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# ON SUPER TOPOLOGICAL MODULES AND SUPER GROUP RINGS

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**Abstract.** Theory of super topological modules and submodules is formulated, which is based on D-supercontinuous functions. In the last section we have used the concept of supertopological groups and supertopological rings defined in [7] with d-compactness and d-separability to stretch the topologies to their group rings under various conditions.

## 1. Introduction

Topological groups and topological rings have been a centre of attraction in algebra as well as topology for a century now. They have been extensively studied in [6], [8] and by I. Kaplansky in [2] and [3]. The theory of topological modules was mainly developed by Arnautov in his book [1]. We have used the concept of D-supercontinuity to study module structure on supertopological rings and the extension problem in super group rings.

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In section 2 we have formulated the study of supertopological modules which is followed by some important results on supertopological submodules in section 3. In section 4, after recapitulating the important concept of group rings we have proved that there exists a d-separable topology on group ring $AG$ in which $AG$ is a super group ring. Last section deals with some other major results that extends supertopology from $A$ and $G$ to group ring $AG$. In this paper we denote super topological group by $G$ and super topological ring with unity by $A$.

**Definition 1.** [5] *A function $f : X \to Y$ from topological space $X$ to topological space $Y$ is said to be D-supercontinuous if for each $x \in X$ and each open set $U \subset Y$ containing $f(x)$ there exists an open $F_\sigma$-set $V \subset X$ containing $x$ such that $f(V) \subset U$.*

**Definition 2.** [4] *A set $U$ in a topological space $X$ is said to be d-open if for each $x \in U$, there exists an open $F_\sigma$-set $F$ such that $x \in F \subset U$. Complement of a d-open set is called d-closed.*

**Definition 3.** [7] *For a set $M \subset X$, the intersection of all the d-closed sets in $X$ containing $M$ is called the d-closure of $M$ which is denoted by $[M]_d$.*

**Remark 4.** *$M$ is d-closed if and only if $M = [M]_d$.*

**Theorem 5.** [5] *The following are equivalent for a function $f : X \to Y$ where $X$ and $Y$ are topological spaces:*

(a) *$f$ is D-supercontinuous at every point in $X$.*
(b) *Inverse image of every open set is open $F_\sigma$-set.*
(c) *Inverse image of every closed set is closed $G\delta$-set.*
(d) *$f([A]_d) \subset [f(A)]_d$.*

**Definition 6.** [7] *A topological space $G$ that is also a group, where $G \times G$ carries product topology, is a supertopological group if the mappings*

$$g_1 : G \times G \to G \text{ such that } (x, y) \to xy$$

*and*

$$g_2 : G \to G \text{ such that } x \to x^{-1}$$

*are D-supercontinuous.*

**Definition 7.** [7] *A topological space $A$ that is also a ring with operations '+' and '.' where $A \times A$ carries product topology, is a supertopological ring if the mappings*

$$a_1 : A \times A \to A \text{ such that } (x, y) \to x + y$$

*,*

$$a_2 : A \to A \text{ such that } x \to -x$$

*and*

$$a_3 : A \to A \text{ such that } (x, y) \to x.y$$

*are D-supercontinuous.*

**Definition 8.** [4] *A topological space $X$ is said to be a d-compact space if every cover of $X$ by open $F_\sigma$-sets has a finite subcover.*

**Definition 9.** *A subset $S$ of topological space $X$ is said to be a d-dense space if $[S]_d = X$.*

**Definition 10.** *A topological space $X$ is said to be a d-separable if it contains a countable, d-dense subset.*

**Definition 11.** *A topological group $X$ is said to be a d-monothetic if it contains a cyclic, d-dense subgroup.*

**Definition 12.** [7] *A subset $S$ of a supertopological ring $A$ is right d-bounded if for any neighborhood $U$ of $0$, there exists a d-neighborhood $V$ such that $V.S \subset U$, where $V.S$ is the set of all products of elements in $V$ and $S$.*

Left d-bounded is defined similarly and a ring that is both left and right d-bounded, is called a d-bounded ring.

