ and $\Phi_{\mathcal{S}}(B) \leq W$. By Theorem 1.4., $cl(B) \leq cl(\Phi_{\mathcal{S}}(B)) = \Phi_{\mathcal{S}}(B) \leq W$. Thus B is gf-closed. □

Theorem 2.16. *Let $(X, \tau, \mathcal{S})$ be a fuzzy $\mathcal{S}$-space and $P, R \in I^X$ such that $P \leq R \leq \tau_{\mathcal{S}}\text{-}cl(P)$. If P is $\mathcal{S}_{gf}$-closed, then R is also so.*

Proof. Suppose $R \le W$, where W is a fuzzy open set. As $P \le R$ and P is $\mathcal{S}_{gf}$-closed, $\Phi_{\mathcal{S}}(P) \le W \Rightarrow \tau_{\mathcal{S}}\text{-}cl(P) \le W$. Now, $P \le R \le \tau_{\mathcal{S}}\text{-}cl(P) \Rightarrow \tau_{\mathcal{S}}\text{-}cl(P) \le \tau_{\mathcal{S}}\text{-}cl(R) \le \tau_{\mathcal{S}}\text{-}cl(\tau_{\mathcal{S}}\text{-}cl(P)) \le \tau_{\mathcal{S}}\text{-}cl(P)$. Thus $\tau_{\mathcal{S}}\text{-}cl(R) \le W$ and hence R is $\mathcal{S}_{gf}$-closed. □

Corollary 2.17. *$\tau_{\mathcal{S}}$-closure of every $\mathcal{S}_{gf}$-closed set is $\mathcal{S}_{gf}$-closed.*

Corollary 2.18. *Let $(X, \tau, \mathcal{S})$ be a fuzzy $\mathcal{S}$-space and $P, R \in I^X$ such that $P \le R \le \Phi_{\mathcal{S}}(P)$. If P is $\mathcal{S}_{gf}$-closed, then R is also so.*

Proof. Follows from the above theorem and the fact that $\Phi_{\mathcal{S}}(P) \le \tau_{\mathcal{S}}\text{-}cl(P)$, for any $P \in I^X$. □

An improved version of the above corollary is given as follows.

Theorem 2.19. *Let $(X, \tau, \mathcal{S})$ be a fuzzy $\mathcal{S}$-space and $P, R \in I^X$ such that $P \le R \le \Phi_{\mathcal{S}}(P)$. If P is $\mathcal{S}_{gf}$-closed, then P and R are gf-closed.*

Proof. If $P \le R \le \Phi_{\mathcal{S}}(P)$ and P is $\mathcal{S}_{gf}$-closed, then by Corollary 2.18., R is $\mathcal{S}_{gf}$-closed. Now, $P \le R \le \Phi_{\mathcal{S}}(P) \Rightarrow \Phi_{\mathcal{S}}(P) \le \Phi_{\mathcal{S}}(R) \le \Phi_{\mathcal{S}}(\Phi_{\mathcal{S}}(P)) \le \Phi_{\mathcal{S}}(P)$ (by Theorem 1.4.), i.e., $\Phi_{\mathcal{S}}(P) = \Phi_{\mathcal{S}}(R)$. Thus P and R are $\tau_{\mathcal{S}}$-dense-in-itself and hence by Theorem 2.15., P and R are gf-closed. □

A dual version of Theorem 2.16 is given below, the proof which follows from the stated theorem and Theorem 1.11.(iv); corresponding dual versions of other results above can similarly be obtained.

Theorem 2.20. *Let $(X, \tau, \mathcal{S})$ be a fuzzy $\mathcal{S}$-space. If $\tau_{\mathcal{S}}\text{-}int(U) \le V \le U$ and U is $\mathcal{S}_{gf}$-open, then V is $\mathcal{S}_{gf}$-open.*

3. $\mathcal{S}_{gf}$-continuity, $\mathcal{S}_{gf}$-irresolute and $\mathcal{S}_{gf}$-strong fuzzy continuity of functions for fuzzy $\mathcal{S}$-spaces

It is found from literature that different related forms of fuzzy continuous function(between fts's) have been studied by many researchers so far. Our aim in this section is to replicate some such functions for fuzzy $\mathcal{S}$-spaces; interrelations among the introduced types of functions and those with regard to fuzzy continuity are also investigated.

