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SOME GEOMETRICAL ASPECTS OF DYNAMICAL SYSTEMS WITH CONTROL

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Abstract. In this article, we discuss geometric controllability conditions for dynamical systems and study some particular cases of dynamical systems with control.

1. INTRODUCTION

A control system is a dynamical system whose dynamical law has the form:

$$\frac{dx}{dt} = F(x,u) \tag{1}$$

where parameters u are called control parameters and have the role of influencing the movement of the system.
It is known that the set of states of a dynamical system is a manifold M with n dimension and the set of parameters U will be the subset $U \subset \mathbb{R}^m$.

The dynamics of the system is described by a function $F : M \times U \to TM$ so that for any $u \in U$, $F_u : M \to TM$ is a smooth vector field.
If the parameters u are differentiable functions of $x \in M$, $u : M \to U$, then we have a feedback control.
If the parameters are seen as functions of t, i.e. $u : \mathbb{R} \to U, t \to u(t)$, then the dynamics is given by the vector field $F(x, u(t))$ which varies with t.

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But if $u : M\times\mathbb{R} \to U, (x,t) \to u(x,t)$, then the dynamics is given by $F(x, u(x,t))$. With the substitution $v(t) = u(x(t),t)$, we find that it is sufficient to limit ourselves to the control functions $u : \mathbb{R} \to U, t \to u(t)$, and (1) will be written explicitly thus:

$$\frac{dx^i(t)}{dt} = F^i(x(t), u(t)), i = 1, 2, ..., n = dim\, M \tag{2}$$

that the solution $x(t, x_0, u_0)$ of the system (2) satisfying the initial condition $x(0, x_0, u_0) = x_0$ represents the trajectory of the system starting at x_0 controlled by u_0.

2. Main results

The geometric study of control systems begins with the simple case where the control functions $u : \mathbb{R} \to U$ are piecewise-constant.
Let $\mathfrak{F} = \{F_u : u \in U\}$ a family of vector fields obtained from $F(x,u)$ by fixing u in various positions in U. Also, let [0,T] interval in the form $0 = t_0 < t_1 < ... < t_m = T$ and the vector fields $X^1, X^2, ..., X^m$ in $\mathfrak{F}$.
We know that a continuous curve $t \to x(t)$ defined on $[0,T]$ is called the integral curve for $\mathfrak{F}$ if it is a differentiable solution on (t_{k-1}, t_k) of the system

$$\frac{dx^i}{dt} = (X^k)^i(x(t)), i = 1, 2, ..., n;\ k = 1, 2, ..., m \tag{3}$$

The vector fields $X^1, X^2, ..., X^m$ being in $\mathfrak{F}$, we have that $X^k = F_{u_k}$, for a certain $u_k \in U$. It follows that the application $t \to x(t)$ which is an integral curve for $\mathfrak{F}$ is also a solution curve of field $F(x, u(t))$, with $u(t)$ a piecewise-constant control function, equal with u_k, for $t \in [t_{k-1}, t_k]$.
For $x_0 \in M$, we denote by $x(t, x_0, u)$ the trajectories that go to $t = 0$ from x_0, obtained for the control u.
For $T > 0$, the point $x(T, x_0, u)$ is called accessible from x_0 by the control u.
Let $A(x_0, T)$ be the set of accessible points in x_0 during T. The reunion of these sets after $T \geq 0$, denoted by $A(x_0)$ is the set of all accessible points in x_0.
We consider the group $G_{\mathfrak{F}}$ of diffeomorphisms generated by the family $\mathfrak{F}$ consisting of compositions of the form:

$$\phi = X^1_{t_1} \circ X^2_{t_2} \circ ... \circ X^j_{t_j}$$

where $X^1_{t_1}, X^2_{t_2}, ..., X^j_{t_j}$ are the 1-parametric groups defined by the fields $X^1, ..., X^j$ in the family $\mathfrak{F}$.

Then, the accessible points from x_0 during T are in the form $\phi(x_0)$, with $t_1 \geq 0, t_2 \geq 0, ..., t_j \geq 0$ and $t_1 + t_2 + ... + t_j = T$.
In addition, $A(x_0)$ is the orbit of semigroup $S_{\mathfrak{F}}$ who has elements of ϕ from $G_{\mathfrak{F}}$ under the form $X^1_{t_1} \circ X^2_{t_2} \circ ... \circ X^j_{t_j}$, with $t_1 \geq 0, ..., t_j \geq 0$ and $X^1, X^2, ..., X^j$ fields in $\mathfrak{F}$.
We denote by $Lie(\mathfrak{F})$ the Lie algebra generated by the vector fields in $\mathfrak{F}$:

