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INVARIANT POINTS AND ε-APPROXIMATIONS FOR MAPPINGS SATISFYING RATIONAL-TYPE CONTRACTIVE CONDITIONS IN TAKAHASHI SPACES

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Abstract. In this paper, we prove some Brosowski-Meinardus type invariant point results for the set of ε-simultaneous approximation and ε-simultaneous coapproximation for rational type contraction mappings defined on Takahashi spaces. Subsequently, we deduce some results on ε-approximation, ε-coapproximation, best approximation and best coapproximation for such class of mappings.

1. Introduction and Preliminaries

The study of best approximation theory plays an important role in nonlinear functional analysis, optimization theory, fixed point theory, nonlinear programming, game theory, variational inequality, complementarity problems, and so forth. The idea of applying fixed point theorems to approximation theory was initiated in normed linear spaces by G. Meinardus [12]. Generalizing the result of Meinardus, Brosowski [1] proved the following theorem on invariant approximation using fixed point theory:

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Theorem 1.1. *Let $\mathscr{T}$ be a linear and nonexpansive operator on a normed linear space E. Let $\mathcal{G}$ be a $\mathscr{T}$-invariant subset of E and $\mathfrak{x}$ a $\mathscr{T}$-invariant point. If the set $P_{\mathcal{G}}(\mathfrak{x})$ of best $\mathcal{G}$-approximants to $\mathfrak{x}$ is nonempty, compact and convex, then it contains a $\mathscr{T}$-invariant point.*

Subsequently, various generalizations of Brosowski's results appeared in the literature. Singh [19] observed that the linearity of the operator $\mathscr{T}$ and convexity of the set $P_{\mathcal{G}}(\mathfrak{x})$ in Theorem 1.1 can be relaxed and proved the following:

Theorem 1.2. *Let $\mathscr{T} : E \to E$ be a nonexpansive self mapping on a normed linear space E. Let $\mathcal{G}$ be a $\mathscr{T}$-invariant subset of E and $\mathfrak{x}$ a $\mathscr{T}$-invariant point. If the set $P_{\mathcal{G}}(\mathfrak{x})$ is nonempty, compact and starshaped, then it contains a $\mathscr{T}$-invariant point.*

Later, Singh [20] demonstrated that Theeorem 1.2 remains valid if $\mathscr{T}$ is assumed to be nonexpansive only on the set $P_{\mathcal{G}}(\mathfrak{x}) \cup \{\mathfrak{x}\}$. Thenceforth, many results have been obtained in this direction by many researchers (see [3, 4, 5, 6, 7, 8, 10, 11, 13, 15, 16, 17, 18] and references cited therein).

In this manuscript, we obtain some similar types of results on $\mathscr{T}$-invariant points for the set of ε-simultaneous approximation and ε-simultaneous coapproximation for a rational type contraction mapping defined on a Takahashi space $(\mathscr{X}, d, W)$. For such class of mappings, we also deduce some results on $\mathscr{T}$-invariant points for the set of ε-approximation, ε-coapproximation, best approximation and best coapproximation.

Definition 1.3. *Let $(\mathscr{X}, d)$ be a metric space, $\emptyset \neq \mathcal{G} \subset \mathscr{X}$, $\mathcal{F}$ a nonempty bounded subset of $\mathscr{X}$. For $\mathfrak{x} \in \mathscr{X}$, assume that*

$$d_{\mathcal{F}}(\mathfrak{x}) = \{\sup d(y, \mathfrak{x}) : y \in \mathcal{F}\},$$
$$D(\mathcal{F}, \mathcal{G}) = \{\inf d_{\mathcal{F}}(\mathfrak{x}) : \mathfrak{x} \in \mathcal{G}\},$$
and
$$P_{\mathcal{G}}(\mathcal{F}) = \{g_0 \in \mathcal{G} : d_{\mathcal{F}}(g_0) = D(\mathcal{F}, \mathcal{G})\}.$$

An element $g_0 \in P_{\mathcal{G}}(\mathcal{F})$ is said to be a ***best simultaneous approximation*** *of $\mathcal{F}$ with respect to $\mathcal{G}$ (see [5]).*