Definition 3.1. *([5]) A map $f : X \to Y$ is called fuzzy continuous if the inverse image of every fuzzy open set in Y is fuzzy open in X.*

Definition 3.2. *([3]) A map $f : X \to Y$ is called generalized fuzzy continuous (gf-continuous, for short) if the inverse image of every fuzzy closed set in Y is gf-closed in X.*

In a similar way as above, we now define:

Definition 3.3. *A map $f : (X, \tau, \mathcal{S}) \to (Y, \sigma)$ is called generalized fuzzy continuous with respect to some fuzzy stack $\mathcal{S}$ on X ($\mathcal{S}_{gf}$-continuous, for short) if the inverse image of every fuzzy closed set in Y is $\mathcal{S}_{gf}$-closed in $(X, \tau, \mathcal{S})$.*

Remark 3.4. *It is shown in ([3]) that every fuzzy continuous function is gf-continuous but not conversely. We now see that every gf-continuous function is clearly $\mathcal{S}_{gf}$-continuous; however the converse may not be true as we show by the following example.*

Example 3.5. *Let $X = Y = \{r, s, t\}$; $\tau = \{0_X, 1_X, A\}$ and $\sigma = \{0_X, 1_X, B\}$, where $A(r) = 0.5, A(s) = 0.4, A(t) = 0.7$ and $B(r) = 0.6, B(s) = 0.8, B(t) = 0.7$. Consider the function $f : (X, \tau) \to (Y, \sigma)$ given by $f(r) = t, f(s) = f(t) = r$. Now, $f^{-1}(B) \notin \tau$ and hence f is not fuzzy continuous. Consider the fuzzy stack $\mathcal{S} = \{G \in I^X : 0.5 \leq G(x) \leq 1,\ \forall x \in X\}$ on X. Now, $f^{-1}(1_X - B) \leq A$ and $f^{-1}(1_X - B) \notin \mathcal{S}$; hence $\Phi_{\mathcal{S}}(f^{-1}(1_X - B)) = 0_X \leq A$. Therefore $f : (X, \tau, \mathcal{S}) \to (Y, \sigma)$ is $\mathcal{S}_{gf}$-continuous.*
Again, we can easily check that $f^{-1}(1_X - B) \leq A$, but $cl(f^{-1}(1_X - B)) = 1_X \nleq A$. Thus f is not gf-continuous on X, although f is $\mathcal{S}_{gf}$-continuous.

Definition 3.6. *A map $f : (X, \tau, \mathcal{S}_1) \to (Y, \sigma, \mathcal{S}_2)$ is called $\mathcal{S}_{gf}$-irresolute if the inverse image of every $\mathcal{S}_{gf}$-closed set in $(Y, \sigma, \mathcal{S}_2)$ is $\mathcal{S}_{gf}$-closed in $(X, \tau, \mathcal{S}_1)$.*

Theorem 3.7. *If $f : (X, \tau, \mathcal{S}_1) \to (Y, \sigma, \mathcal{S}_2)$ is $\mathcal{S}_{gf}$-irresolute, then it is $\mathcal{S}_{gf}$-continuous.*

Remark 3.8. *That the converse of the above theorem is false is shown by the following example.*

Example 3.9. *Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ and $\sigma = \{0_X, 1_X, B\}$, where $A(a) = 0.5, A(b) = 0.4$ and $B(a) = 0.6, B(b) = 0.8$, be two fuzzy topologies on X. Let $\mathcal{S}_1 = \{G_1 \in I^X : 0.2 < G_1(x) \leq 1,\ \forall x \in X\}$ and $\mathcal{S}_2 = \{G_2 \in I^X : 0.7 < G_2(x) \leq 1,\ \forall x \in X\}$ be two fuzzy stacks on X.*

Let $f : (X, \tau, \mathcal{S}_1) \to (X, \sigma, \mathcal{S}_2)$ *be the identity function. Then* f *is* $\mathcal{S}_{gf}$*-continuous, since* $f^{-1}(1_X - B)$ *is* $\mathcal{S}_{gf}$*-closed in* $(X, \tau, \mathcal{S}_1)$ *[by Remark 2.3.(i)]. Let us take a fuzzy set* U *in* $(X, \sigma, \mathcal{S}_2)$ *such that* $U(a) = 0.3, U(b) = 0.4$*. Then* U *is* $\mathcal{S}_{gf}$*-closed in* $(X, \sigma, \mathcal{S}_2)$ *[by Remark 2.3.(i)]. Now* $f^{-1}(U) = U$*,* $U \leq A$ *but* $\Phi_{\mathcal{S}}(U) \not\leq A$*; in fact, the only open q-neighbourhood of the fuzzy point* $b_{0.6}$ *in* $(X, \tau, \mathcal{S}_1)$ *is* 1_X *and* $U * 1_X \in \mathcal{S}_1$*, so that* $b_{0.6} \leq \Phi_{\mathcal{S}}(U)$*, but* $b_{0.6} \not\leq A$*. Thus* $f^{-1}(U)$ *is not* $\mathcal{S}_{gf}$*-closed in* $(X, \tau, \mathcal{S}_1)$*, showing that* f *is not* $\mathcal{S}_{gf}$*-irresolute.*

