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SYSTEMS WITH CONTROL IN DIFFERENTIABLE TANGENT BUNDLE

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Abstract. We study differentiable systems with control in tangent bundle, including the feedback of these systems.

1. Introduction

We recall that a control system is a dynamical system whose dynamical law has the form

$$\frac{dx}{dt} = F(x, u) \tag{1}$$

where the coordinates of the vector u are called control parameters and have the role of influencing the movement of the system [1]. We consider the case where $u : M \times \Re \to U$ with $(x, t) \to u(x, t)$, where M is a smooth $n-$ dimensional manifold and the set of parameters U is a subset of $\Re^m$. Then the dynamics is given by $F(x, u(x, t))$, and (1) becomes:

$$\frac{dx^i(t)}{dt} = F^i(x(t), u(t)), i = 1, 2, ..., n. \tag{2}$$

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2. Control systems with feedback

To a control system there is an associated set of functions:

$$y^\alpha = h^\alpha(x^1, x^2, ..., x^n), \alpha = 1, ..., p, \tag{3}$$

where $h^\alpha : M \to \Re$ are smooth functions called output functions. On the solution $x(t, x_0, u)$ of the system (2) these functions become $h^\alpha(x(t, x_0, u))$ and one can pose the problem of choosing the control functions such that the values of the output functions have certain properties.

In (2), the functions (F^i) are the components of a vector field on M, parameterized by the controls u.Â A change of coordinates $S : (x^i) \to (\tilde{x}^i), \tilde{x}^i = \tilde{x}^i(x^j), rang(\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n$ on the manifold M is called state transformation of the system (2). In the new coordinates, the system (2) has the form $\frac{d\tilde{x}^i}{dt} = \tilde{F}^i(\tilde{x}^i(x), n)$
or

$$\frac{d\tilde{x}^i}{dt} = \frac{\partial \tilde{x}^i}{\partial x^j}(S^{-1}(x))F^j(S^{-1}(x), u) \tag{4}$$

where S^{-1} is the inverse of S transformation.
The system (4) is a system with control in the variables $(\tilde{x}^i)$.

It is possible that with a transformation, the right-hand side of equation (4) becomes linear in x and u or the system (4) becomes a controllable system.

Problems of this type for affine systems are solved in [4].

We assume that the function $F : M \times U \to TM$ is smooth on $M \times U$. A local transformation on $M \times U$ is of the form $\begin{cases} \tilde{x}^i = \tilde{x}^i(x, u) \\ \tilde{u}^a = \tilde{u}^a(x, u) \end{cases}$.

We will consider only particular transformations, which will be called state and feedback transformations, as follows:

$$\begin{cases} \tilde{x}^i = \tilde{x}^i(x^j), \text{ with } rank\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n \\ \tilde{u}^a = \tilde{u}^a(x, u), \text{ with } rank\left(\frac{\partial u^a}{\partial u^b}\right) = m \end{cases} \tag{5}$$

The transformation $\tilde{u}^a = \tilde{u}^a(x, u)$ with $rank\left(\frac{\partial \tilde{u}^a}{\partial u^b}\right) = m$ is called the feedback transformation. It is reversible in the form $u = u(\tilde{u}, x)$.

For the affine systems shape transformations are considered as feedback transformations:

$$u^a = A^a(x) + B^a_b(x)\tilde{u}^b, \text{ where } rank(B^a_b(x) = m \tag{6}$$

Next, we will present a more general way of looking at the systems with control, regarding state transformations and feedback.

Let (E, π, M) a local tangent bundle with $\pi : E \to M$ a differentiable submersion. There is an open covering $\{O_\alpha\}_{a \in A}$ of manifold M and the diffeomorphisms $\phi_\alpha : \pi^{-1}(O_\alpha) \to O_\alpha$, with the property that the diagram below is commutative, that is the E manifold is locally diffemorphic with $O_\alpha \times U$:

$$\begin{array}{ccc} \pi^{-1}(O\alpha) & \longrightarrow & O\alpha \times U \\ \pi \Big\downarrow & pr_1 \swarrow & \\ O\alpha & & \end{array}$$

The open covering $\{O_\alpha\}$ can also be the covering of an atlas on M.