For $\varepsilon > 0$, we define

$$\begin{aligned} P_{\mathcal{G}(\varepsilon)}(\mathcal{F}) =& \{g_0 \in \mathcal{G} : d_{\mathcal{F}}(g_0) \leq D(\mathcal{F}, \mathcal{G}) + \varepsilon\} \\ =& \{g_0 \in \mathcal{G} : \sup_{y \in \mathcal{F}} d(y, g_0) \leq \inf_{g \in \mathcal{G}} \sup_{y \in \mathcal{F}} d(y, g) + \varepsilon\}. \end{aligned}$$

An element $g_0 \in P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ *is said to be a* ε*-simultaneous approximation of* $\mathcal{F}$ *with respect to* $\mathcal{G}$ *(see* [5]*).*

It can be easily seen that for $\varepsilon > 0$*, the set* $P_{\mathcal{G}}(\varepsilon)(\mathcal{F})$ *is always a nonempty bounded set and is closed if* $\mathcal{G}$ *is closed.*

In case $\mathcal{F} = \{p\}$*,* $p \in \mathscr{X}$*, then elements of* $P_{\mathcal{G}}(p)$ *are called* ***best approximations*** *to* p *in* $\mathcal{G}$ *and of* $P_{\mathcal{G}(\varepsilon)}(p)$ *are called* ε-***approximation*** *to* p *in* $\mathcal{G}$*.*

For $\varepsilon > 0$*, we define*

$$R_{\mathcal{G}(\varepsilon)}(\mathcal{F}) = \{g_0 \in \mathcal{G} : \sup_{g \in \mathcal{G}} d(g, g_0) + \varepsilon \leq \inf_{g \in \mathcal{G}} \sup_{y \in \mathcal{F}} d(y, g)\}.$$

An element $g_0 \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ *is said to be a* ε*-simultaneous coapproximation of* $\mathcal{F}$ *with respect to* $\mathcal{G}$ *(see* [5]*).*

Let $\mathscr{T}$ *be a self mapping defined on a subset* $\mathcal{G}$ *of a metric space* $\mathscr{X}$*. A best approximant* $\mathfrak{y}$ *in* $\mathcal{G}$ *to an element* $\mathfrak{x}_0$ *in* $\mathscr{X}$ *with* $\mathscr{T}\mathfrak{x}_0 = \mathfrak{x}_0$ *is an invariant approximation in* $\mathscr{X}$ *to* $\mathfrak{y}$ *if* $\mathscr{T}\mathfrak{y} = \mathfrak{y}$*.*

Example 1.4. *Let* $\mathscr{X} = \mathbb{R}$ *and* $\mathcal{G} = [0, 1] \subset \mathscr{X}$*. Define* $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ *as*

$$\mathscr{T}\mathfrak{x} = \begin{cases} \mathfrak{x}, \mathfrak{x} < 0 \\ \mathfrak{x} + 1, 0 \leq \mathfrak{x} \leq 1 \\ \frac{\mathfrak{x}+2}{2}, \mathfrak{x} > 1. \end{cases}$$

Clearly, $\mathscr{T}(\mathcal{G}) = \mathcal{G}$ *and* $\mathscr{T}(2) = 2$*. Also,* $P_{\mathcal{G}}(2) = \{1\}$*. Hence* $\mathscr{T}$ *has a fixed point in* $P_{\mathcal{G}}(2)$ *which is a best approximation to* 2 *in* $\mathcal{G}$*. Thus,* 2 *is an invariant approximation.*

Definition 1.5. *A sequence* $\{y_n\}$ *in* $\mathcal{G}$ *is called a* ε-***minimizing sequence*** *for* $\mathcal{F}$*, if*

$$\lim \sup_{\mathfrak{x} \in \mathcal{F}} d(\mathfrak{x}, y_n) \leq D(\mathcal{F}, \mathcal{G}) + \varepsilon.$$

The set $\mathcal{G}$ *is said to be* ε-***simultaneous approximatively compact with respect to*** $\mathcal{F}$ *(see* [5]*) if for every* $\mathfrak{x} \in \mathcal{F}$*, each* ε*-minimizing sequence* $\{y_n\}$ *in* $\mathcal{G}$ *has a subsequence* $\{y_{n_i}\}$ *converging to an element of* $\mathcal{G}$*.*

Inspired by the work of Takahashi [21] and Guay et al. [9], we have the following definition.