**Definition 13.** [7] *A topological space $X$ is D-disconnected if it can be expressed as a union of two disjoint non-empty open $F_\sigma$-sets. Otherwise, $X$ is said to be D-connected.*

**Definition 14.** [7] *A topological space $X$ is totally (or completely) D-disconnected if $D_x = \{x\}$ for each $x \in X$, where $D_x$ is the maximal D-connected set containing $x$, also known as D-component of $x$ or equivalently if it has no nontrivial D-connected subsets.*

Throughout this paper a *d-neighborhood* of a point $x$ will mean an open $F_\sigma$-set containing $x$.

## 2. Topological super-Modules

We introduce topological super-modules using supertopological rings in this section. Supertopological rings were introduced and studied in [7]. We will denote a topological ring by $R$ and a supertopological ring by $A$ and a supertopological group by $G$.

**Definition 15.** *A supertopological ring $A$ is called a super-normed ring if there exist a non-negative real function $\xi$ such that:*

(1) $\xi(a) = 0$ *iff* $a = 0$,
(2) $\xi(a_1 - a_2) \leq \xi(a_1) + \xi(a_2)$ *for any* $a \in A$,
(3) $\xi(a_1.a_2) = \xi(a_1).\xi(a_2)$, *for any* $a_1, a_2 \in A$
*It is called pseudonorm on A if condition* 3 *is replaced by* $3'$ *which states that* $\xi(a_1.a_2) \leq \xi(a_1).\xi(a_2) \forall\ a_1, a_2 \in A$.

**Example 16.** *$X = \mathbb{C}$ where for an element $a + \iota b$, $|z| = \sqrt{a^2 + b^2}$ is a normed super topological ring.*

**Example 17.** *$X = \mathbb{Q}$ under p-adic topology is a normed super topological ring.*

**Remark 18.** *The multiplicative group of non-zero elements of a super topological ring $A$ is a super topological group $G$ that carries the same topology as $A$.*

**Definition 19.** *Let $A$ be a supertopological ring. A left $A$ module $M$ is called a left $A$ supertopological module if on $M$ is a specified topology that makes it a super topological abelian group and the mapping $(a, m) \to am$ of supertopological space $M$ is D-supercontinuous. Right $A$ super-modules are defined similarly.*

**Remark 20.** *Any supertopological abelian group $G$ in a canonical way is a $\mathbb{Z}$ - supertopological module with discrete topology. The same works for anti-discrete topology on $G$.*

**Example 21.** *The ring $C([0, 1], \mathbb{R})$ with ordinary addition and multiplication by real numbers, is a super-topological vector space over $\mathbb{R}$ with usual topology.*

**Theorem 22.** *Let $M$ and $N$ be subsets of supertopological abelian group $G$, the following are true:*

(1) *if $M$ and $N$ are d-compact, then $B + C$ is d-compact in $G$.*
(2) *if $M$ is d-closed and $N$ is d-compact subset then $M + N$ is a d-closed subset in $G$.*

*Proof.*

(1) Let $\phi : G \times G \to G$ where $\phi(g, h) = g + h$ for $g, h \in G$. The mapping is continuous and hence we restrict it onto subspace $M \times N$ of $G \times G$ is continuous too. As a continuous image of d-compact space $M \times N$, $M + N$ is also d-compact subspace.
(2) Assume the contrary that $M + N$ is not d-closed subspace. Let $x \in [M + N]_d$ but $x \in M + N$. Then $(x - N) \bigcap M = \phi$. As $M$ is d-closed in $G$, we have $G \setminus M$ is a d-open subset in $G$, hence

$G \setminus M$ is a d-neighbourhood in $G$ of any element of the type of $x - n$, where $n \in N$.
We can choose two d-neighbourhoods $U_x$ and $V_n$ of $x$ and $n$ in $G$ such that $U_x - V_n \subseteq G \setminus M$. Thus, $\{V_n : n \in N\}$ is a d-open cover of d-compact subset $N$ and hence there exist a finite set of elements $n_1, n_2, n_t$ from $N$ such that

$$V = \bigcup_1^n V_{n_i}$$

contains $N$.
As a consequence $U = \bigcap_1^n U_{n_i}$ is a d-neighbourhood of $x$. Thus $U - V \subseteq G \setminus M$. Therefore it is easily seen that $U \bigcap (M + N) = \phi$, hence a contradiction.

