“Vasile Alecsandri” University of Bacău
Faculty of Sciences
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COMPLEX DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS OF FINITE $(\alpha, \beta)-$ORDER

BENHARRAT BELAÏDI AND TANMAY BISWAS

Abstract. In this paper, we wish to investigate the complex higher order linear differential equations in which the coefficients are entire functions of (α, β)-order and obtain some results which improve and generalize some previous results of Tu et al. [33] as well as Belaïdi [2, 3, 4].

1. Introduction

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of entire and meromorphic functions and the theory of complex linear differential equations which are available in [15, 24, 38] and therefore we do not explain those in details. To study the generalized growth properties of entire and meromorphic functions, the concepts of different growth indicators such as the iterated p-order (see [23, 29]), the (p, q)-th order (see [20, 21]), (p, q)-φ order (see [30]) etc. are very useful and during the past decades, several authors made close investigations on the generalized growth properties of entire and meromorphic functions related to the above growth indicators in some different directions. The theory of complex linear equations has been developed since 1960s. Many authors have investigated the complex linear differential equations

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$$f^{(k)}(z) + A_{k-1}(z) f^{(k-1)}(z) + \cdots + A_0(z) f(z) = 0 \tag{1}$$

and

$$f^{(k)}(z) + A_{k-1}(z) f^{(k-1)}(z) + \cdots + A_0(z) f(z) = F(z) \tag{2}$$

and achieved many valuable results when the coefficients $A_0(z), ...,$ $A_{k-1}(z)$, $F(z)$ $(k \geq 2)$ in (1) or (2) are entire or meromorphic functions of finite order or finite iterated p-order or (p,q)-th order or (p,q)-φ order (e.g. [1], [9], [10], [13], [18], [23], [25]-[27], [30], [33]-[35], [37]).

In [12], Chyzhykov and Semochko showed that both definitions of iterated p-order and the (p,q)-th order have the disadvantage that they do not cover arbitrary growth (see [12, Example 1.4]). They used more general scale, called the φ-order (see [12]). In recent times, the concept of φ-order is used to study the growth of solutions of complex differential equations which extend and improve many previous results (see [5, 6, 12, 22]).

In [28], Mulyava et al. have used the concept of (α,β)-order or generalized order of an entire function in order to investigate the properties of solutions of a heterogeneous differential equation of the second order and obtained several interesting results. For details one may see [28].

In this paper, our aim is to make use of the concepts of entire functions of (α,β)-order or generalized order after giving a minor modification to the original definition (e.g. see, [28, 31]) in order to investigate the complex linear differential equations (1) or (2).

2. **Definitions and Notations**

First of all, let L be a class of continuous non-negative on $(-\infty,+\infty)$ function α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L_1$, if $\alpha \in L$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$. Further, we say that $\alpha \in L_2$, if $\alpha \in L$ and $\alpha(x+O(1)) = (1+o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x \to +\infty$ for each fixed $c \in (0,+\infty)$, i.e., α is slowly increasing function. Clearly $L_{si} \subset L_1$ and $L_2 \subset L_1$. Moreover, we assume that throughout the present paper α, β always denote the functions belonging to L_{si}, L_1 respectively and for an integer $p \geq 1$, $\alpha(\log^{[p]} x) = o(\beta(x))$, $\alpha(\log x) = o(\alpha(x))$ and $\alpha^{-1}(kx) = o(\alpha^{-1}(x))$ $(0 < k < 1)$ as $x \to +\infty$ unless otherwise specifically stated. The value

$$\sigma_{\alpha\beta}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M(r,f))}{\beta(\log r)} \ (\alpha \in L,\ \beta \in L)$$

is called [28, 31] (α, β)-order or generalized order of an entire function $f(z)$. For details about (α, β)-order one may see [28, 31].

If $\alpha \in L_{si}$ and $\beta \in L_2$ and $f(z)$ is an entire transcendental function, then (see [28])

$$\sigma_{\alpha\beta}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M(r,f))}{\beta(\log r)} = \limsup_{r\to+\infty} \frac{\alpha(T(r,f))}{\beta(\log r)}.$$

Now we rewrite the definition of the (α, β)-order of an entire function in the following way after giving a minor modification to the original definition (e.g. see, [28, 31]):

Definition 1. ([7]) *The (α, β)-order denoted by $\sigma_{(\alpha,\beta)}[f]$ of an entire function $f(z)$ is defined by*

$$\sigma_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} M(r,f))}{\beta(\log r)}.$$

Proposition 2. *If $f(z)$ is an entire function, then*

$$\sigma_{(\alpha(\log),\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T(r,f))}{\beta(\log r)} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[3]} M(r,f))}{\beta(\log r)}.$$

Proof. By the inequality $T(r,f) \le \log^+ M(r,f) \le \frac{R+r}{R-r} T(R,f)$ $(0 < r < R)$ (cf. [15]) for an entire function f, set $R = 2r$, we have

$$T(r,f) \le \log^+ M(r,f) \le 3T(2r,f).$$

By using $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ and $\beta((1+o(1))x) = (1+o(1))\beta(x)$ as $x \to +\infty$, one can easily obtain the proposition. □

Definition 3. *If $f(z)$ is an entire function satisfying $0 < \sigma_{(\alpha,\beta)}[f] < +\infty$, then for any $\gamma \in L$ and $\gamma(r) \neq r$,*

$$\begin{cases} \sigma_{(\gamma(\alpha),\beta)}[f] = +\infty \quad \text{when } \gamma(\alpha) \in L_{si} \text{ and } \lim\limits_{r\to+\infty} \frac{\gamma(\sigma\beta(\log r))}{\beta(\log r)} = +\infty \\ \text{for any } \sigma < \sigma_{(\alpha,\beta)}[f]; \\ \sigma_{(\gamma(\alpha),\beta)}[f] = 0 \quad \text{when } \gamma(\alpha) \in L_{si} \text{ and } \lim\limits_{r\to+\infty} \frac{\gamma(\sigma_1\beta(\log r))}{\beta(\log r)} = 0 \\ \text{for any } \sigma_1 > \sigma_{(\alpha,\beta)}[f]; \\ \sigma_{(\alpha,\gamma(\beta))}[f] = +\infty \quad \text{when } \gamma(\beta) \in L_1 \text{ and } \lim\limits_{r\to+\infty} \frac{\sigma\beta(\log r)}{\gamma(\beta(\log r))} = +\infty \\ \text{for any } \sigma < \sigma_{(\alpha,\beta)}[f]; \\ \sigma_{(\alpha,\gamma(\beta))}[f] = 0 \quad \text{when } \gamma(\beta) \in L_1 \text{ and } \lim\limits_{r\to+\infty} \frac{\sigma_1\beta(\log r)}{\gamma(\beta(\log r))} = 0 \\ \text{for any } \sigma_1 > \sigma_{(\alpha,\beta)}[f]; \\ \sigma_{(\gamma(\alpha),\gamma(\beta))}[f] = 1 \quad \text{when } \gamma \in L_{si}. \end{cases}$$

Remark 4. *An entire function $f(z)$ is said to have generalized index-pair (α, β) if $0 < \rho_{(\alpha,\beta)}[f] < +\infty$ and $\rho_{(\exp\alpha,\exp\beta)}[f]$ is not a nonzero finite number.*

Remark 5. *Definition 3 and Remark 4 extend the definition of index pair (p, q) of an entire function introduced by Juneja et al.* [20].

Remark 6. *Let $f(z)$ be an entire function of (α, β)-order σ and $f_1(z)$ be an entire function of (α_1, β_1)-order σ_1 and let either $\alpha(r) = \alpha_1(r)$ or $\lim\limits_{r\to+\infty} \frac{\alpha(r)}{\alpha_1(r)} = +\infty$. The following results about their comparative growth can be easily deduced:*
(i) If $\frac{\alpha_1(r)}{\alpha(r)} < \frac{\beta_1(r)}{\beta(r)}$, then the growth of $f(z)$ is slower than the growth of $f_1(z)$.
(ii) If $\frac{\alpha_1(r)}{\alpha(r)} > \frac{\beta_1(r)}{\beta(r)}$, then $f(z)$ grows faster than $f_1(z)$.
(iii) If $\alpha(r) = \alpha_1(r)$ and $\beta(r) = \beta_1(r)$, then $f(z)$ and $f_1(z)$ are of the same generalized index-pair (α, β). If $\sigma > \sigma_1$, then $f(z)$ grows faster than $f_1(z)$, and if $\sigma < \sigma_1$, then $f(z)$ grows slower than $f_1(z)$. If $\sigma = \sigma_1$, Definition 1 does not give any precise estimate about the relative growth of $f(z)$ and $f_1(z)$.