Proposition 3.10. *Let* $f : (X, \tau_1, \mathcal{S}_1) \to (Y, \sigma, \mathcal{S}_2)$ *and* $g : (Y, \sigma, \mathcal{S}_2) \to (Z, \tau, \mathcal{S}_3)$ *be two functions. If* f *is* $\mathcal{S}_{gf}$*-irresolute and* g *is* $\mathcal{S}_{gf}$*-continuous, then the composition* $g \circ f : (X, \tau_1, \mathcal{S}_1) \to (Z, \tau, \mathcal{S}_3)$ *is also* $\mathcal{S}_{gf}$*-continuous.*

Proof. Straightforward. □

Theorem 3.11. *Let* $f : (X, \tau_1, \mathcal{S}_1) \to (Y, \sigma, \mathcal{S}_2)$ *and* $g : (Y, \sigma, \mathcal{S}_2) \to (Z, \tau, \mathcal{S}_3)$ *be two functions. If* g *is* $\mathcal{S}_{gf}$*-continuous and* f *is fuzzy continuous, then the composition* $g \circ f : (X, \tau_1, \mathcal{S}_1) \to (Z, \tau, \mathcal{S}_3)$ *is also* $\mathcal{S}_{gf}$*-continuous.*

Definition 3.12. *Let* $f : (X, \tau, \mathcal{S}_1) \to (Y, \sigma, \mathcal{S}_2)$ *be a function. Then* f *is said to be an*

(i) $\mathcal{S}_{gf}$*-closed function if* $f(P)$ *is* $\mathcal{S}_{gf}$*-closed in* Y*, for each fuzzy closed set* P *in* X*.*
(ii) $\mathcal{S}_{gf}$*-open function if* $f(P)$ *is* $\mathcal{S}_{gf}$*-open in* Y*, for each fuzzy open set* P *in* X*.*

Remark 3.13. *Every fuzzy closed (open) function is* $\mathcal{S}_{gf}$*-closed (*$\mathcal{S}_{gf}$*-open) but converse may not be true, as shown by the following example.*

Example 3.14. *Let* $X = \{x_1, x_2\}$*,* $\tau = \{0_X, 1_X, A\}$ *and* $\sigma = \{0_X, 1_X, B\}$*, where* $A(x_1) = 0.3, A(x_2) = 0.4$ *and* $B(x_1) = 0.2, B(x_2) = 0.5$*, be two fuzzy topologies on* X*. Let* $\mathcal{S}_1 = \{G_1 \in I^X : 0.2 < G_1(x) \leq 1,\ \forall x \in X\}$ *and* $\mathcal{S}_2 = \{G_2 \in I^X : 0.4 < G_2(x) \leq 1,\ \forall x \in X\}$ *be two fuzzy stacks on* X*. Clearly* $(X, \tau, \mathcal{S}_1)$ *and* $(X, \sigma, \mathcal{S}_2)$ *are two fuzzy* $\mathcal{S}$*-spaces.*
Let $f : (X, \tau, \mathcal{S}_1) \to (X, \sigma, \mathcal{S}_2)$ *be the identity function. Then* $f(1_X - A)$ *is not a fuzzy closed set in* $(X, \sigma, \mathcal{S}_2)$*. Thus* f *is not a fuzzy closed function. But* $f(1_X - A)$ *is* $\mathcal{S}_{gf}$*-closed in* $(X, \sigma, \mathcal{S}_2)$*(as* 1_X *is the only fuzzy open set in* $(X, \sigma, \mathcal{S}_2)$ *containing* $f(1_X - A)$*). Therefore* f *is an* $\mathcal{S}_{gf}$*-closed function.*

Theorem 3.15. *A map* $f:(X,\tau,\mathcal{S}_1)\to(Y,\sigma,\mathcal{S}_2)$ *is an* $\mathcal{S}_{gf}$*-closed function iff for each fuzzy set* P *in* Y *and each fuzzy open set* Q *in* X *with* $f^{-1}(P)\leq Q$*, there is an* $\mathcal{S}_{gf}$*-open set* R *in* Y *such that* $P\leq R$ *and* $f^{-1}(R)\leq Q$.