Definition 1.6. *Let* $\mathscr{X}$ *be a nonempty set,* d *be a metric on* $\mathscr{X}$ *and* $W : \mathscr{X} \times \mathscr{X} \times [0, 1] \to \mathscr{X}$ *be a continuous mapping satisfying, for all* $\mathfrak{x}, \mathfrak{y}, \mathfrak{u} \in \mathscr{X}$ *and* $\lambda \in [0, 1]$*,*

(i) $d(\mathfrak{u}, W(\mathfrak{x}, \mathfrak{y}, \lambda)) \leq \lambda d(\mathfrak{u}, \mathfrak{x}) + (1 - \lambda) d(\mathfrak{u}, \mathfrak{y})$*,*

(ii) $d(W(\mathfrak{x},\mathfrak{u},\lambda), W(\mathfrak{y},\mathfrak{u},\lambda)) \leq d(\mathfrak{x},\mathfrak{y})$.

Then the triple $(\mathscr{X}, d, W)$ *is called a* ***Takahashi space****.*

A normed linear space and each of its convex subset are simple examples of Takahashi spaces with W given by $W(\mathfrak{x},\mathfrak{y},\lambda) = \lambda\mathfrak{x} + (1-\lambda)\mathfrak{y}$ for $\mathfrak{x},\mathfrak{y} \in \mathscr{X}$ and $0 \leq \lambda \leq 1$. For definition of convex set, p-starshaped set and starshaped set see [5] and references cited therein.

Definition 1.7. *Let* $\mathcal{G}$ *be a nonempty subset of a metric space* $(\mathscr{X}, d)$ *and* $\mathscr{T} : \mathcal{G} \to \mathcal{G}$ *be a self map. Then* $\mathscr{T}$ *is said to be* ***asymptotically regular*** *(see, [2]) if for all* $\mathfrak{x} \in \mathcal{G}$, $d(\mathscr{T}^n(\mathfrak{x}), \mathscr{T}^{n+1}(\mathfrak{x})) \to 0$ *as* $n \to \infty$.

Definition 1.8. *A mapping* $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ *satisfies* ***condition (A)*** *(see [14]) if*

$$d(\mathscr{T}^n\mathfrak{x}, \mathfrak{y}) \leq d(\mathfrak{x}, \mathfrak{y}),$$

for all $\mathfrak{x},\mathfrak{y} \in \mathscr{X}$ *and for some positive integer* n.

2. Main Results

To start with, we give the following definition of H_C-contraction.

Definition 2.1. *Let* $(\mathscr{X}, d)$ *be a metric space. A mapping* $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ *is called a* H_C*-contraction if there exist* $\alpha, \beta \in [0,1)$ *with* $\alpha+\beta < 1$ *such that for all* $\mathfrak{x},\mathfrak{y} \in \mathscr{X}$*, we have*

$$d(\mathscr{T}\mathfrak{x}, \mathscr{T}\mathfrak{y}) \leq \alpha\frac{d(\mathfrak{x}, \mathscr{T}\mathfrak{x})d(\mathfrak{y}, \mathscr{T}\mathfrak{y})}{1+d(\mathfrak{x},\mathfrak{y})} + \beta(d(\mathfrak{x},\mathfrak{y})). \tag{2.1}$$

Remark 2.2. *On a metric space, every* H_C*-contraction has at most one fixed point. Indeed, let* $\mathfrak{x}$ *and* $\mathfrak{y}$ *be two distinct fixed points of* $\mathscr{T}$*, which is a* H_C*-contraction. Then*

$$\begin{aligned} d(\mathfrak{x},\mathfrak{y}) = d(\mathscr{T}\mathfrak{x}, \mathscr{T}\mathfrak{y}) &\leq \alpha\frac{d(\mathfrak{x}, \mathscr{T}\mathfrak{x})d(\mathfrak{y}, \mathscr{T}\mathfrak{y})}{1+d(\mathfrak{x},\mathfrak{y})} + \beta(d(\mathfrak{x},\mathfrak{y})) \\ &= \beta(d(\mathfrak{x},\mathfrak{y})), \end{aligned}$$

which is a contradiction as $0 \leq \beta < 1$ *and* $d(\mathfrak{x},\mathfrak{y}) > 0$.