Similarly to Definition 1, we can also define the (α, β)-exponent of convergence of the zero sequence of a meromorphic function $f(z)$ in the following way:

Definition 7. ([7]) *The (α, β)-exponent of convergence of the zero sequence denoted by $\lambda_{(\alpha,\beta)}[f]$ of a meromorphic function $f(z)$ is defined* *by*

$$\lambda_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log n(r,1/f))}{\beta(\log r)}.$$

Analogously, the (α,β)-exponent of convergence of the distinct zero sequence denoted by $\overline{\lambda}_{(\alpha,\beta)}[f]$ of f is defined by

$$\overline{\lambda}_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log \overline{n}(r,1/f))}{\beta(\log r)}.$$

Accordingly, the values

$$\lambda_{(\alpha(\log),\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} n(r,1/f))}{\beta(\log r)}$$

and

$$\overline{\lambda}_{(\alpha(\log),\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \overline{n}(r,1/f))}{\beta(\log r)}$$

are respectively called as $(\alpha(\log),\beta)$-exponent of convergence of the zero sequence and $(\alpha(\log),\beta)$-exponent of convergence of the distinct zero sequence of a meromorphic function $f(z)$.

The linear measure of a set $E \subset [0,+\infty)$ is defined as $m(E) = \int_0^{+\infty} \chi_E(t)dt$. The logarithmic measure of a set $F \subset [1,+\infty)$ is defined by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t}dt$, where $\chi_G(t)$ is the characteristic function of a set G. The upper and lower densities of E are

$$\overline{dens}E = \limsup_{r\to+\infty} \frac{m(E\cap[0,r])}{r} \text{ and } \underline{dens}E = \liminf_{r\to+\infty} \frac{m(E\cap[0,r])}{r}.$$

Proposition 8. ([7]) *If $f(z)$ is a meromorphic function, then*

$$\begin{aligned}
\lambda_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(\log n(r,1/f))}{\beta(\log r)} = \limsup_{r\to+\infty} \frac{\alpha(\log N(r,1/f))}{\beta(\log r)}, \\
\overline{\lambda}_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(\log \overline{n}(r,1/f))}{\beta(\log r)} = \limsup_{r\to+\infty} \frac{\alpha(\log \overline{N}(r,1/f))}{\beta(\log r)}.
\end{aligned}$$

Proposition 9. *If $f(z)$ is a meromorphic function, then*

$$\begin{aligned}
\lambda_{(\alpha(\log),\beta)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} n(r,1/f))}{\beta(\log r)} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} N(r,1/f))}{\beta(\log r)}, \\
\overline{\lambda}_{(\alpha(\log),\beta)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \overline{n}(r,1/f))}{\beta(\log r)} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \overline{N}(r,1/f))}{\beta(\log r)}.
\end{aligned}$$

Proof. Without loss of generality, assume that $f(0) \neq 0$, then $N(r, 1/f) = \int_0^r \frac{n(t,1/f)}{t} dt$. We have

$$N(r,1/f) - N(r_0,1/f) = \int_{r_0}^{r} \frac{n(t,1/f)}{t} dt \leq n(r,1/f) \log \frac{r}{r_0} \ (0 < r_0 < r),$$

that is

$$N(r,1/f) \leq N(r_0,1/f) + n(r,1/f) \log \frac{r}{r_0} \ (0 < r_0 < r),$$

$$i.e., \ N(r,1/f) \leq \left(1 + \frac{N(r_0,1/f)}{n(r,1/f) \log \frac{r}{r_0}}\right) n(r,1/f) \log \frac{r}{r_0} \ (0 < r_0 < r),$$

which implies

$$(3) \qquad \log^{[2]} N(r,1/f) \leq (1 + o(1)) \left(\log^{[2]} n(r,1/f) + \log^{[3]} r\right),$$

then by (3), we have

$$\limsup_{r \to +\infty} \frac{\alpha(\log^{[2]} N(r,1/f))}{\beta(\log r)}$$

$$\leq \limsup_{r \to +\infty} \frac{\alpha\left((1 + o(1)) \left(\log^{[2]} n(r,1/f) + \log^{[3]} r\right)\right)}{\beta(\log r)}$$

$$= \limsup_{r \to +\infty} \frac{(1 + o(1))\, \alpha(\log^{[2]} n(r,1/f) + \log^{[3]} r)}{\beta(\log r)}$$

$$\leq \limsup_{r \to +\infty} \frac{\alpha(2 \max\{\log^{[2]} n(r,1/f), \ \log^{[3]} r\})}{\beta(\log r)}$$

$$= \limsup_{r \to +\infty} \frac{(1 + o(1)) \max\{\alpha(\log^{[2]} n(r,1/f)), \ \alpha(\log^{[3]} r)\}}{\beta(\log r)}$$

$$\leq \limsup_{r \to +\infty} \frac{\alpha(\log^{[2]} n(r,1/f)) + \alpha(\log^{[3]} r)}{\beta(\log r)}$$

$$\leq \limsup_{r \to +\infty} \frac{\alpha(\log^{[2]} n(r,1/f))}{\beta(\log r)} + \limsup_{r \to +\infty} \frac{\alpha(\log^{[3]} r)}{\beta(\log r)}$$

$$(4) \qquad = \limsup_{r \to +\infty} \frac{\alpha(\log^{[2]} n(r,1/f))}{\beta(\log r)},$$

since $\alpha(\log^{[p]} x) = o(\beta(x))$, $p \geq 1$ as $x \to +\infty$, we have $\frac{\alpha(\log^{[3]} r)}{\beta(\log r)} \to 0$ as $r \to +\infty$.

On the other hand, we have

$$N(er,1/f)=\int_0^{er}\frac{n(t,1/f)}{t}dt\geq\int_r^{er}\frac{n(t,1/f)}{t}dt$$

$$\geq n(r,1/f)\log e=n(r,1/f). \tag{5}$$

By (5), we obtain

$$\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}N(er,1/f))}{\beta(\log r)}\geq\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}n(r,1/f))}{\beta(\log r)}.$$

By using the condition $\beta((1+o(1))x)=(1+o(1))\beta(x)$ as $x\to+\infty$, we can write

$$\begin{aligned}\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}N(er,1/f))}{\beta(\log r)} &= \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}N(er,1/f))}{\beta(\log er-\log e)}\\ &= \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}N(er,1/f))}{\beta\left(\left(1-\frac{1}{\log er}\right)\log er\right)}\\ &= \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}N(er,1/f))}{\beta((1+o(1))\log er)}\\ &= \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}N(er,1/f))}{(1+o(1))\,\beta(\log er)}\\ &= \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}N(r,1/f))}{\beta(\log r)},\end{aligned}$$

it follows that

$$\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}N(r,1/f))}{\beta(\log r)}\geq\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}n(r,1/f))}{\beta(\log r)}. \tag{6}$$

By (4) and (6), it is easy to see that

$$\lambda_{(\alpha(\log),\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}n(r,1/f))}{\beta(\log r)}=\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}N(r,1/f))}{\beta(\log r)}.$$

By the same proof above, we can obtain the conclusion

$$\overline{\lambda}_{(\alpha(\log),\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\overline{n}(r,1/f))}{\beta(\log r)}=\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\overline{N}(r,1/f))}{\beta(\log r)}.$$

□

Proposition 10. *Let $f_1(z)$, $f_2(z)$ be non-constant meromorphic functions with $\sigma_{(\alpha(\log),\beta)}[f_1]$ and $\sigma_{(\alpha(\log),\beta)}[f_2]$ as their $(\alpha(\log),\beta)$-order. Then*
(i) $\sigma_{(\alpha(\log),\beta)}[f_1 \pm f_2] \leq \max\{\sigma_{(\alpha(\log),\beta)}[f_1], \sigma_{(\alpha(\log),\beta)}[f_2]\}$;
(ii) $\sigma_{(\alpha(\log),\beta)}[f_1 \cdot f_2] \leq \max\{\sigma_{(\alpha(\log),\beta)}[f_1], \sigma_{(\alpha(\log),\beta)}[f_2]\}$;
(iii) If $\sigma_{(\alpha(\log),\beta)}[f_1] \neq \sigma_{(\alpha(\log),\beta)}[f_2]$, *then*

$$\sigma_{(\alpha(\log),\beta)}[f_1 \pm f_2] = \max\{\sigma_{(\alpha(\log),\beta)}[f_1], \sigma_{(\alpha(\log),\beta)}[f_2]\};$$