Proof. Let us consider that f is an $\mathcal{S}_{gf}$-closed function. Let P be a fuzzy set in Y and Q be a fuzzy open set in X such that $f^{-1}(P)\leq Q$. Take $R=1_Y-f(1_X-Q)$. Since f is an $\mathcal{S}_{gf}$-closed function, $f(1_X-Q)$ is an $\mathcal{S}_{gf}$-closed set in Y and hence R is an $\mathcal{S}_{gf}$-open set in Y. We first prove that $P\leq R$. Indeed, $f^{-1}(P)\leq Q\Rightarrow 1_X-Q\leq 1_X-f^{-1}(P)=f^{-1}(1_Y-P)\Rightarrow f(1_X-Q)\leq ff^{-1}(1_Y-P)\leq(1_Y-P)\Rightarrow P\leq 1_Y-f(1_X-Q)=R$.
Next we have to prove that $f^{-1}(R)\leq Q$. Now, $R=1_Y-f(1_X-Q)\Rightarrow f(1_X-Q)=1_Y-R\Rightarrow(1_X-Q)\leq f^{-1}(1_Y-R)=1_X-f^{-1}(R)\Rightarrow f^{-1}(R)\leq Q$.

Conversely, let A is a fuzzy closed set in X. Now $A\leq f^{-1}f(A)\Rightarrow 1_X-A\geq 1_X-f^{-1}f(A)=f^{-1}(1_Y-f(A))$ and (1_X-A) is a fuzzy open set in X. By hypothesis, there is an $\mathcal{S}_{gf}$-open set R in Y such that $1_Y-f(A)\leq R$........(i)
and $f^{-1}(R)\leq 1_X-A\Rightarrow A\leq 1_X-f^{-1}(R)=f^{-1}(1_Y-R)\Rightarrow f(A)\leq ff^{-1}(1_Y-R)\leq 1_Y-R$....(2)
From (1) and (2), $f(A)=1_Y-R$ and (1_Y-R) is an $\mathcal{S}_{gf}$-closed set. Thus f is an $\mathcal{S}_{gf}$-closed function. □

We now define a stronger form of fuzzy continuous function from an fts to a fuzzy $\mathcal{S}$-space.

Definition 3.16. *A map* $f:(X,\tau)\to(Y,\sigma,\mathcal{S})$ *is called* $\mathcal{S}_{gf}$*-strongly fuzzy continuous if the inverse image of every* $\mathcal{S}_{gf}$*-closed set in* $(Y,\sigma,\mathcal{S})$ *is fuzzy closed in* (X,τ).

Theorem 3.17. *If* $f:(X,\tau)\to(Y,\sigma,\mathcal{S})$ *is* $\mathcal{S}_{gf}$*-strongly fuzzy continuous, then* f *is fuzzy continuous.*

Proof. The proof follows from the fact that every fuzzy closed set in a fuzzy $\mathcal{S}$-space is $\mathcal{S}_{gf}$-closed. □

Remark 3.18. *The converse of the above theorem is not true in general as is seen from the example below.*

Example 3.19. *Let* $X=\{a,b\}$, $Y=\{p,r\}$. *Consider* $\tau=\{0_X,1_X,A\}$, *where* $A(a)=0.8, A(b)=0.6$. *Then* τ *is a fuzzy topology*

on X. Again, $\sigma = \{0_Y, 1_Y, B\}$, where $B(p) = 0.6, B(r) = 0.8$, is a fuzzy topology on Y. Let $\mathcal{S} = \{S \in I^Y : 0.7 < S(y) \leq 1,\ \forall y \in Y\}$ be a fuzzy stack on Y.
Consider the function $f : (X, \tau) \to (Y, \sigma, \mathcal{S})$ given by $f(a) = r, f(b) = p$. Then f is clearly fuzzy continuous (as $f^{-1}(B) = A$), but f is not $\mathcal{S}_{gf}$-strongly fuzzy continuous because if we take the fuzzy set U in $(Y, \sigma, \mathcal{S})$ such that $U(p) = 0.3, U(r) = 0.4$, then U is $\mathcal{S}_{gf}$-closed in $(Y, \sigma, \mathcal{S})$ [by Remark 2.3.(i)] but $f^{-1}(U)$ is not fuzzy closed in (X, τ).