We prove the following result needed in the sequel:

Proposition 2.3. *Let* $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ *be a* H_C-*contraction on a metric space* $(\mathscr{X}, d)$*. Then for all* $\mathfrak{x} \in \mathscr{X}$*, the sequence* $\{d(\mathscr{T}^n\mathfrak{x}, \mathscr{T}^{n+1}\mathfrak{x})\}$ *is decreasing and* $\mathscr{T}$ *is asymptotically regular.*

Proof. Let $\mathfrak{x}_0$ be an arbitrary point in $\mathscr{X}$ and $\{\mathfrak{x}_n\}$ be the Picard sequence in $\mathscr{X}$ such that $\mathfrak{x}_{n+1} = \mathscr{T}\mathfrak{x}_n = \mathscr{T}^n\mathfrak{x}_0$, for every $n \geq 0$. Using (2.1), we have

$$\begin{aligned} d(\mathfrak{x}_{n+2}, \mathfrak{x}_{n+1}) &= d(\mathscr{T}\mathfrak{x}_{n+1}, \mathscr{T}\mathfrak{x}_n) \\ &\leq \alpha \frac{d(\mathfrak{x}_{n+1}, \mathscr{T}\mathfrak{x}_{n+1}) d(\mathfrak{x}_n, \mathscr{T}\mathfrak{x}_n)}{1 + d(\mathfrak{x}_{n+1}, \mathfrak{x}_n)} + \beta(d(\mathfrak{x}_{n+1}, \mathfrak{x}_n)) \\ &= \alpha \frac{d(\mathfrak{x}_{n+1}, \mathfrak{x}_{n+2}) d(\mathfrak{x}_n, \mathfrak{x}_{n+1})}{1 + d(\mathfrak{x}_{n+1}, \mathfrak{x}_n)} + \beta(d(\mathfrak{x}_{n+1}, \mathfrak{x}_n)) \\ &\leq \alpha(d(\mathfrak{x}_{n+1}, \mathfrak{x}_{n+2})) + \beta(d(\mathfrak{x}_{n+1}, \mathfrak{x}_n)). \end{aligned}$$

This implies

$$d(\mathfrak{x}_{n+2}, \mathfrak{x}_{n+1}) \leq \frac{\beta}{1-\alpha} d(\mathfrak{x}_{n+1}, \mathfrak{x}_n). \tag{2.2}$$

Since $\frac{\beta}{1-\alpha} < 1$, the sequence $\{d(\mathscr{T}^n\mathfrak{x}_0, \mathscr{T}^{n+1}\mathfrak{x}_0)\}$ is a decreasing sequence. Using mathematical induction, we have

$$d(\mathfrak{x}_{n+2}, \mathfrak{x}_{n+1}) \leq \left(\frac{\beta}{1-\alpha}\right)^{n+1} d(\mathfrak{x}_1, \mathfrak{x}_0). \tag{2.3}$$

Taking the limit $n \to \infty$, we have $d(\mathfrak{x}_{n+2}, \mathfrak{x}_{n+1}) \to 0$, that is, $d(\mathscr{T}^n\mathfrak{x}_0, \mathscr{T}^{n+1}\mathfrak{x}_0) \to 0$. Hence the result. □

Using the above proposition, we prove the following:

Theorem 2.4. *Every H_C-contraction on a complete metric space has unique fixed point.*

Proof. Using Proposition 2.3, the sequence $\{d(\mathscr{T}^n\mathfrak{x}_0, \mathscr{T}^{n+1}\mathfrak{x}_0)\}$ is decreasing and $d(\mathscr{T}^n\mathfrak{x}_0, \mathscr{T}^{n+1}\mathfrak{x}_0) \to 0$ as $n \to \infty$ for all $\mathfrak{x}_0 \in \mathscr{X}$. We claim that $\{\mathfrak{x}_n\}$ is a Cauchy sequence. For $m > n$, and $k = \dfrac{\beta}{1-\alpha}$ we have

$$\begin{aligned} d(\mathfrak{x}_n, \mathfrak{x}_m) &\leq d(\mathfrak{x}_n, \mathfrak{x}_{n+1}) + d(\mathfrak{x}_{n+1}, \mathfrak{x}_{n+2}) + \ldots + d(\mathfrak{x}_{m-1}, \mathfrak{x}_m) \\ &\leq (k^n + k^{n+1} + \ldots + k^{m-1}) d(\mathfrak{x}_0, \mathfrak{x}_1) \\ &\leq \frac{k^n(1-k^{m-n})}{1-k} d(\mathfrak{x}_0, \mathfrak{x}_1). \end{aligned}$$