$$[X_k, X_{k-1}, ..., [X_3, [X_2, X_1], ...], ...]$$

or

$$adX_k \circ ... \circ adX_2 \circ adX_2(X_1),$$

where $adX(Y) = [X, Y]$.
If $dimLie_x(\mathfrak{F}) = k$, the tangent space in x to the orbit of $G_{\mathfrak{F}}$ may coincide with $Lie_x(\mathfrak{F})$, and in this case each orbit has dimension k.
For a family $\mathfrak{F}$ that is Lie determined, introduces saturated Lie $L_S(\mathfrak{F})$ noted that in the largest subset $\mathfrak{F}$ of $Lie(\mathfrak{F})$ so that the closing of the set $A_{\hat{\mathfrak{F}}}(x)$ is equal to the closing of the set $A_{\mathfrak{F}}(x)$, $\forall x \in M$.
Similarly, define strong saturate Lie of $Lie(\mathfrak{F})$ as the largest set which the closing of the set $A_{\hat{\mathfrak{F}}}(x, \leq T)$ for every $x \in M$ and for all $T > 0$.
Definition 1. The control system (1) or (2) the associated family $\mathfrak{F}$ called controllable if any point of M is accessible from any point of M and is called highly controllable if it happens strongly controllable $\leq T$ for every $T > 0$
Theorem 1[1]. i) The system (1) is controllable if and only if $\mathfrak{F}$ coincides with Lie saturate $LS(F)$ and $Lie_x(\mathfrak{F}) = T_xM$.
ii) The system (1) is strongly controllable if and only if $\mathfrak{F}$ coincides with Lie strong saturate and $Lie_x(\mathfrak{F}) = T_xM$.
A control system of form

$$\frac{dx^i}{dt} = X^i_0(x) + \sum_{a=1}^{m} u^a X^i_a(x) \tag{4}$$

with $X_0, X_a, a = 1, ..., m$, smooth vector fields in M and (u^a), $a = 1, 2, ..., m$ are controls.
The X_0-field is called *drift* and the vector fields (X_a) are called *control fields.*
The control functions are considered as a m-tuple $(u^1, u^2, ..., u^m) \in U \subset \Re^m$ and if $U = \mathbb{R}^m$ it is said that the system (4) is under *non-restrictive control.*

A particular case is that where vectors (X_a) are constant and the drift is linear, so denoting $b_1, b_2, ..., b_m$ the constant values of the vector fields $X_1, X_2, ..., X_m$ and $X_0(x) = Ax$, the form of the linear system with control is:

$$\frac{dx^i}{dt} = A^i_j x^j + \sum_a u^a b^i_a \tag{5}$$

which with the notation:

$$u = \begin{bmatrix} u^1 \\ \vdots \\ u^m \end{bmatrix} \text{ and } B = \begin{bmatrix} b^1_1 b^1_2 ... b^1_m \\ b^2_1 b^2_2 ... b^2_m \\ ------- \\ b^n_1 b^n_2 ... b^n_m \end{bmatrix}$$

will be written:

$$\frac{dx}{dt} = Ax + B \tag{6}$$

For this type of system, a necessary and sufficient condition of controllability is:

$$rank(B/AB/A^2B/.../A^{n-1}B) = n$$

If $Lie_x\{X_1, X_2, ..., X_m\} = T_xM$, for $\forall x \in M$, then we have:

Theorem 2[9]. a) The system (4) is strongly controllable if there are no restrictions on the controls (u_a) i.e. $U = \mathbb{R}^m$.
b) if the system (4) is without drift $(X_0 = 0)$, it remains controllable with $U \subset \mathbb{R}^m$, provided that the convex hull of U is a neighborhood of the origin $O \in \mathbb{R}^m$.

If H is the distribution generated by $\{X_1, X_2, ..., X_m\}$, the condition $Lie_x\{X_1, ..., X_m\} = T_xM$, is the completely non-holonomic distribution.

We recall Chow's theorem: if H is a completely non-holonomic distribution on convex manifold M, then any two points in M can be joined by a horizontal curve.

Application.

Let $\Re^3$ be defined as the kernel of the Pfaff system defined by 1-form $w = dz - y^2dx$.

This distribution is generated by vector fields $X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$ and $Y = \frac{\partial}{\partial y}$
.

Calculate $w(X) = y^2 - y^2 = 0$, $w(Y) = 0$, so X and Y are linear independent.

Calculate $[X, Y] = -2y\frac{\partial}{\partial z}, [[X, Y], Y] = 2\frac{\partial}{\partial z}$

For $y \neq 0$, X, Y and $[X, Y]$ generate T_xM and for $y = 0$, X, Y and $[X, Y]$ generated also T_xM .

So, $Lie_x\{X, Y\} = T_xM$ and according to the Chow's theorem, any

two points from $\mathbb{R}^3$ can be joined by a horizontal curve.
These curves are the solutions of the system of differential equations:

$$\frac{dx}{dt} = \lambda, \frac{dy}{dt} = \mu, \frac{dz}{dt} = \lambda y^2$$

with $(\lambda, \mu) \in \mathbb{R}^2$.
If λ and μ are constant functions, then the solution is given by

$$\begin{cases} x = x_0 + \lambda t \\ y = y_0 + \mu t \\ z = z_0 + \lambda y_0^2 t + \lambda \mu t^2 + \frac{\lambda \mu^2}{3} t^3 \end{cases}$$

and $\frac{dz}{dt} = \lambda(y_0^2 + 2\mu t + \mu^2 t^2), (\lambda, \mu) \in \mathbb{R}^2$
So, the horizontal curves are contained in the plane
$\mu(x - x_0) = \lambda(y - y_0)$

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