From what has gone so far in this section, we have the following implications:

$$\mathcal{S}_{gf}\text{-strong fuzzy continuity} \Rightarrow \text{fuzzy continuity} \Rightarrow \\ \Rightarrow gf\text{-continuity} \Rightarrow \mathcal{S}_{gf}\text{-continuity}$$

where none of the implications is reversible, in general.

We now wish to look for conditions which will ensure the above converses to hold. In this connection we recall from ([3]) that a fuzzy topological space (X, τ) is said to be fuzzy $\tau_{1/2}$ if every generalized fuzzy closed set in X is fuzzy closed in X. Analogously we define:

Definition 3.20. *A fuzzy $\mathcal{S}$-space $(X, \tau, \mathcal{S})$ is called a fuzzy $\mathcal{S}\tau_{1/2}$-space if every $\mathcal{S}_{gf}$-closed set in $(X, \tau, \mathcal{S})$ is fuzzy closed.*

Remark 3.21. *Every fuzzy $\mathcal{S}\tau_{1/2}$-space is clearly fuzzy $\tau_{1/2}$ but the converse need not be true, as is shown by the following example.*

Example 3.22. *Let us consider the fuzzy $\mathcal{S}$-space $(X, \tau, \mathcal{S})$, where $X = \{a, b\}$, $\tau = \{0_X, 1_X\} \vee \{U \in I^X : 0 < U(a) \leq 0.5;\ 0 \leq U(b) \leq 1\}$, and $\mathcal{S} = \{G \in I^X : 0.3 < G(x) \leq 1,\ \forall x \in X\}$. We first verify that (X, τ) is fuzzy $\tau_{1/2}$. In fact, if F is a gf-closed set in X, then $F(a) \geq 0.5$. For, if $F(a) < 0.5$, then we consider any fuzzy open set U with $F(a) \leq U(a) < 0.5$ and $U(b) \geq F(b)$. Then $F \leq U$, but $(clF)(a) = 0.5 > U(a)$, i.e., $clF \not\leq U$, so that F is not gf-closed. Hence for any gf-closed set F in X, $F(a) \geq 0.5$ and $0 \leq F(b) \leq 1$, and so F is clearly fuzzy closed. Now, consider the fuzzy set A in $(X, \tau, \mathcal{S})$ such that $A(a) = 0.1$ and $A(b) = 0.4$. It can be checked that A is $\mathcal{S}_{gf}$-closed but not fuzzy closed. Hence $(X, \tau, \mathcal{S})$ is not a fuzzy $\mathcal{S}\tau_{1/2}$-space.*

Theorem 3.23. *If $f : (X, \tau, \mathcal{S}_1) \to (Y, \sigma, \mathcal{S}_2)$ is $\mathcal{S}_{gf}$-continuous and $(X, \tau, \mathcal{S}_1)$ is a fuzzy $\mathcal{S}\tau_{1/2}$-space, then it is fuzzy continuous.*

Proof. The proof follows from the definitions of fuzzy $\mathcal{S}\tau_{1/2}$-space and $\mathcal{S}_{gf}$-continuity. □

Theorem 3.24. *Let $f:(X,\sigma,\mathcal{S}_1)\to(Y,\tau,\mathcal{S}_2)$ and $g:(Y,\tau,\mathcal{S}_2)\to(Z,\tau_1,\mathcal{S}_3)$ be two functions and Y be a fuzzy $\mathcal{S}\tau_{1/2}$-space. If f and g are both $\mathcal{S}_{gf}$-continuous, then the composition $g\circ f:(X,\sigma,\mathcal{S}_1)\to(Z,\tau_1,\mathcal{S}_3)$ is also $\mathcal{S}_{gf}$-continuous.*

Proof. Straightforward. □

However, we show that the above theorem is not true if Y is not a fuzzy $\mathcal{S}\tau_{1/2}$-space.