∎

**Corollary 23.** *The sum $B + C$ of a d-closed subset $B$ with a finite subset $C$ of any topological groups $G$ is its d-closed subset.*

**Theorem 24.** *Let $B$ and $C$ be subsets of supertopological abelian group $G$. Then the following are true:*

(1) $[B]_d + [C]_d \subseteq [B + C]_d$
(2) $[-B]_d = -[B]_d$
(3) $[B]_d - [C]_d \subseteq [B - C]_d$
(4) *if $C$ is d-compact subset* $[B + C]_d = [B]_d + [C]_b = [B]_d + C$ *and* $[B = C]_d = [B]_d - [C]_d = [B]_d - C$.

*Proof.*

(1) Let $x \in [B]_d + [C]_d$ and $U$ be a d-neighbourhood of $x$. Then $x = b + c$ where $b \in [B]_d$ and $c \in [C]_d$, thus there exist d-neighbourhoods $V$ and $W$ in $G$ of $b$ and $c$ respectively such that $V + W \subseteq U$. As $V \bigcap B \neq \phi$ and $W \bigcap C \neq \phi$, elements $b_1$ and $c_1$ can be found. Thus $b_1 + c_1 \in B + C$ and $b_1 + c_1 \in V + W \subseteq U$, that is $(B + C) \bigcap U \neq \phi$, hence $[B + C]_d$ contains $[B]_d + [C]_d$.
(2) It is a consequence of the fact $x \to -x$ is a homeomorphism of $G$ onto $G$.
(3) Follows from (1) and (2).
(4) $[B]_d + C \subseteq [B]_d + [C_d] \subseteq [B + C]_d$ follows from (1). For other containment notice that $[B]_d + C$ is a d-closed subset of $G$.

∎

**Theorem 25.** *Let $A$ be a supertopological ring, $M$ be a $A$ supermodule, $a \in A, Q \subset A, B \subset M$, then the following are true:*

(1) *the mapping $\phi_a : M \to M$ where $\phi_a(x) = a.x$ is a D-supercontinuous mapping.*
(2) *the mapping $\phi_m : A \to M$, where $\phi_m(x) = x.m, x \in R$ is a D-supercontinuous mapping.*
(3) *if the subsets $Q$ and $B$ are d-compact, then $Q.A$ is a d-compact subset.*

**Theorem 26.** *Let $A$ be a supertopological ring with unitary element and $a \in A$ be an invertible element and $x \in A$. Then the following are equivalent:*

(1) *$U$ is a d-neighbourhood of $x \in A$*
(2) *$U.a$ is a d-neighbourhood of the element $x.a \in A$*
(3) *$a.U$ is a d-neighbourhood of the element $a.x \in A$.*

**Corollary 27.** *Let $K$ be a supertopological skew field. Then the mapping $\theta : K - \{0\} \to K - \{0\}$, where $\theta(x) = x^{-1}$ for $x \neq 0$, is a homeomorphism.*

**Theorem 28.** *Let a family $\mathfrak{S}_0$ of subsets of a supertopological abelian group $G$ be a basis of d-neighbourhoods of zero in $G$. Then following are true:*

1 $0 \in \bigcap_{V \in \mathfrak{S}_o} V$,
2 *for any subsets $U$ and $V$ from $\mathfrak{S}_o$ there exists a subset $W \in \mathfrak{S}_o$ such that $W \subseteq U \bigcap V$,*
3 *for any $U \in \mathfrak{S}_o$ there exists a subset $V \in \mathfrak{S}_o$ such that $V + V \subseteq U$, for any subsets $U \in \mathfrak{S}_o$ there exists a subset $V \in \mathfrak{S}_o$ such that $-V \subseteq U$,*
4 *for any subset $U \in \mathfrak{S}_o$ there exists a subset $V \in \mathfrak{S}_o$ such that $V.V \subseteq U$.*