(iv) If $\sigma_{(\alpha(\log),\beta)}[f_1] \neq \sigma_{(\alpha(\log),\beta)}[f_2]$, *then*

$$\sigma_{(\alpha(\log),\beta)}[f_1 \cdot f_2] = \max\{\sigma_{(\alpha(\log),\beta)}[f_1], \sigma_{(\alpha(\log),\beta)}[f_2]\}.$$

Proof. Without loss of generality, we assume that

$$\sigma_{(\alpha(\log),\beta)}[f_1] \leq \sigma_{(\alpha(\log),\beta)}[f_2] < +\infty.$$

From the definition of $(\alpha(\log),\beta)$-order, for any given $\varepsilon > 0$, we obtain for all sufficiently large values of r that

$$T(r, f_1) < \exp^{[2]}(\alpha^{-1}((\sigma_{(\alpha(\log),\beta)}[f_1] + \varepsilon)\beta(\log r))) \tag{7}$$

and

$$T(r, f_2) < \exp^{[2]}(\alpha^{-1}((\sigma_{(\alpha(\log),\beta)}[f_2] + \varepsilon)\beta(\log r))). \tag{8}$$

Since $T(r, f_1 \pm f_2) \leq T(r, f_1) + T(r, f_2) + \log 2$ for all large r, we get from (7) and (8) for all sufficiently large values of r that

$$\begin{aligned} T(r, f_1 \pm f_2) &< 2\exp^{[2]}(\alpha^{-1}((\sigma_{(\alpha(\log),\beta)}[f_2] + \varepsilon)\beta(\log r))) + \log 2 \\ \text{i.e., } T(r, f_1 \pm f_2) &< 3\exp^{[2]}(\alpha^{-1}((\sigma_{(\alpha(\log),\beta)}[f_2] + \varepsilon)\beta(\log r))) \\ \text{i.e., } \frac{1}{3}T(r, f_1 \pm f_2) &< \exp^{[2]}(\alpha^{-1}((\sigma_{(\alpha(\log),\beta)}[f_2] + \varepsilon)\beta(\log r))) \end{aligned}$$

$$\begin{aligned} \text{i.e., } (1 + o(1))\log^{[2]} T(r, f_1 \pm f_2) &< \alpha^{-1}((\sigma_{(\alpha(\log),\beta)}[f_2] + \varepsilon)\beta(\log r)) \\ \text{i.e., } \alpha((1 + o(1))\log^{[2]} T(r, f_1 \pm f_2)) &< (\sigma_{(\alpha(\log),\beta)}[f_2] + \varepsilon)\beta(\log r) \\ \text{i.e., } (1 + o(1))\alpha(\log^{[2]} T(r, f_1 \pm f_2)) &< (\sigma_{(\alpha(\log),\beta)}[f_2] + \varepsilon)\beta(\log r), \end{aligned}$$

which implies that

$$\limsup_{r\to+\infty} \frac{(1 + o(1))\alpha(\log^{[2]} T(r, f_1 \pm f_2))}{\beta(\log r)} \leq \sigma_{(\alpha(\log),\beta)}[f_2] + \varepsilon$$

holds for any given $\varepsilon > 0$. Hence

$$\sigma_{(\alpha(\log),\beta)}[f_1 \pm f_2] \leq \max\{\sigma_{(\alpha(\log),\beta)}[f_1], \sigma_{(\alpha(\log),\beta)}[f_2]\}. \tag{9}$$

Further without loss of any generality, let

$$\sigma_{(\alpha(\log),\beta)}[f_1] < \sigma_{(\alpha(\log),\beta)}[f_2] < +\infty$$

and $f(z) = f_1(z) \pm f_2(z)$. Then in view of (9) we get that $\sigma_{(\alpha(\log),\beta)}[f] \leq \sigma_{(\alpha(\log),\beta)}[f_2]$. As, $f_2(z) = \pm(f(z) - f_1(z))$ and in this case we obtain that $\sigma_{(\alpha(\log),\beta)}[f_2] \leq \max\ \{\sigma_{(\alpha(\log),\beta)}[f],\ \sigma_{(\alpha(\log),\beta)}[f_1]\}$.As we assume that $\sigma_{(\alpha(\log),\beta)}[f_1] < \sigma_{(\alpha(\log),\beta)}[f_2]$, therefore we have $\sigma_{(\alpha(\log),\beta)}[f_2] \leq \sigma_{(\alpha(\log),\beta)}[f]$ and hence

$$\sigma_{(\alpha(\log),\beta)}[f] = \sigma_{(\alpha(\log),\beta)}[f_2] = \max\{\sigma_{(\alpha(\log),\beta)}[f_1], \sigma_{(\alpha(\log),\beta)}[f_2]\}.$$

Similarly, from $T(r, f_1 \cdot f_2) \leq T(r, f_1) + T(r, f_2)$ for all large r, we can also get

$$\sigma_{(\alpha(\log),\beta)}[f_1 \cdot f_2] \leq \max\{\sigma_{(\alpha(\log),\beta}[f_1],\ \sigma_{(\alpha(\log),\beta)}[f_2]\}$$

and if $\sigma_{(\alpha(\log),\beta)}[f_1] \neq \sigma_{(\alpha(\log),\beta)}[f_2]$, then

$$\sigma_{(\alpha(\log),\beta)}[f_1 \cdot f_2] = \max\{\sigma_{(\alpha(\log),\beta)}[f_1],\ \sigma_{(\alpha(\log),\beta)}[f_2]\},$$

which completes the proof of Proposition 10. □

3. **Main Results**

In this section we present our main results which considerably extend the results of Tu et al. [33] as well as Belaïdi [2, 3, 4].

Theorem 11. *Let* $A_0(z),\ A_1(z), ..., A_{k-1}(z)$ *be entire functions with* $A_0(z) \not\equiv 0$ *such that for real constants* $a,\ b,\ \mu,\ \theta_1,\ \theta_2$ *with* $0 \leq b < a, \mu > 0,\ \theta_1 < \theta_2$*, we have*

$$|A_0(z)| \geq \exp\left\{a \exp\left(\alpha^{-1}\left(\mu\beta\left(\log|z|\right)\right)\right)\right\} \tag{10}$$

and

$$|A_j(z)| \leq \exp\left\{b \exp\left(\alpha^{-1}\left(\mu\beta\left(\log|z|\right)\right)\right)\right\},\quad j = 1, ..., k-1 \tag{11}$$

as $z \to \infty$ *with* $\theta_1 \leq \arg z \leq \theta_2$. *Then* $\sigma_{(\alpha(\log),\beta)}[f] \geq \mu$ *holds for all non-trivial solutions of* (1).

Theorem 12. *Let* H *be a set of complex numbers satisfying* $\overline{dens}\{|z| : z \in H\} > 0$, *and let* $A_0(z),\ A_1(z), ..., A_{k-1}(z)$ *be entire functions that satisfy* (10) *and* (11) *as* $z \to \infty$ *for* $z \in H$, *where* $0 \leq b < a,\ \mu > 0$. *Then every solution* $f(z) \not\equiv 0$ *of* (1) *satisfies* $\sigma_{(\alpha(\log),\beta)}[f] \geq \mu$.