Let $(x^1, ..., x^n)$ be local coordinates on O_α and $(u^1, u^2 ..., u^m)$ be local coordinates on $\pi^{-1}(x)$, $x \in O_\alpha$. Then $(x^i \circ \phi_\alpha \circ \pi, u^a \circ \phi_\alpha)$ are local coordinates on E. We will identify $x^i \circ \phi_\alpha \circ \pi \equiv x^i, u^i \circ \phi_\alpha \equiv u^i$ and we can write $(x^i, u^a) \equiv (x, u)$ for local coordinates on E.

On the intersection $O_\alpha \cap O_\beta$, we have:

$$\begin{array}{ccc} \pi^{-1}(O\alpha \cap O\beta) & \xrightarrow{\phi\alpha} & O\alpha \cap O\beta \times U \\ \phi\beta \Big\downarrow & \swarrow & \\ O\alpha \cap O\beta \times U & & \end{array}$$

The diagram is closed by the application $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$, that is a diffeomorphism.

With x fixed in $O_\alpha \cap O_\beta$, the application $\phi_{\alpha\beta} : U \to U$ has the form: $\tilde{u}^a = \tilde{u}^a(x, u^b)$, where $rank\left(\frac{\partial \tilde{u}^a}{\partial u^b}(x, u)\right) = m$.

Let (TM, τ_M, M) be the tangent bundle of the manifold M.

Definition 2.1 A *differential system with control* in the tangent bundle (E, π, M, U) is a morphism of bundles $F : B \to TM$ such that the following diagram is commutative:

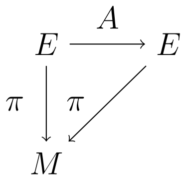
$$\begin{array}{ccc} B & \xrightarrow{F} & TM \\ \pi \Big\downarrow & \tau_M \swarrow & \\ M & & \end{array}$$

For an open set $O \subset M$ we have $\tau_M^{-1}(O) \simeq O \times U$ and $\pi_M^{-1}(O) \simeq O \times \Re^n$. The restriction of the application F to $\pi^{-1}(0)$ i.e. $F_0 : O \times U \to O \times \Re^n$ from the commutativity of the diagram (Definition 2.1) has the form $(x, u) \to (x, f(x, u))$, where $f(x, u)$ is a vector field on O that depends differentiably on the control variables $(u^1, u^2, ..., u^m)$.

One looks for the integral curves, which are the solutions of the controlled system

$$\frac{dx^i}{dt} = f^i(x^j(t), u^a(t)) \tag{7}$$

Definition 2.2. Let (E, M, F) a system with control. We call a feedback of the system with control (E, M, F) an isomorphism of bundles $A : E \to E$ such that the following diagram is commutative:



Locally, the application A has the form $(x, u) \to A(x, u) = (x, \alpha(x, u))$ and is called feedback transformation for system (7).

We consider in Definition 2.1 the vectorial bundle $(E, \pi, M, \Re^m)$. The applications $\phi_{\alpha\beta}$ are linear on $\Re^m$.

An affine system with control consists in a vectorial bundle $(E, \pi, M, \Re^m)$ and the morphism $F : E \to TM$ with the property that is affine form $\pi^{-1}(x) \to \tau_M^{-1}(x)$, for $x \in M$. Locally, F has the form $f(x, u) = (x, f_0(x) + g_a(x)u^a)$ and system (7) takes the particular form:

$$\frac{dx^i}{dt} = f_0^i(x(t)) + g_a^i(x(t))u^a(t), \tag{8}$$

where $f_0(x)$ and $g_a(x)$ are local vector fields on M.

A feedback transformation will be of the above form under the condition to be affine. So, locally, we will have $A(x, u) = (x^i, A_b^a(x) + B_b^a(x)u^b$ with $rank(B_b^a(x)) = m$.

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