*Proof.* (1) and (2) follows from definition of a basis of d-neighbourhoods of an element in a topological space. (3) and (4) follows from the fact that $\mathfrak{B}_o$ is a basis of d-neighbourhoods of zero in $G$ and thus $0 + 0 = 0$ and $0 = -0$. ∎

**Theorem 29.** *Let $S$ be a subset of a supertopological abelian group $A$ with a basis $\mathfrak{B}_o$ of d-neighbourhoods of zero. Then $[S]_d = \bigcap_{V \in \mathfrak{B}_o}(S + V)$.*

*Proof.* Let $x \in [S]_d$ and $V \in \mathfrak{B}_o$. Let also $V' \in \mathfrak{B}_o$ and $-V' \subseteq V$, then $(x + V') \bigcap S \neq \phi$, because $x \in [S]_d$. Then $x \in S - V' \subseteq S + V$. Therefore $[S]_d \subseteq S + V$ and hence $[S]_d \subseteq \bigcap_{V \in \mathfrak{B}_o}(S + V)$. The other containment is clear by choosing a d-neighbourhood $U$ of zero and another d-neighbourhood $-V \in \mathfrak{B}_o$ of zero such that $-V \subseteq U$. ∎

**Corollary 30.** *Let $\mathfrak{B}_o$ be a basis of d-neighbourhoods of zero of topological abelian group $G$. Then $\bigcap_{V\in\mathfrak{B}_o} V$ is a d-closed set.*

*Proof.* $\bigcap_{V\in\mathfrak{B}_o} V$ is a closure of a one element subset $\{0\}$ in $G$ . ∎

**Theorem 31.** *Let $G$ be a supertopological abelian group, then $G$ has a basis of d-neighbourhoods of zero consisting of symmetric d-open (or d-closed) d-neighbourhoods.*

**Corollary 32.** *Let $a$ be an invertible element of a supertopological ring $A$ with the unitary element, $\mathfrak{B}_o(A)$ be a basis of the d-neighbourhoods of the ring $A$. Then $\{a.U \mid U \in \mathfrak{B}_o(A)\}$ and $\{U.a \mid U \in \mathfrak{B}_o(A)\}$ are bases of d-neighbourhoods of zero of the ring $A$.*

## 3. Super-Subrings and Super-submodules

**Definition 33.** *Let $A$ be a supertopological ring, $M$ be a $A$ super-module. A subset $Q$ of $A$ (a subset $N$ of $M$) is called super-subring of $A$ (a super-submodule of $M$) if $Q$ is a subring of $R$ (if $N$ is a submodule of ${}_RM$) endowed with the topology induced from $A$ (or ${}_RM$).*

**Remark 34.** *A subring $Q$ of a supertopological ring $A$ is a supertopological ring. Similarly for super-modules. Closures of subrings or submodules are also super-subrings or super-submodules respectively.*

**Remark 35.** *Let $B$ be a subgroup of topological abelian group $G$. Then the d-closure of $B$ in $G$ is a subgroup of topological group $G$.*

**Theorem 36.** *Let $A$ be a super topological ring and $M$ be a $A$ super module. Let $Q$ be a super subring of $A$, and $N$ be a $Q$ super submodule of $A$ super module $M$, then by denoting d-closure of $Q$ in $A$ by $[Q]_A$ and the d-closure of $N$ in $M$ by $[N]_M$, we have*

(1) *$[Q]_A$ is a super subring of $A$,*
(2) *$[N]_M$ is a $[Q]_A$-super module.*

*Proof.* $[Q]_A$ and $[N]_M$ are subgroups of topological abelian group $A$ and $M$ by above remark, and thus the result follows. ∎

**Corollary 37.** *Let $\mathfrak{B}_o$ be a basis of d-neighbourhoods of zero of a $R$ super module $M$, then $M_0 = \bigcap_{\mathfrak{B}_o} U$ is the smallest d-closed super submodule of $M$.*

The following remark is a direct consequence of above corollary.