Theorem 13. *Let* H *be a set of complex numbers satisfying* $\overline{dens}\{|z| : z \in H\} > 0$, *and let* $A_0(z),\ A_1(z), ..., A_{k-1}(z)$ *be entire functions of* (α, β)*-order with* $\max\{\sigma_{(\alpha,\beta)}[A_j] : j = 1, ..., k-1\} \leq \sigma_{(\alpha,\beta)}[A_0] = \sigma < +\infty$ *such that for some constants* $0 \leq b < a$ *and for any given* $\varepsilon > 0$, *we have*

$$|A_0(z)| \geq \exp\left\{a \exp\left(\alpha^{-1}\left(\left(\sigma - \varepsilon\right)\beta\left(\log|z|\right)\right)\right)\right\} \tag{12}$$

and

(13)
$$|A_j(z)| \leq \exp\left\{b \exp\left(\alpha^{-1}\left((\sigma-\varepsilon)\,\beta\left(\log|z|\right)\right)\right)\right\}, \quad j=1,...,k-1$$

as $z \to \infty$ *for* $z \in H$. *Then every solution* $f(z) \not\equiv 0$ *of* (1) *satisfies* $\sigma_{(\alpha(\log),\beta)}[f] = \sigma_{(\alpha,\beta)}[A_0] = \sigma$.

Theorem 14. *Let* H, $A_0(z)$, $A_1(z), ..., A_{k-1}(z)$ *satisfy the hypotheses of Theorem 13, and let* $F(z) \not\equiv 0$ *be an entire function of* (α,β)*-order.*
(i) If $\sigma_{(\alpha(\log),\beta)}[F] < \sigma_{(\alpha,\beta)}[A_0]$, *then every solution* $f(z)$ *of* (2) *satisfies* $\overline{\lambda}_{(\alpha(\log),\beta)}[f] = \lambda_{(\alpha(\log),\beta)}[f] = \sigma_{(\alpha(\log),\beta)}[f] = \sigma$, *with at most one exceptional solution* $f_0(z)$ *satisfying* $\sigma_{(\alpha(\log),\beta)}[f_0] < \sigma$.
(ii) If $\sigma_{(\alpha,\beta)}[A_0] \leq \sigma_{(\alpha(\log),\beta)}[F] < +\infty$, *then every solution* $f(z)$ *of* (2) *satisfies* $\sigma_{(\alpha(\log),\beta)}[f] = \sigma_{(\alpha(\log),\beta)}[F]$.

Remark 15. *For some related results in the whole complex plane for the* (α,β,γ)*-order, see* [8].

4. Some Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 16. *(*[14]*) Let* $f(z)$ *be a nontrivial entire function, and let* $\kappa > 1$ *and* $\varepsilon > 0$ *be given constants. Then there exist a constant* $c > 0$ *and a set* $E_1 \subset [0,+\infty)$ *having finite linear measure such that for all* z *satisfying* $|z| = r \notin E_1$, *we have*

(14)
$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq c[T(\kappa r,f)r^{\varepsilon}\log T(\kappa r,f)]^k \quad (k \in \mathbb{N}).$$

Lemma 17. *(*[16, 36]*) Let* $f(z)$ *be a transcendental entire function, and let* z *be a point with* $|z| = r$ *at which* $|f(z)| = M(r,f)$. *Then, for all* $|z|$ *outside a set* E_2 *of* r *of finite logarithmic measure, we have*

(15)
$$\frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu(r,f)}{z}\right)^k (1+o(1)) \quad (k \in \mathbb{N},\ r \notin E_2),$$

where $\nu(r,f)$ *is the central index of* $f(z)$.

Lemma 18. *(*[17]*, Theorems 1.9 and 1.10, or* [19]*, Satz 4.3 and 4.4) Let* $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ *be any entire function,* $\mu(r,f)$ *be the maximum term, i.e.,* $\mu(r,f) = \max\{|a_n|r^n; n=0,1,...\}$, *and* $\nu(r,f)$ *be the central index of* $f(z)$.

(i) If $|a_0| \neq 0$, *then*

$$\log \mu(r,f) = \log|a_0| + \int_0^r \frac{\nu(t,f)}{t} dt. \tag{16}$$

(ii) For $r < R$, *we have*

$$M(r,f) < \mu(r,f)\left(\nu(R,f) + \frac{R}{R-r}\right). \tag{17}$$

Lemma 19. *Let* $f(z)$ *be an entire function satisfying* $\sigma_{(\alpha(\log),\beta)}[f] = \sigma_1$, *and let* $\nu(r,f)$ *be the central index of* $f(z)$. *Then*

$$\limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \nu(r,f))}{\beta(\log r)} = \sigma_1.$$

Proof. In view of the first part of Lemma 18, one may obtain that

$$\log \mu(2r,f) = \log|a_0| + \int_0^{2r} \frac{\nu(t,f)}{t} dt$$

$$\geq \log|a_0| + \int_r^{2r} \frac{\nu(t,f)}{t} dt \geq \log|a_0| + \nu(r,f)\log 2. \tag{18}$$

Also by Cauchy's inequality, it is well known that (cf. [32])

$$\mu(r,f) \leq M(r,f). \tag{19}$$

Therefore, one may obtain from (18) and (19) that

$$\nu(r,f)\log 2 \leq \log M(2r,f) - \log|a_0|.$$

Thus from above, we get that

$$\log \nu(r,f) + \log^{[2]} 2 \leq \log^{[2]} M(2r,f) + \log\left(1 - \frac{\log|a_0|}{\log M(2r,f)}\right)$$

$$i.e.,\ \log^{[2]} \nu(r,f) + \log\left(1 + \frac{\log^{[2]} 2}{\log^{[2]} \nu(r,f)}\right) \leq \log^{[3]} M(2r,f)$$

$$+ \log\left(1 + \frac{\log\left(1 - \frac{\log|a_0|}{\log M(2r,f)}\right)}{\log^{[3]} M(2r,f)}\right)$$

$$i.e.,\ \limsup_{r\to+\infty} \frac{\alpha((1+o(1))\log^{[2]} \nu(r,f))}{\beta(\log r)}$$

$$\leq \limsup_{r\to+\infty}\frac{\alpha((1+o(1))\log^{[3]}M(2r,f))}{\beta(\log 2r-\log 2)}$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{(1+o(1))\alpha(\log^{[2]}\nu(r,f))}{\beta(\log r)}$$

$$\leq \limsup_{r\to+\infty}\frac{(1+o(1))\alpha(\log^{[3]}M(2r,f))}{\beta((1+o(1))\log 2r)}$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\nu(r,f))}{\beta(\log r)}\leq \limsup_{r\to+\infty}\frac{\alpha(\log^{[3]}M(2r,f))}{(1+o(1))\beta(\log 2r)}$$

$$i.e.,\ \sigma_1=\limsup_{r\to+\infty}\frac{\alpha(\log^{[3]}M(2r,f))}{\beta(\log 2r)}\geq \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\nu(r,f))}{\beta(\log r)}. \tag{20}$$

Further for any constant K, one may get from the second part of Lemma 18, that (cf. [11])

$$\log M(r,f)<\nu(r,f)\log r+\log\nu(2r,f)+K.$$

Therefore from above we obtain that

$$\begin{aligned}\log M(r,f) &< \nu(2r,f)\log r+\nu(2r,f)+K\\ i.e.,\ \log M(r,f) &< \nu(2r,f)(1+\log r)+K\\ i.e.,\ \log M(r,f) &< \nu(2r,f)\log(e\cdot r)+K\end{aligned}$$

$$i.e.,\ \log^{[2]}M(r,f)<\log\nu(2r,f)+\log^{[2]}(e\cdot r)$$
$$+\log\left(1+\frac{K}{\nu(2r,f)\log(e\cdot r)}\right)$$

$$i.e.,\ \log^{[3]}M(r,f) < (1+o(1))\log^{[2]}\nu(2r,f)$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(\log^{[3]}M(r,f))}{\beta(\log r)} \leq \limsup_{r\to+\infty}\frac{\alpha((1+o(1))\log^{[2]}\nu(2r,f))}{\beta(\log 2r-\log 2)}$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(\log^{[3]}M(r,f))}{\beta(\log r)}\leq \limsup_{r\to+\infty}\frac{(1+o(1))\alpha(\log^{[2]}\nu(r,f))}{(1+o(1))\beta(\log r)}$$

$$i.e.,\ \sigma_1=\limsup_{r\to+\infty}\frac{\alpha(\log^{[3]}M(r,f))}{\beta(\log r)}\leq \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\nu(r,f))}{\beta(\log r)}. \tag{21}$$

Combining (20) and (21), we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\nu(r,f))}{\beta(\log r)}=\sigma_1.$$

This proves the lemma. □

Lemma 20. *Let f be a transcendental entire function. Then* $\sigma_{(\alpha(\log),\beta)}[f] = \sigma_{(\alpha(\log),\beta)}[f']$.