Therefore, $d(\mathfrak{x}_m, \mathfrak{x}_n) \to 0$, when $m, n \to \infty$. Thus $\{\mathfrak{x}_n\}$ is a Cauchy sequence in a complete metric space $\mathscr{X}$ and so there exists $u \in \mathscr{X}$ such that $\lim_{n\to\infty} \mathfrak{x}_n = u$.

Now, we'll show that the point u is a fixed point of $\mathscr{T}$. On the contrary, suppose that $\mathscr{T}u \neq u$, then $d(u, \mathscr{T}u) > 0$. Consider

$$\begin{aligned} d(\mathfrak{x}_{n+1}, \mathscr{T}u) =& d(\mathscr{T}\mathfrak{x}_n, \mathscr{T}u) \leq \alpha \frac{d(\mathfrak{x}_n, \mathscr{T}\mathfrak{x}_n) d(u, \mathscr{T}u)}{1 + d(\mathfrak{x}_n, u)} + \beta(d(\mathfrak{x}_n, u)) \\ =& \alpha \frac{d(\mathfrak{x}_n, \mathfrak{x}_{n+1}) d(u, \mathscr{T}u)}{1 + d(\mathfrak{x}_n, u)} + \beta(d(\mathfrak{x}_n, u)). \end{aligned}$$

Taking $n \to \infty$, we have $d(u, \mathscr{T}u) \leq 0$, it implies that $d(u, \mathscr{T}u) = 0$. Hence u is a fixed point of $\mathscr{T}$. Using Remark 2.2, we obtain that $\mathscr{T}$ has unique fixed point. □

Example 2.5. *Let* $\mathscr{X} = [0, 1]$ *and* d *be the usual metric on* $\mathscr{X}$. *Define* $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ *as* $\mathscr{T}\mathfrak{x} = \begin{cases} \frac{2}{5}, & \mathfrak{x} \in [0, \frac{2}{3}) \\ \frac{1}{5}, & \mathfrak{x} \in [\frac{2}{3}, 1]. \end{cases}$ *Suppose* $\alpha = \frac{1}{7}, \beta = \frac{1}{7} \in [0, 1)$ *with* $\alpha + \beta = \frac{2}{7} < 1$. *We may check that*

$$d(\mathscr{T}\mathfrak{x}, \mathscr{T}y) \leq \frac{1}{7} \frac{d(\mathfrak{x}, \mathscr{T}\mathfrak{x}) d(y, \mathscr{T}y)}{1 + d(\mathfrak{x}, y)} + \frac{1}{7}(d(\mathfrak{x}, y)),$$

for all $\mathfrak{x}, y \in \mathscr{X}$. *Thus using Theorem 2.4, we notice that* $\frac{2}{5} \in \mathscr{X}$ *is a fixed point of* $\mathscr{T}$.

Example 2.6. *Let* $\mathscr{X} = [0, 1]$ *and* d *be the usual metric on* $\mathscr{X}$. *Define* $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ *as* $\mathscr{T}\mathfrak{x} = \begin{cases} \frac{\mathfrak{x}}{10}, & \mathfrak{x} \in [0, \frac{1}{2}] \\ \frac{\mathfrak{x}}{5} - \frac{1}{20}, & \mathfrak{x} \in (\frac{1}{2}, 1]. \end{cases}$ *Suppose* $\alpha = \frac{1}{8}, \beta = \frac{1}{4} \in [0, 1)$ *with* $\alpha + \beta = \frac{3}{8} < 1$. *We may check that*

$$d(\mathscr{T}\mathfrak{x}, \mathscr{T}y) \leq \frac{1}{8} \frac{d(\mathfrak{x}, \mathscr{T}\mathfrak{x}) d(y, \mathscr{T}y)}{1 + d(\mathfrak{x}, y)} + \frac{1}{4}(d(\mathfrak{x}, y)),$$

for all $\mathfrak{x}, y \in \mathscr{X}$. *Thus using Theorem 2.4,* $\mathscr{T}$ *has unique fixed point. Notice that* $0 \in \mathscr{X}$ *is the fixed point of* $\mathscr{T}$.