**Remark 38.** *Let $\mathfrak{B}_o$ be a basis of d-neighbourhoods of $0$ of topological ring $A$, then $A_0 = \bigcap_{V\in\mathfrak{B}_o} V$ is the smallest d-closed two sided ideal of $A$.*

## 4. Super Group Rings

Let $G$ be a super topological group and $A$ be a super topological ring then its super group ring $AG$ is the set of all the formal linear combinations of the form

$$\alpha = \sum_{g \in G} a_g g$$

where $a_g \in A$ and $a_g = 0$ almost everywhere. The addition and multiplication on $AG$ is defined as follows:

$$\alpha + \beta = \sum_{\substack{g \in G \\ a_g \in A}} a_g g + \sum_{\substack{g \in G \\ b_g \in A}} b_g g = \sum_{g \in G} (a_g + b_g) g$$

$$\alpha . \beta = (\sum_{g \in G} a_g g)(\sum_{h \in G} b_h h) = \sum_{g,h \in G} a_g b_h gh = \sum_{u \in G} c_u u$$

where $c_u = \sum_{gh=u} a_g b_h$, and both these operations are D-supercontinuous.

In case $A$ is commutative with identity, we say $AG$ is a *super group algebra* over $A$.

The map $\varepsilon : AG \to A$ given by

$$\varepsilon(\sum a_g g) = \sum a_g$$

is called the *super augmentation map*; it is a ring homomorphism and its kernel, denoted by $\Delta_A(G)$ [or simply $\Delta(G)$, in case super topological ring $A$ is obvious from the context] is called the *super augmentation ideal* of $AG$. Notice that the set $\{g - 1 : g \in G, g \neq 1\}$ is a *basis* of $\Delta(G)$ over $A$.

If $G$ is finite, then the rank of $AG$ over $A$ is $|G|$. It also is obvious that $ag = ga$ in $AG$.

Now we shift our focus to the problem of extending super topologies of $A$ and $G$ to group ring $AG$ which makes it a super topological ring.

In this section and next we will show how topologies can be extended from a supertopological group and supertopological ring to their super group ring under various conditions. We will denote a supertopological group by $G$ and a supertopological ring by $A$.

**Remark 39.** *If both $A$ and $G$ are supertopological, and operations $+$ and $.$ are D-supercontinuous in group ring $AG$, then we call $AG$ a super group ring. If any one or both of $G$ or $A$ are not supertopological (which will be clear from the context) , but addition and multiplication* *in group ring AG are D-supercontinuous, then the group ring AG is called a supertopological group ring.*

Let $\mathcal{G}$ be the class of all super topological groups $G$ such that $G = \bigcup_{i\in\mathbb{N}} \mathcal{O}_i$ where the $\mathcal{O}_i$ are d-open symmetric subsets of $G$ and $\bigcap_{g\in G} g^{-1}Vg$ is a b-neighborhood of the identity for each $\mathcal{O}_i$ and any d-neighbourhood of $V$ of the identity in $G$. Clearly, locally compact groups that are unions of countably many d-compact subsets lie in $\mathcal{G}$. By $\mathcal{A}$ we denote the class of locally bounded super topological rings with identities that are unions of countably many bounded subsets. This class includes locally bounded connected rings and d-compact rings.

**Theorem 40.** *Let $(A,\tau_0) \in \mathcal{A}$ and $(G,\tau_1) \in \mathcal{G}$. Then there exist a d-separable topology $\tau$ on the group ring AG in which AG is a super group ring. $\tau/A = \tau_0$ and $\tau/G = \tau_1$ and A and G are d-closed subsets of $(AG,\tau)$.*

*Proof.* In $A$ we can find a multiplicative subset which is a super semi topological group $U_0$ and a base $\mathcal{B}_0(A)$ of symmetric d-neighbourhoods of zero that are ideals of $U_0$ and such that $U_0 \in \mathcal{B}_0(A)$. Let $A = \bigcup_{i\in\mathbb{N}} D_i$ where $D_i$ are bounded subsets. Since $G \in \mathcal{G}$ there exist d-open symmetric subsets $O_i$ such that $G = \bigcup_{i\in\mathbb{N}} O_i$ and $\bigcap_{g\in O_i} g^{-1}Vg$ is a d-neighbourhood of the identity in $G$ for each $O_i$ and any d-neighbourhood $V$ of the identity of $G$. Let $\mathcal{B}_1(G)$ be a base of d-neighbourhoods of the identity of $G$.