Proof. By Cauchy's integral formula, we have

$$f'(z) = \frac{1}{2\pi i} \oint\limits_{\Gamma} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta,$$

where $\Gamma = \{\zeta : |\zeta - z| = R - r\}$, $|z| = r < R$. Set $\zeta - z = (R - r)\, e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$, $d\zeta = (R - r)\, ie^{i\theta} d\theta$. Since $\max\{|f\,(\zeta)\,| : \zeta \in \Gamma\} \leq M(R, f)$, then we obtain

$$M(r, f') = |f'(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(\zeta)|}{|\zeta - z|^2}(R - r) d\theta \leq \frac{M(R, f)}{R - r}.$$

Set $R = r + 1$, it follows that

$$M(r, f') \leq M(r + 1, f).$$

Then

$$\sigma_{(\alpha(\log),\beta)}[f'] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[3]} M(r, f'))}{\beta(\log r)}$$

$$\leq \limsup_{r\to+\infty} \left(\frac{\alpha(\log^{[3]} M(r + 1, f))}{\beta(\log\,(r + 1))} \cdot \frac{\beta(\log\,(r + 1))}{\beta(\log r)} \right)$$

$$= \limsup_{r\to+\infty} \left(\frac{\alpha(\log^{[3]} M(r + 1, f))}{\beta(\log\,(r + 1))} \cdot \frac{\beta\left(\left(1 + \frac{\log\left(1+\frac{1}{r}\right)}{\log r}\right) \log r\right)}{\beta(\log r)} \right)$$

$$= \limsup_{r\to+\infty} \left(\frac{\alpha(\log^{[3]} M(r + 1, f))}{\beta(\log\,(r + 1))} \cdot \frac{\beta\,((1 + o\,(1)) \log r)}{\beta(\log r)} \right).$$

Since $\beta((1 + o(1))x) = (1 + o(1))\beta(x)$ as $x \to +\infty$, from above we obtain that

$$\sigma_{(\alpha(\log),\beta)}[f'] \leq \sigma_{(\alpha(\log),\beta)}[f]. \tag{22}$$

On the other hand, for an entire function $f(z)$, we have $f(z) - f(0) = \int\limits_0^z f'(t)dt$, where the integral being taken along the straight line from 0 to z, so we obtain that

$$M(r, f) \leq \left| \int\limits_0^z f'(t)dt \right| + |f(0)| \leq rM(r, f') + |f(0)|\,.$$

Therefore from above we get that

$$\log^{[3]} M(r,f) \leq (1+o(1)) \log^{[3]} M(r,f')$$

$$i.e., \ \sigma_{(\alpha(\log),\beta)}[f] \leq \sigma_{(\alpha(\log),\beta)}[f']. \tag{23}$$

Hence the lemma follows from (22) and (23). □

Remark 21. *In the line of Lemma 20 one can easily deduce that* $\sigma_{(\alpha,\beta)}[f] = \sigma_{(\alpha,\beta)}[f']$, *where* $f(z)$ *is an entire transcendental function.*

Lemma 22. *Let* $f(z)$ *be an entire function of* (α,β)*-order satisfying* $\sigma_{(\alpha,\beta)}[f] = \sigma$. *Then there exists a set* $E_3 \subset (1,+\infty)$ *having infinite logarithmic measure such that, we have*

$$\lim_{\substack{r\to+\infty \\ r\in E_3}} \frac{\alpha(\log T(r,f))}{\beta(\log r)} = \sigma.$$

Proof. By Definition 1, there exists an increasing sequence $\{r_n\}_{n=1}^{+\infty}$ tending to $+\infty$ that satisfying $(1+\frac{1}{n})r_n < r_{n+1}$ and

$$\lim_{r_n\to+\infty} \frac{\alpha(\log T(r_n,f))}{\beta(\log r_n)} = \sigma_{(\alpha,\beta)}[f] = \sigma.$$

So, there exists an n_1 $(\in \mathbb{N})$ such that for $n \geq n_1$ and for any $r \in E_3 = \bigcup_{n=n_1}^{+\infty} [r_n, (1+\frac{1}{n})r_n]$, we have

$$\frac{\alpha(\log T(r_n,f))}{\beta(\log((1+\frac{1}{n})r_n))} \leq \frac{\alpha(\log T(r,f))}{\beta(\log r)} \leq \frac{\alpha(\log T((1+\frac{1}{n})r_n,f))}{\beta(\log r_n)}. \tag{24}$$

By (24), we get that

$$\lim_{r_n\to+\infty} \left(\frac{\alpha(\log T(r_n,f))}{\beta(\log r_n)} \cdot \frac{\beta(\log r_n)}{\beta(\log((1+\frac{1}{n})r_n))} \right) \leq \lim_{\substack{r\to+\infty \\ r\in E_3}} \frac{\alpha(\log T(r,f))}{\beta(\log r)}. \tag{25}$$

By (25) and $\beta((1+o(1))x) = (1+o(1))\beta(x)$ as $x \to +\infty$, we obtain that

$$\lim_{\substack{r\to+\infty \\ r\in E_3}} \frac{\alpha(\log T(r,f))}{\beta(\log r)} \geq \sigma. \tag{26}$$

On the other hand, by (24) and $\beta((1+o(1))x) = (1+o(1))\beta(x)$ as $x \to +\infty$, we have

$$\lim_{\substack{r\to+\infty \\ r\in E_3}} \frac{\alpha(\log T(r,f))}{\beta(\log r)}$$

$$(27)\qquad \leq \lim_{r_n\to+\infty}\left(\frac{\alpha(\log T((1+\frac{1}{n})r_n,f))}{\beta(\log((1+\frac{1}{n})r_n))}\cdot\frac{\beta(\log((1+\frac{1}{n})r_n))}{\beta(\log r_n)}\right)\leq\sigma.$$

Therefore, by (26) and (27), we get that

$$\lim_{\substack{r\to+\infty\\ r\in E_3}}\frac{\alpha(\log T(r,f))}{\beta(\log r)}=\sigma,$$

where

$$lm(E_3)=\sum_{n=n_1}^{+\infty}\int_{r_n}^{(1+\frac{1}{n})r_n}\frac{dt}{t}=\sum_{n=n_1}^{+\infty}\log(1+\frac{1}{n})=+\infty.$$

This complete the proof. □

Lemma 23. *Let $f(z)$ be an entire function of (α,β)-order with $\sigma_{(\alpha,\beta)}[f]=\sigma>0$, and let $f_1(z)$ be an entire function of (α_1,β_1)-order with $\sigma_{(\alpha_1,\beta_1)}[f_1]=\sigma_1<+\infty$. If $\sigma_{(\alpha,\beta)}[f]$ and $\sigma_{(\alpha_1,\beta_1)}[f_1]$ satisfy one of the following conditions:*
(i) $\alpha(r)=\alpha_1(r)$, $\beta(r)=\beta_1(r)$ and $\sigma_{(\alpha_1,\beta_1)}[f_1]<\sigma_{(\alpha,\beta)}[f]$;
(ii) $\lim\limits_{r\to+\infty}\frac{\alpha_1^{-1}(r)}{\alpha^{-1}(r)}=0$, $\beta(r)=\beta_1(r)$ and $\sigma_{(\alpha_1,\beta_1)}[f_1]<\sigma_{(\alpha,\beta)}[f]$;
then there exists a set $E_4\subset(1,+\infty)$ having infinite logarithmic measure such that, we have

$$\lim_{\substack{r\to+\infty\\ r\in E_4}}\frac{T(r,f_1)}{T(r,f)}=0.$$

Proof. (i) By Definition 1, we obtain for all sufficiently large values of r that

$$(28)\qquad T(r,f_1)\leq\exp\left\{\alpha^{-1}((\sigma_1+\varepsilon)\beta(\log r))\right\}.$$

By $\sigma_{(\alpha,\beta)}[f]=\sigma$ and Lemma 22, there exists a set E_4 of infinite logarithmic measure satisfying

$$\lim_{\substack{r\to+\infty\\ r\in E_4}}\frac{\alpha(\log T(r,f))}{\beta(\log r)}=\sigma.$$