Theorem 2.7. *Let* $\mathscr{T}$ *be a asymptotically regular and continuous self mapping on a complete Takahashi space* $(\mathscr{X}, d, W)$ *and satisfying the following for some positive integer* n,

(2.4)
$$d(\mathscr{T}^n\mathfrak{x}, \mathscr{T}^n y) \leq \alpha \left(\frac{dist(\mathfrak{x}, [\mathscr{T}^n\mathfrak{x}, q]) dist(y, [\mathscr{T}^n y, q])}{1 + d(\mathfrak{x}, y)} \right) + \beta(d(\mathfrak{x}, y)),$$

for all $\mathfrak{x}, y, q \in \mathscr{X}$, *where* $\alpha, \beta \in [0, 1)$ *with* $\alpha + \beta < 1$. *Suppose that* $\mathcal{G}$ *is a* $\mathscr{T}$*-invariant subset of* $\mathscr{X}$ *and* $\mathcal{F}$ *a nonempty bounded subset of*

$\mathscr{X}$ such that $\mathscr{T}\mathfrak{x} = \mathfrak{x}$ for all $\mathfrak{x} \in \mathcal{F} \setminus \mathcal{G}$. If $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ is compact, and q-starshaped, then it contains a $\mathscr{T}$-invariant point.

Proof. Let $z \in P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ be arbitrary. Then using (2.4) and the fact that for all $\mathfrak{x} \in \mathcal{F} \setminus \mathcal{G}$, $\mathscr{T}\mathfrak{x} = \mathfrak{x} = \mathscr{T}^n\mathfrak{x}$, we have

$$\begin{aligned} d(\mathfrak{x}, \mathscr{T}^n z) &= d(\mathscr{T}^n\mathfrak{x}, \mathscr{T}^n z) \\ &\leq \alpha\left(\frac{dist\big(\mathfrak{x}, [\mathscr{T}^n\mathfrak{x}, q]\big)dist(z, [\mathscr{T}^n z, q]\big)}{1 + d(\mathfrak{x}, z)}\right) + \beta(d(\mathfrak{x}, z)) \\ &= \beta(d(\mathfrak{x}, z)). \end{aligned}$$

Hence,

$$d(\mathfrak{x}, \mathscr{T}^n z) \leq \beta(d(\mathfrak{x}, z)). \tag{2.5}$$

since $\beta < 1$. Therefore, using definition of $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$, we get

$$\sup_{\mathfrak{x} \in \mathcal{F}}\{d(\mathfrak{x}, \mathscr{T}^n z)\} \leq \sup_{\mathfrak{x} \in \mathcal{F}}\{d(\mathfrak{x}, z)\} \leq D(\mathcal{F}, \mathcal{G}) + \varepsilon. \tag{2.6}$$

Hence $\mathscr{T}^n z \in P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Therefore $\mathscr{T}^n$ is a self map on $P_{\mathcal{G}}(\varepsilon)(\mathcal{F})$. Define $\mathscr{T}_n : P_{\mathcal{G}(\varepsilon)}(\mathcal{F}) \to P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ as $\mathscr{T}_n z = W(\mathscr{T}^n z, q, \lambda_n)$, $z \in P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ where $\{\lambda_n\}$ is a sequence in $(0, 1)$ such that $\lambda_n \to 1$. Consider

$$\begin{aligned} d(\mathscr{T}_n z, \mathscr{T}_n y) &= d(W(\mathscr{T}^n z, q, \lambda_n), W(\mathscr{T}^n y, q, \lambda_n)) \\ &\leq \lambda_n d(\mathscr{T}^n z, \mathscr{T}^n y) \\ &\leq \lambda_n\left[\alpha\left(\frac{d(z, [\mathscr{T}^n z, q])d(y, [\mathscr{T}^n y, q])}{1 + d(z, y)}\right) + \beta(d(z, y))\right] \\ &\leq \lambda_n\left[\alpha\left(\frac{d(z, \mathscr{T}_n z)d(y, \mathscr{T}_n y)}{1 + d(z, y)}\right) + \beta(d(z, y))\right], \end{aligned}$$