If $\mathfrak{R} = \{\mathcal{U} = (U_1, U_2, \ldots \mid U_i \in \mathcal{B}_0(A))\}$ and $\mathcal{R} = \{\mathcal{V} = (V_1, V_2, \ldots \mid V_i \in \mathcal{B}_1(G))\}$ then we put

$$W(\mathcal{U},\mathcal{V}) = \{z \in AG \mid z = \sum_{i=1}^{m} a_i b_i + \sum_{i=1}^{n} q_i(g_i - h_i)\}$$

where $b_i, g_i, h_i \in O_i \subseteq G, g_i h_i^{-1} \in V_i, a_i \in U_i, q_i \in D_i \subseteq A$. The family $\{W(\mathcal{U},\mathcal{V}) \mid \mathcal{U} \in \mathcal{G}, \mathcal{V} \in \mathcal{A}\}$ defines a super topological ring topology $\tau$ on the super group ring $AG$ in which $\tau/A = \tau_0, \tau/G = \tau_1$ and $A$ and $G$ are d-closed subsets of super group ring $(AG,\tau)$. ∎

## 5. Extending topologies to super group rings

**Theorem 41.** *Let $(A,\tau_0)$ be a supertopological ring and $(G,\tau_1)$ be a supertopological group. Let $B_1(G)$ be a basis of d-neighbourhoods of identity in G, composed of normal subgroups. If for any normal subgroup $N \in B_1(G)$ the discrete topology of $G/N$ and $\tau_0$ of A extend* *to a ring topology for $AG/N$, then $\tau_0$ and $\tau_1$ extend to a ring topology for super group ring $AG$.*

*Proof.* Let $N \in B_1(G)$ and $\eta_N : G \to G/N$ be identity homomorphism and $\bar{\eta}_N : AG \to AG/N$ be corresponding homomorphism of super group rings. By assumption, there exists a ring topology $\tau_N^{'}$ on $AG/N$ such that $\tau_N^{'}|A = \tau_0$ and $\tau_N^{'}|G/N$ is discrete topology. The preimage $\bar{\tau}_N$ of topology $\tau_N^{'}$ with respect to $\bar{\eta}_N$ is a ring topology on $AG$ which may not be d-separable.

$$\hat{\tau} = \sup \{\bar{\tau}_N : N \in B_1(G)\}$$

is a ring topology on $AG$.

To verify that $\hat{\tau}$ is d-separable, we need to find a d-neighbourhood $N \in B_1(G)$ such that $g_i g_j^{-1} \in N$ for all $1 \leq i, j \leq n$ and $i \neq j$, then $\eta_N(g_i) \neq \eta_N(g_j)$ for $i \neq j$ and hence $\bar{\eta}_N(\sum_1^n r_i g_i) = \sum_1^n r_i \eta_N(g_i) \neq 0$.

As there exists a d-neighbourhood of 0, say $U_0$ such that $\bar{\eta}_N(\sum_1^n r_i g_i) \notin U_0$ in $(AG/N, \tau_N^{'})$, then $U = (\hat{\eta}_N)^{-1}(U_0)$ is a d-neighbourhood of zero in $(AG, \eta_N)$, hence it is a d-neighbourhood in $(AG, \tau)$ such that $\sum_1^n r_i g_i \notin U$. Now, it is clear that the topology $\hat{\tau}$ is d-separable by showing $\tau|A = \tau_0$ and $\tau|G = \tau_1$. ∎

**Theorem 42.** *Let $D$ be a supertopological division ring, $\tau_0$ be a ring topology on $D$ in which the intersection of any countable set of d-neighbourhoods of zero is nonzero, and $(G, \tau_1)$ a supertopological group which is countable union of d-compact subsets. If $\tau_0$ and $\tau_1$ extend a ring topology $\tau$ of the super group ring $AG$ then the supertopological group $(G, \tau_1)$ satisfies the following condition:*
*for any d-neighbourhood $V$ of the identity of $G$, $\exists$ a d-neighborhood $V_1$ of the identity of $G$ such that $\{x^n \ : n \in \mathbb{Z}\} \subseteq V$ for all $x \in V_1$.*