Then

$$(29)\qquad T(r,f)\geq\exp\left\{\alpha^{-1}((\sigma-\varepsilon)\beta(\log r))\right\}\quad(r\in E_4),$$

where $0<2\varepsilon<\sigma-\sigma_1$. Now by (28) and (29), we obtain that

$$\frac{T(r,f_1)}{T(r,f)}\leq\frac{\exp\left\{\alpha^{-1}\left(\left(\sigma_1+\varepsilon\right)\beta\left(\log r\right)\right)\right\}}{\exp\left\{\alpha^{-1}\left(\left(\sigma-\varepsilon\right)\beta\left(\log r\right)\right)\right\}}$$

$$= \exp\left\{\alpha^{-1}\left(\left(\sigma_1+\varepsilon\right)\beta\left(\log r\right)\right)-\alpha^{-1}\left(\left(\sigma-\varepsilon\right)\beta\left(\log r\right)\right)\right\}$$
$$= \exp\left\{\alpha^{-1}\left(\left(\sigma-\varepsilon\right)\beta\left(\log r\right)\right)\left(\frac{\alpha^{-1}\left(\left(\sigma_1+\varepsilon\right)\beta\left(\log r\right)\right)}{\alpha^{-1}\left(\left(\sigma-\varepsilon\right)\beta\left(\log hr\right)\right)}-1\right)\right\}$$
$$= \exp\left\{\alpha^{-1}\left(\left(\sigma-\varepsilon\right)\beta\left(\log r\right)\right)\left(\frac{\alpha^{-1}\left(\frac{\sigma_1+\varepsilon}{\sigma-\varepsilon}\left(\sigma-\varepsilon\right)\beta\left(\log r\right)\right)}{\alpha^{-1}\left(\left(\sigma-\varepsilon\right)\beta\left(\log r\right)\right)}-1\right)\right\}$$
$$= \exp\left\{\alpha^{-1}\left(\left(\sigma-\varepsilon\right)\beta\left(\log r\right)\right)\left(\frac{\alpha^{-1}\left(k\left(\sigma-\varepsilon\right)\beta\left(\log r\right)\right)}{\alpha^{-1}\left(\left(\sigma-\varepsilon\right)\beta\left(\log r\right)\right)}-1\right)\right\}$$
$$\to 0, r\to+\infty \quad (r\in E_4),\ 0<k=\frac{\sigma_1+\varepsilon}{\sigma-\varepsilon}<1.$$

Therefore, we get from above that

$$\lim_{r\to+\infty}\frac{T(r,f_1)}{T(r,f)}=0 \quad (r\in E_4).$$

(ii) By definition, we obtain for all sufficiently large values of r that

$$T(r,f_1)\le \exp\left\{\alpha_1^{-1}\left(\left(\sigma_1+\varepsilon\right)\beta\left(\log r\right)\right)\right\}. \tag{30}$$

Now by (29) and (30), for any given ε with $0<2\varepsilon<\sigma-\sigma_1.$ we obtain that

$$\frac{T(r,f_1)}{T(r,f)}\le\frac{\exp\left\{\alpha_1^{-1}\left(\left(\sigma_1+\varepsilon\right)\beta\left(\log r\right)\right)\right\}}{\exp\left\{\alpha^{-1}\left(\left(\sigma-\varepsilon\right)\beta\left(\log r\right)\right)\right\}}$$
$$=\frac{\exp\left\{\alpha_1^{-1}\left(\left(\sigma_1+\varepsilon\right)\beta\left(\log r\right)\right)\right\}}{\exp\left\{\alpha^{-1}\left(\left(\sigma_1+\varepsilon\right)\beta\left(\log r\right)\right)\right\}}\cdot\frac{\exp\left\{\alpha^{-1}\left(\left(\sigma_1+\varepsilon\right)\beta\left(\log r\right)\right)\right\}}{\exp\left\{\alpha^{-1}((\sigma-\varepsilon)\beta(\log r))\right\}}$$
$$= \exp\left\{\alpha^{-1}\left(\left(\sigma_1+\varepsilon\right)\beta\left(\log r\right)\right)\left(\frac{\alpha_1^{-1}\left(\left(\sigma_1+\varepsilon\right)\beta\left(\log r\right)\right)}{\alpha^{-1}\left(\left(\sigma_1+\varepsilon\right)\beta\left(\log r\right)\right)}-1\right)\right\}$$
$$\times\frac{\exp\left\{\alpha^{-1}\left(\left(\sigma_1+\varepsilon\right)\beta\left(\log r\right)\right)\right\}}{\exp\left\{\alpha^{-1}((\sigma-\varepsilon)\beta(\log r))\right\}}.$$

Since $\lim\limits_{r\to+\infty}\frac{\alpha_1^{-1}(r)}{\alpha^{-1}(r)}=0$ and $\lim\limits_{r\to+\infty}\frac{\alpha^{-1}(kr)}{\alpha^{-1}(r)}=0$ $(0<k<1)$, then by the inequality obove, we obtain

$$\lim_{r\to+\infty}\frac{T(r,f_1)}{T(r,f)}=0 \quad (r\in E_4).$$

□

Lemma 24. *Let* $F(z)\not\equiv 0$, $A_j(z)$ $(j=0,...,k-1)$ *be entire functions. Also let* $f(z)$ *be a solution of* (2) *satisfying* $\max\{\sigma_{(\alpha(\log),\beta)}[A_j]$ $(j=0,...,k-1),\sigma_{(\alpha(\log),\beta)}[F]\}<\sigma_{(\alpha(\log),\beta)}[f]$. *Then we have*

$$\overline{\lambda}_{(\alpha(\log),\beta)}[f]=\lambda_{(\alpha(\log),\beta)}[f]=\sigma_{(\alpha(\log),\beta)}[f].$$

Proof. By (2) we get that

$$\frac{1}{f}=\frac{1}{F}\left(\frac{f^{(k)}}{f}+A_{k-1}(z)\frac{f^{(k-1)}}{f}+\cdots+A_1(z)\frac{f'}{f}+A_0\left(z\right)\right). \tag{31}$$

Now, it is easy to see that if $f(z)$ has a zero at z_0 of order a $(a>k)$, and $A_0,...,A_{k-1}$ are analytic at z_0, then $F(z)$ must have a zero at z_0 of order $a-k$, hence

$$n\left(r,\frac{1}{f}\right)\leq k\overline{n}\left(r,\frac{1}{f}\right)+n\left(r,\frac{1}{F}\right) \tag{32}$$

and

$$N\left(r,\frac{1}{f}\right)\leq k\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{F}\right). \tag{33}$$

By the lemma on logarithmic derivative and (31), we have

(34)

$$m\left(r,\frac{1}{f}\right)\leq m\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k-1}m(r,A_j)+O(\log T(r,f)+\log r)\quad (r\notin E_5),$$

where E_5 is a set of r of finite linear measure. By (33) and (34), we obtain that

$$T(r,f)=T\left(r,\frac{1}{f}\right)+O(1)\leq k\overline{N}\left(r,\frac{1}{f}\right)+T(r,F)$$

$$+\sum_{j=0}^{k-1}T(r,A_j)+O(\log(rT(r,f)))\ (r\notin E_5). \tag{35}$$

Since $\max\{\sigma_{(\alpha(\log),\beta)}[A_j]\quad (j=0,1,...,k-1)\,,\sigma_{(\alpha(\log),\beta)}[F]\}<\sigma_{(\alpha(\log),\beta)}[f]$, then by Lemma 23, there exists a set E_4 having infinite logarithmic measure such that

$$\max_{j=0,\dots,k-1}\left\{\frac{T(r,F)}{T(r,f)},\frac{T(r,A_j)}{T(r,f)}\right\}\to 0,\ r\to+\infty,\ r\in E_4. \tag{36}$$

Since f is transcendental, we have

$$O(\log(rT(r,f)))=o(T(r,f)). \tag{37}$$

Therefore, by (35), (36) and (37), for all $|z|=r\in E_4\backslash E_5$, we get that

$$T(r,f)\leq O\left(\overline{N}\left(r,\frac{1}{f}\right)\right).$$

Hence, from above we have

$$\sigma_{(\alpha(\log),\beta)}[f]\leq\overline{\lambda}_{(\alpha(\log),\beta)}[f]\leq\lambda_{(\alpha(\log),\beta)}[f].$$