where $\lambda_n(\alpha + \beta) < 1, z, y \in P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Therefore by Theorem 2.4, each $\mathscr{T}_n$ has a unique fixed point z_n in $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Since $\{\mathscr{T}^n z_n\}$ is a sequence in the compact set $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$, there exists a subsequence $\{\mathscr{T}^{n_i} z_{n_i}\}$ of $\{\mathscr{T}^n z_n\}$ such that $\{\mathscr{T}^{n_i} z_{n_i}\} \to z \in P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Moreover,

$$z_{n_i} = \mathscr{T}_{n_i} z_{n_i} = W[\mathscr{T}^{n_i} z_{n_i}, q, \lambda_{n_i}] \to z.$$

As $\mathscr{T}$ is continuous, $\mathscr{T}^{n_i} z_{n_i} \to \mathscr{T}^{n_i} z$. By the uniqueness of the limit, we have $\lim_{n\to\infty} \mathscr{T}^{n_i} z = z$ and so $\lim_{n\to\infty} \mathscr{T}^{n_i+1} z = \mathscr{T} z$.

Now, we show that $d(z, \mathscr{T} z) = 0$. Consider

$$d(z, \mathscr{T} z) \leq d(z, \mathscr{T}^{n_i} z) + d(\mathscr{T}^{n_i} z, \mathscr{T}^{n_i+1} z) + d(\mathscr{T}^{n_i+1} z, \mathscr{T} z).$$

Letting $n \to \infty$, in the above inequality, and using $\mathscr{T}$ is asymptotically regular, we have $d(z, \mathscr{T}z) \to 0$. Therefore $\mathscr{T}z = z$. i.e. z is $\mathscr{T}$-invariant. □

Using Proposition 2.1 of Chandok and Narang [5], we have the following result.

Corollary 2.8. *Let $\mathscr{T}$ be asymptotically regular and continuous self map on a complete Takahashi space $(\mathscr{X}, d, W)$ satisfying inequality (2.4). Suppose that F is a nonempty bounded subset of $\mathscr{X}$ such that $\mathscr{T}\mathfrak{x} = \mathfrak{x}$ for all $\mathfrak{x} \in F \setminus \mathcal{G}$, where $\mathcal{G}$ a $\mathscr{T}$-invariant subset of $\mathscr{X}$. If $\mathcal{G}$ is ε-simultaneous approximatively compact with respect to $\mathcal{F}$ and $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ is starshaped, then it contains a $\mathscr{T}$-invariant point.*

For $\mathcal{F} = \{\mathfrak{x}\}$ and $\varepsilon = 0$, we have the following result on the set of best approximation.

Corollary 2.9. *Let $\mathscr{T}$ be asymptotically regular and continuous self mapping on a Takahashi space $(\mathscr{X}, d, W)$ and satisfying inequality (2.4). If $\mathcal{G}$ is an approximatively compact, p-starshaped, $\mathscr{T}$-invariant subset of $\mathscr{X}$ and $\mathfrak{x}$ a $\mathscr{T}$-invariant point and $P_{\mathcal{G}}(\mathfrak{x})$ is starshaped, then $P_{\mathcal{G}}(\mathfrak{x})$ contains a $\mathscr{T}$-invariant point.*

We now prove a result for $\mathscr{T}$-invariant points from the set of ε-simultaneous coapproximations.

Theorem 2.10. *Let $\mathscr{T}$ be asymptotically regular and continuous self mapping satisfying condition (A) and inequality (2.4) on a Takahashi space $(\mathscr{X}, d, W)$. Assume that $\emptyset \neq \mathcal{G} \subset \mathscr{X}$ and $\mathcal{F}$ is a nonempty bounded subset of $\mathscr{X}$ such that $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ is compact and q-starshaped. Then $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ contains a $\mathscr{T}$-invariant point.*

Proof. Let $g_0 \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Consider

$$d(\mathscr{T}^n g_0, g) + \varepsilon \leq d(g_0, g) + \varepsilon \leq \inf_{g \in \mathcal{G}} sup_{y \in \mathcal{F}} d(y, g),$$

and so $\mathscr{T}^n g_0 \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ i.e. $\mathscr{T}^n : R_{\mathcal{G}(\varepsilon)}(\mathcal{F}) \to R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Since $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ is q-starshaped, $W(z, q, \lambda) \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ for all $z \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$, $\lambda \in [0, 1]$. Let $\{\lambda_n\}$, $0 \leq \lambda_n < 1$, be a sequence of real numbers such that $\lambda_n \to 1$ as $n \to \infty$. Define $\mathscr{T}_n$ as $\mathscr{T}_n(z) = W(\mathscr{T}^n z, q, \lambda_n)$, $z \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Since $\mathscr{T}$ is a self mapping on $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ and $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ is starshaped, each $\mathscr{T}_n$ is a well defined and maps $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ into $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$.