Example 3.25. *Let $X=\{x_1,x_2\}$, $\sigma=\{0_X,1_X,A\}$ and $\tau=\{0_X,1_X,B\}$, where $A(x_1)=0.4, A(x_2)=0.6$ and $B(x_1)=0.3, B(x_2)=0.4$, be two fuzzy topologies on X. Let $\mathcal{S}_1=\{G_1\in I^X: 0.2<G_1(x)\le 1,\ \forall x\in X\}$ and $\mathcal{S}_2=\{G_2\in I^X: 0.7<G_2(x)\le 1,\ \forall x\in X\}$ be two fuzzy stacks on X. Clearly $(X,\sigma,\mathcal{S}_1)$ and $(X,\tau,\mathcal{S}_2)$ are two fuzzy $\mathcal{S}$-spaces.*

Let $f:(X,\sigma,\mathcal{S}_1)\to(X,\tau,\mathcal{S}_2)$ be the identity function. Then f is $\mathcal{S}_{gf}$-continuous, since $f^{-1}(1_X-B)$ is $\mathcal{S}_{gf}$-closed in $(X,\sigma,\mathcal{S}_1)$.

Let us consider the fuzzy topology $\tau_1=\{0_X,1_X,P\}$ on X, where $P(x_1)=0.5, P(x_2)=0.6$ and let $\mathcal{S}_3=\{U\in I^X: 0.6<U(x)\le 1,\ \forall x\in X\}$ be a fuzzy stack on X.

Then the mapping $g:(X,\tau,\mathcal{S}_2)\to(X,\tau_1,\mathcal{S}_3)$ defined by $g(x_1)=x_2, g(x_2)=x_1$ is $\mathcal{S}_{gf}$-continuous, since $g^{-1}(1_X-P)$ is $\mathcal{S}_{gf}$-closed in $(X,\tau,\mathcal{S}_2)$ [as 1_X is the only fuzzy open set in $(X,\tau,\mathcal{S}_2)$ containing $g^{-1}(1_X-P)$]. But we show that the composition $g\circ f:(X,\sigma,\mathcal{S}_1)\to(X,\tau_1,\mathcal{S}_3)$ is not $\mathcal{S}_{gf}$-continuous.

In fact, $(g\circ f)^{-1}(1_X-P)(x_1)=1-(g\circ f)^{-1}P(x_1)=1-P((g\circ f)(x_1))=1-P(g(f(x_1)))=1-P(g(x_1))=1-P(x_2)=0.4$ *and similarly* $(g\circ f)^{-1}(1_X-P)(x_2)=0.5$.

Hence $(g\circ f)^{-1}(1_X-P)=R$(say)$\le A$. But $\Phi_{\mathcal{S}_1}(R)\not\le A$; indeed $(x_1)_{0.5}\not\le A$, but $(x_1)_{0.5}\le\Phi_{\mathcal{S}_1}(R)$, since 1_X is the only open q-nbd of $(x_1)_{0.5}$ in (X,σ) and $1_X * A\in\mathcal{S}_1$. Thus R is not an $\mathcal{S}_{gf}$-closed set in $(X,\sigma,\mathcal{S}_1)$ and hence $g\circ f$ is not $\mathcal{S}_{gf}$-continuous.

Remark 3.26. *Let $f:(X,\sigma,\mathcal{S}_1)\to(Y,\tau,\mathcal{S}_2)$ be a function and Y be a fuzzy $\mathcal{S}\tau_{1/2}$-space. Then $\mathcal{S}_{gf}$-strong fuzzy continuity and fuzzy continuity of f coincide.*

Theorem 3.27. *Let* $f : (X, \sigma, \mathcal{S}_1) \to (Y, \tau, \mathcal{S}_2)$ *and* $g : (Y, \tau, \mathcal{S}_2) \to (Z, \tau_1, \mathcal{S}_3)$ *be both* $\mathcal{S}_{gf}$*-closed functions and* Y *be a fuzzy* $\mathcal{S}\tau_{1/2}$*-space. Then the composition* $g \circ f : (X, \sigma, \mathcal{S}_1) \to (Z, \tau_1, \mathcal{S}_3)$ *is also an* $\mathcal{S}_{gf}$*-closed function.*

Proof. It follows from the definitions of $\mathcal{S}_{gf}$-closed functions and fuzzy $\mathcal{S}\tau_{1/2}$-space. □

Remark 3.28. *Let* $f : (X, \sigma, \mathcal{S}_1) \to (Y, \tau, \mathcal{S}_2)$ *be a function and* Y *be a fuzzy* $\mathcal{S}\tau_{1/2}$*-space, then*

(i) f *is* $\mathcal{S}_{gf}$*-closed iff it is fuzzy closed.*
(ii) f *is* $\mathcal{S}_{gf}$*-open iff it is fuzzy open.*

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