By definition, we have $\overline{\lambda}_{(\alpha(\log),\beta)}[f] \leq \lambda_{(\alpha(\log),\beta)}[f] \leq \sigma_{(\alpha(\log),\beta)}[f]$. Therefore

$$\overline{\lambda}_{(\alpha(\log),\beta)}[f] = \lambda_{(\alpha(\log),\beta)}[f] = \sigma_{(\alpha(\log),\beta)}[f].$$

Hence, the lemma follows. □

Lemma 25. *Let f be a meromorphic function. If $\sigma_{(\alpha,\beta)}[f] = \sigma < +\infty$, then $\sigma_{(\alpha(\log),\beta)}[f] = 0$.*

Proof. Suppose that $\sigma_{(\alpha,\beta)}[f] = \sigma < +\infty$. Then, for any given $\varepsilon > 0$ and sufficiently large r, we have

$$T(r,f) \leq \exp\left\{\alpha^{-1}\left((\sigma+\varepsilon)\,\beta\left(\log r\right)\right)\right\}.$$

Then, we immediately get

$$\sigma_{(\alpha(\log),\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T(r,f))}{\beta\left(\log r\right)}$$

$$\leq \limsup_{r\to+\infty} \frac{\alpha\left(\log^{[2]}\left(\exp\left\{\alpha^{-1}\left((\sigma+\varepsilon)\,\beta\left(\log r\right)\right)\right\}\right)\right)}{\beta\left(\log r\right)}$$

$$= \limsup_{r\to+\infty} \frac{\alpha\left(\log \alpha^{-1}\left((\sigma+\varepsilon)\,\beta\left(\log r\right)\right)\right)}{\beta\left(\log r\right)}$$

$$= \limsup_{x\to+\infty} \frac{\alpha\left(\log \alpha^{-1}\left((\sigma+\varepsilon)\,x\right)\right)}{x} = (\sigma+\varepsilon)\limsup_{x\to+\infty} \frac{\alpha\left(\log x\right)}{\alpha\left(x\right)} = 0.$$

□

5. **Proof of the Main Results**

Proof of Theorem 11. Let $f(z) \not\equiv 0$ be a solution of (1) and rewritten (1) as

$$A_0(z) = -\left(\frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z)\frac{f^{(k-1)}(z)}{f(z)} + \cdots + A_1(z)\frac{f'(z)}{f(z)}\right).$$

Therefore

$$|A_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_1(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{38}$$

By Lemma 16, there exist a constant $c > 0$ and a set $E_1 \subset [0,+\infty)$ having finite linear measure such that $|z| = r \notin E_1$ for all $z = re^{i\theta}$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq c[rT(2r,f)]^{2k}, \quad j = 1, ..., k-1. \tag{39}$$

By (38), (39) and the hypotheses of Theorem 11, we get that

$$\exp\left\{a\exp\left(\alpha^{-1}\left(\mu\beta\left(\log|z|\right)\right)\right)\right\} \leq |A_0(z)|$$

$$\leq k\exp\left\{b\exp\left(\alpha^{-1}\left(\mu\beta\left(\log|z|\right)\right)\right)\right\} c[rT(2r,f)]^{2k} \tag{40}$$

as $z \to \infty$ with $|z| = r \notin E_1$, $\theta_1 \leq \arg z = \theta \leq \theta_2$.
Now from (40) we have

$$\exp\left\{(a-b)\exp\left(\alpha^{-1}\left(\mu\beta\left(\log|z|\right)\right)\right)\right\} \leq kc[rT(2r,f)]^{2k}$$

$$\begin{aligned} i.e.,\ (a-b)\exp\left(\alpha^{-1}\left(\mu\beta\left(\log|z|\right)\right)\right) &\leq 2k(\log r + \log T(2r,f)) \\ &\quad + \log(kc) \end{aligned}$$

$$\begin{aligned} i.e.,\ \exp\left(\alpha^{-1}\left(\mu\beta\left(\log|z|\right)\right)\right) &\leq \frac{2k}{a-b}(\log r + \log T(2r,f)) \\ &\quad + \frac{\log(kc)}{a-b}. \end{aligned}$$

By using $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$, we get from above that

$$\begin{aligned} \alpha^{-1}\left(\mu\beta\left(\log r\right)\right) &\leq (1+o(1))\left(\log^{[2]} T(2r,f) + \log^{[2]} r\right) \\ i.e.,\ \mu\beta\left(\log r\right) &\leq \alpha\left((1+o(1))\left(\log^{[2]} T(2r,f) + \log^{[2]} r\right)\right) \\ i.e.,\ \mu\beta\left(\log r\right) &\leq (1+o(1))\alpha\left(\log^{[2]} T(2r,f) + \log^{[2]} r\right) \\ i.e.,\ \mu\beta\left(\log r\right) &\leq \alpha(2\max\{\log^{[2]} T(2r,f),\ \log^{[2]} r\}) \\ i.e.,\ \mu\beta\left(\log r\right) &\leq (1+o(1))\alpha\left(\max\{\log^{[2]} T(2r,f),\ \log^{[2]} r\}\right) \end{aligned}$$

$$i.e.,\ \mu\beta\left(\log r\right) \leq \alpha\left(\log^{[2]} T(2r,f)\right) + \alpha\left(\log^{[2]} r\right). \tag{41}$$

Since $\beta((1+o(1))x) = (1+o(1))\beta(x)$ as $x \to +\infty$ and $\frac{\alpha(\log^{[2]} r)}{\beta(\log r)} \to 0$ as $r \to +\infty$, then by (41) and Proposition 2, we have $\sigma_{(\alpha(\log),\beta)}[f] \geq \mu$. Thus Theorem 11 follows.

Proof of Theorem 12. Let $f(z) \not\equiv 0$ be a solution of (1). By the hypotheses of Theorem 12, there exists a set H with $\overline{dens}\{|z| : z \in H\} > 0$ such that for all z satisfying $z \in H$, we have

$$|A_0(z)| \geq \exp\left\{a\exp\left(\alpha^{-1}\left(\mu\beta\left(\log|z|\right)\right)\right)\right\} \tag{42}$$

and

$$|A_j(z)| \leq \exp\left\{b\exp\left(\alpha^{-1}\left(\mu\beta\left(\log|z|\right)\right)\right)\right\},\quad j = 1,...,k-1 \tag{43}$$

as $z \to \infty$. Set $H_1 = \{|z| = r : z \in H\}$, since $\overline{dens}\{|z| : z \in H\} > 0$, then H_1 is a set with $\int_{H_1} dr = +\infty$. Therefore from, by substituting (39), (42) and (43) into (38), it follows that for all z satisfying $|z| = r \in H_1 \setminus E_1$, we obtain that

$$\exp\left\{a \exp\left(\alpha^{-1}\left(\mu\beta\left(\log|z|\right)\right)\right)\right\}$$
$$\leq k \exp\left\{b \exp\left(\alpha^{-1}\left(\mu\beta\left(\log|z|\right)\right)\right)\right\} c[rT(2r,f)]^{2k}$$

as $|z| = r \to +\infty$. Thus

$$\exp\left\{(a-b)\exp\left(\alpha^{-1}\left(\mu\beta\left(\log|z|\right)\right)\right)\right\} \leq kc[rT(2r,f)]^{2k} \tag{44}$$

as $|z| = r \in H_1 \setminus E_1$, $r \to +\infty$. Therefore, by (44) and Proposition 2, we obtain that $\sigma_{(\alpha(\log),\beta)}[f] \geq \mu$.