Moreover,

$$\begin{aligned} d(\mathscr{T}_n y, \mathscr{T}_n z) &= d(W(\mathscr{T}^n y, q, \lambda_n), W(\mathscr{T}^n z, q, \lambda_n)) \\ &\leq \lambda_n d(\mathscr{T}^n y, \mathscr{T}^n z) \\ &\leq \lambda_n \left[\alpha \left(\frac{d(y, [\mathscr{T}^n y, q]) d(z, [\mathscr{T}^n z, q])}{1 + d(y, z)} \right) + \beta(d(y, z)) \right] \\ &\leq \lambda_n \left[\alpha \left(\frac{d(y, \mathscr{T}^n y) d(z, \mathscr{T}^n z)}{1 + d(y, z)} \right) + \beta(d(y, z)) \right], \end{aligned}$$

where $\lambda_n[\alpha + \beta] < 1$. So by Theorem 2.4 each $\mathscr{T}_n$ has a unique fixed point $u_n \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ i.e. $\mathscr{T}_n u_n = u_n$ for each n. Since $\{\mathscr{T}^n u_n\}$ is a sequence in the compact set $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$, there exists a subsequence $\{\mathscr{T}^{n_i} u_{n_i}\}$ of $\{\mathscr{T}^n u_n\}$ such that $\{\mathscr{T}^{n_i} u_{n_i}\} \to u \in R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$. Moreover,

$$u_{n_i} = \mathscr{T}_{n_i} u_{n_i} = W[\mathscr{T}^{n_i} u_{n_i}, q, \lambda_{n_i}] \to u.$$

As $\mathscr{T}$ is continuous, $\mathscr{T}^{n_i} u_{n_i} \to \mathscr{T}^{n_i} u$. By the uniqueness of the limit, we have $\lim_{n\to\infty} \mathscr{T}^{n_i} u = u$ and so $\lim_{n\to\infty} \mathscr{T}^{n_i+1} u = \mathscr{T} u$.

Now, we show that $d(u, \mathscr{T} u) = 0$. Since $\mathscr{T}$ is asymptotically regular, we have

$$d(u, \mathscr{T} u) \leq d(u, \mathscr{T}^{n_i} u) + d(\mathscr{T}^{n_i} u, \mathscr{T}^{n_i+1} u) + d(\mathscr{T}^{n_i+1} u, \mathscr{T} u) \to 0.$$

Therefore $\mathscr{T} u = u$. i.e. u is $\mathscr{T}$-invariant. □

Remark 2.11. *(1) By taking $\mathcal{F} = \{\mathfrak{x}_1, \mathfrak{x}_2\}$, $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathscr{X}$, the set $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ (respectively, $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$) is the set of ε-simultaneous approximation (respectively, ε-simultaneous coapproximation) to the pair of points $\mathfrak{x}_1, \mathfrak{x}_2$ and so we can obtain the results for such pair of points $P_{\mathcal{G}(\varepsilon)}(\mathcal{F})$ or $R_{\mathcal{G}(\varepsilon)}(\mathcal{F})$.*

(2) By taking $\mathcal{F} = \{\mathfrak{x}\}$, $\mathfrak{x} \in \mathscr{X}$, the set $P_{\mathcal{G}(\varepsilon)}(\mathfrak{x})$ (respectively, $R_{\mathcal{G}(\varepsilon)}(\mathfrak{x})$) is the set of ε-approximation(respectively, ε-coapproximation) to point $\mathfrak{x}$ and so we can obtain the results on the set of ε-approximation (or ε-coapproximation).

(3) By taking $\mathcal{F} = \{\mathfrak{x}\}$ and $\varepsilon = 0$, we can obtain the results on the set of best approximation (or best coapproximation).

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