*Proof.* Assume the contrary that there exist a symmetric d-neighbourhood $V_0$ of identity $e$ of the $G$ such that for any d-neighbourhood $V$ of identity there can be found $x \in V$ and $n = n(x) \in \mathbb{Z}$, for which $x^n \notin V_0$. Since $\tau|G = \tau_1$, there exists a d-neighbourhood $W$ of zero in $AG$ such that $(1 + W) \cap G = V_0$. By choosing neighbourhoods $W_i$, where $i \in \mathbb{N} \cup \{0\}$, such that $W_0.W_0 \subseteq W$, $W_i + W_i \subseteq W_{i-1}$ for $i \geq 1$. By induction on $m$,

$$(\sum_{i=1}^{m} W_i) + W_m \subseteq W_0 \text{ for any } m \in \mathbb{N}.$$

Thus, $\sum_{i=1}^m W_i \subseteq W_0$. Let $G = \bigcup G_i$ where $G_i$ is a d-compact hence d-bounded subset of $AG$. Assuming $G_i \subseteq G_{i+1}$ for all $i \in \mathbb{N}$, for any $i$

there exists a d-neighbourhood of zero, say $V_i$ in $AG$ such that $V_i.G_i \subseteq W_i$. Since $V_i \bigcap A$ is a d-neighbourhood of zero in $A$ then by assumption $\bigcap_i V_i$ contains a nonzero element $a$. Since $A$ is a division ring (or skew-field), $a^{-1}$ exists. As $AG$ with topology $\tau$ is a topological ring, there can be found a d-neighbourhood $V'$ in $AG$ such that $V'^{-1} \subseteq W_0$. Then $D' = (1+W') \cap G$ is a d-neighbourhood of identity in $G$. By assumption there exists an element $x \in D'$ and an integer $n$ such that $x^n \notin V_0$. As $V_0$ is symmetric, $n$ can be assumed as natural number. There can also be found a natural number $k$ such that $\{x, x^2, \ldots x^{n-1}\} \subseteq G_k$. The rest follows, as $x^n \in V_0$ which is a contradiction. ∎

**Lemma 43.** *If $(A, \tau)$ is a super topological ring and $G$ is a finite monoid, then $\tau$ extends to a ring topology $\tau'$ for the semi super topological group ring $AG$.*

**Lemma 44.** *If the d-connected component of identity of a d-compact supertopological group $G$ is different from the identity, then there exists a d-neighbourhood $U_e$ of the identity in $G$ such that for any neighbourhood of the identity $U$ in $G$ there can be found $x \in U$ and a natural number $n$ such that $x^n \notin U_e$.*

**Theorem 45.** *Let $(G, \tau)$ be a d-compact supertopological group, then the following are equivalent:*

(1) *for any supertopological ring $(A, \tau_1)$ with identity, the topologies $\tau$ and $\tau_1$ extend to a ring topology on $AG$;*
(2) *$(G, \tau)$ is a completely D-disconnected supertopological group.*

*Proof.* (1) implies (2) is clear. For converse, consider that completely D-disconnected d-compact group $(G, \tau)$ possesses a basis of d-neighbourhoods of identity composed of normal subgroups and thus by theorem 41 and lemma 43, condition (1) is satisfied. ∎

**Theorem 46.** *Let $(G, \tau)$ be nondiscrete supertopological group and $D$ is a d-dense cyclic subgroup of $(G, \tau)$. Then the following are equivalent:*

(1) *for any supertopological ring $(A, \tau_1)$ with identity, the topologies $\tau$ and $\tau_1$ extend to a ring topology on $AG$;*
(2) *for any supertopological ring $(A, \tau_1)$ with identity, the topologies $\tau|D$ and $\tau_1$ extend to a ring topology on $AH$;*
(3) *the supertopological group $(D, \tau|D)$ possesses a basis of d-neighbourhoods of the identity composed of subgroups.*

*Proof.* (1) implies (2) and (2) implies (3) is obvious. For (3) implies (1), the completion $\hat{G}$ of $G$ is a completely D-disconnected d-compact group and thus the result follows by theorem 45. ∎

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