Proof of Theorem 13. By Theorem 12, we have $\sigma_{(\alpha(\log),\beta)}[f] \geq \sigma - \varepsilon$, since $\varepsilon > 0$ is arbitrary, we get $\sigma_{(\alpha(\log),\beta)}[f] \geq \sigma_{(\alpha,\beta)}[A_0] = \sigma$. On the other hand, by Lemma 17, there exists a set $E_2 \subset [1, +\infty)$ having finite logarithmic measure such that (15) holds for all z satisfying $|z| = r \notin [0,1] \cup E_2$ and $|f(z)| = M(r,f)$. Now for any given $\varepsilon > 0$ and for sufficiently large r, we obtain

$$|A_j(z)| \leq \exp^{[2]}\left\{\alpha^{-1}\left((\sigma+\varepsilon)\beta\left(\log r\right)\right)\right\}, \quad j = 0, 1, ..., k-1. \tag{45}$$

Substituting (15) and (45) into (1), for all z satisfying $|z| = r \notin [0,1] \cup E_2$ and $|f(z)| = M(r,f)$, we have

$$\left(\frac{\nu(r,f)}{|z|}\right)^k |1+o(1)|$$
$$\leq \; k\left(\frac{\nu(r,f)}{|z|}\right)^{k-1} |1+o(1)| \exp^{[2]}\left\{\alpha^{-1}\left((\sigma+\varepsilon)\beta\left(\log r\right)\right)\right\}.$$

It follows that

$$\nu(r,f) \leq kr\,|1+o(1)| \exp^{[2]}\left\{\alpha^{-1}\left((\sigma+\varepsilon)\beta\left(\log r\right)\right)\right\}. \tag{46}$$

Therefore in view of (46), $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$ and $\frac{\alpha(\log^{[2]} r)}{\beta(\log r)} \to 0$ as $r \to +\infty$, we get that

$$\limsup_{r\to+\infty} \frac{\alpha(\log^{[2]}\nu(r,f))}{\beta(\log r)} \leq \sigma + \varepsilon. \tag{47}$$

Since $\varepsilon > 0$ is arbitrary, by (47) and Lemma 19, we obtain that $\sigma_{(\alpha(\log),\beta)}[f] \leq \sigma$. This and the fact that $\sigma_{(\alpha(\log),\beta)}[f] \geq \sigma$ yield $\sigma_{(\alpha(\log),\beta)}[f] = \sigma$. The proof is complete.

Proof of Theorem 14. (i) Suppose that $\sigma_{(\alpha(\log),\beta)}[F] < \sigma_{(\alpha,\beta)}[A_0]$. First, we show that (2) can possess at most one exceptional solution

$f_0(z)$ satisfying $\sigma_{(\alpha(\log),\beta)}[f_0] < \sigma$. In fact, if $f^*(z)$ is a second solution with $\sigma_{(\alpha(\log),\beta)}[f^*] < \sigma$, then $\sigma_{(\alpha(\log),\beta)}[f_0 - f^*] < \sigma$. But $f_0(z) - f^*(z)$ is a solution of the corresponding homogeneous equation (1) of (2), this contradicts Theorem 13. We assume that $f(z)$ is a solution with $\sigma_{(\alpha(\log),\beta)}[f] \geq \sigma$, and $f_1(z)$, $f_2(z)$, . . . , $f_k(z)$ is a solution base of the corresponding homogeneous equation (1). Then, $f(z)$ can be expressed in the form

$$f(z) = B_1(z)f_1(z) + B_2(z)f_2(z) + \cdots + B_k(z)f_k(z), \tag{48}$$

where $B_1(z)$, $B_2(z)$, . . . , $B_k(z)$ are determined by

$$\begin{aligned} B_1'(z)f_1(z) + B_2'(z)f_2(z) + \cdots + B_k'(z)f_k(z) &= 0, \\ B_1'(z)f_1'(z) + B_2'(z)f_2'(z) + \cdots + B_k'(z)f_k'(z) &= 0, \\ &\;\;\vdots \\ B_1'(z)f_1^{(k-1)}(z) + B_2'(z)f_2^{(k-1)}(z) + \cdots + B_k'(z)f_k^{(k-1)}(z) &= F(z). \end{aligned} \tag{49}$$

As the Wronskian $W(f_1, f_2, ..., f_k)$ is a differential polynomial in f_1, f_2, ...,f_k with constant coefficients, it is easy to deduce that

$$\sigma_{(\alpha(\log),\beta)}[W] \leq \sigma_{(\alpha(\log),\beta)}[f_j] = \sigma_{(\alpha,\beta)}[A_0] = \sigma. \tag{50}$$

From (49) we get that,

$$B_j' = F \cdot G_j(f_1, f_2, ..., f_k) \cdot W(f_1, f_2, ..., f_k)^{-1}, \;\; j = 1, ..., k, \tag{51}$$

where $G_j(f_1, f_2, ..., f_k)$ are differential polynomials in $f_1, f_2, ..., f_k$ with constant coefficients. Therefore

$$\sigma_{(\alpha(\log),\beta)}[G_j] \leq \sigma_{(\alpha(\log),\beta)}[f_j] = \sigma_{(\alpha,\beta)}[A_0] = \sigma, \;\; j = 1, ..., k. \tag{52}$$

Since $\sigma_{(\alpha(\log),\beta)}[F] < \sigma_{(\alpha,\beta)}[A_0]$, by Lemma 20, (50)$-$(52), for $j = 1, ..., k$, we obtain that

$$\begin{aligned} \sigma_{(\alpha(\log),\beta)}[B_j] = \sigma_{(\alpha(\log),\beta)}[B_j'] &\leq \max\{\sigma_{(\alpha(\log),\beta)}[F], \sigma_{(\alpha,\beta)}[A_0]\} \\ &= \sigma_{(\alpha,\beta)}[A_0] = \sigma. \end{aligned} \tag{53}$$

Now from (48) and (53), we obtain that

$$\begin{aligned} \sigma_{(\alpha(\log),\beta)}[f] &\leq \max\{\sigma_{(\alpha(\log),\beta)}[f_j], \sigma_{(\alpha(\log),\beta)}[B_j] \;\; (j = 1, ..., k)\} \\ &= \sigma_{(\alpha,\beta)}[A_0] = \sigma. \end{aligned} \tag{54}$$

This and the assumption $\sigma_{(\alpha(\log),\beta)}[f] \geq \sigma$ yield $\sigma_{(\alpha(\log),\beta)}[f] = \sigma$. by Lemma 25, we have

$$\max\left\{\sigma_{(\alpha(\log),\beta)}[F], \sigma_{(\alpha(\log),\beta)}\left[A_j\right] \quad (j = 0, 1, ..., k-1)\right\} = \sigma_{(\alpha(\log),\beta)}(F)$$
$$< \sigma_{(\alpha,\beta)}[A_0] = \sigma_{(\alpha(\log),\beta)}[f].$$

So, if $f(z)$ is a solution of equation (2) satisfying $\sigma_{(\alpha(\log),\beta)}[f] = \sigma$, then by Lemma 24, we get that

$$\overline{\lambda}_{(\alpha(\log),\beta)}[f] = \lambda_{(\alpha(\log),\beta)}[f] = \sigma_{(\alpha(\log),\beta)}[f] = \sigma.$$

(ii) Suppose that $\sigma_{(\alpha,\beta)}[A_0] \leq \sigma_{(\alpha(\log),\beta)}[F] < +\infty$. Then, by (53), for $j = 1, ..., k$, we obtain that

$$\sigma_{(\alpha(\log),\beta)}[B_j] = \sigma_{(\alpha(\log),\beta)}[B_j^{'}] \leq \max\{\sigma_{(\alpha(\log),\beta)}[F], \sigma_{(\alpha,\beta)}[A_0]\}$$
$$= \sigma_{(\alpha(\log),\beta)}[F]. \tag{55}$$

Now from (48) and (55), we obtain that

$$\sigma_{(\alpha(\log),\beta)}[f] \leq \max\{\sigma_{(\alpha(\log),\beta)}[f_j], \sigma_{(\alpha(\log),\beta)}[B_j] \ \ (j = 1, ..., k)\}$$
$$\leq \sigma_{(\alpha(\log),\beta)}[F]. \tag{56}$$

From (2), a simple consideration of $(\alpha(\log), \beta)$-order implies that

$$\sigma_{(\alpha(\log),\beta)}[f] \geq \sigma_{(\alpha(\log),\beta)}[F].$$

By the above inequality and (56), we get that

$$\sigma_{(\alpha(\log),\beta)}[f] = \sigma_{(\alpha(\log),\beta)}[F]$$

which completes the proof.

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B. Belaïdi: Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (UMAB), B. P. 227 Mostaganem-Algeria.
e-mail: benharrat.belaidi@univ-mosta.dz
T. Biswas: Rajbari, Rabindrapally, R. N. Tagore Road, P.O.- Krishnagar, P.S. Kotwali, Dist-Nadia, PIN-741101, West Bengal, India.
e-mail: tanmaybiswas_math@rediffmail.com