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ON THE DENSITY OF LIPSCHITZ FUNCTIONS IN NEWTONIAN SPACES

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Abstract. Let $\mathbf{E}$ be a rearrangement invariant Banach function space over a metric measure space X, where the measure of X is doubling and X supports a $(1, \mathbf{E})$ −Poincaré inequality. We provide sufficient conditions for the local Hölder continuity of a representative of each function in $N^{1,\mathbf{E}}(X)$, using a quasiconcavity property of a certain power of the fundamental function of $\mathbf{E}$. Using the properties of a non-centered maximal operator based on $\mathbf{E}$, we give a simple proof for the density of Lipschitz functions in a Newtonian space $N^{1,\mathbf{E}}(X)$, under the assumptions that $\mathbf{E}$ has an absolutely continuous norm and its fundamental function satisfies a certain lower estimate.

1. Introduction

The theory of Sobolev spaces on metric measure spaces, that emerged in the late 1990s, has reached an advanced stage of development, see the monographs [7] and [16]. The Newtonian spaces are first-order Sobolev spaces on metric measure spaces, based on upper gradients.

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In analysis on metric measure spaces Lipschitz continuous functions are a natural substitute for smooth functions. The notion of Lipschitz function, a purely metric one, is pervasive in Analysis and its applications, from differential equations to machine learning [10].

Several extensions of Sobolev spaces to metric measure spaces have been introduced in the mid-late 90s: Hajlasz-Sobolev spaces (based on Hajlasz gradients) [14], Cheeger spaces (defined as completions of Lipschitz class) [9] and Newtonian spaces (based on weak upper gradients) [33], [16].

In this paper we consider a metric measure space (X, d, μ) and a Newtonian space $N^{1,\mathbf{E}}(X)$, where $\mathbf{E}$ is a Banach function space in the sense of [6]. For $\mathbf{E} = L^p(X)$, $1 \leq p < \infty$, the space $N^{1,\mathbf{E}}(X) = N^{1,p}(X)$ was the first extension based on upper gradients of Sobolev spaces to metric measure spaces, introduced and studied by Shanmugalingam [33]. The theory of Newtonian spaces $N^{1,p}(X)$ was further generalized by Tuominen [34] and Aïssaoui [1], who studied the case where $\mathbf{E} = L^{\Psi}(X)$ is an Orlicz space, then by Costea and Miranda [11], who developed the theory for the case where $\mathbf{E} = L^{p,q}(X)$ is a Lorentz space. The case $\mathbf{E} = L^{\infty}(X)$ has been studied by Durand-Cartagena and Jaramillo [13]. The case where $\mathbf{E}$ is a Banach function space was approached in [32], using Banach function spaces as an unifying framework for Orlicz spaces and Lorentz spaces. The more general case of Newtonian spaces based on quasi-Banach function lattices has been studied by L. Malý in several papers [24], [25], [23].

Note that there are plenty of quasi-Banach function lattices that are not Banach function spaces. In Functional Analysis, quasi-Banach (function) spaces which are not normed spaces are the subject of an active research [22], [20]. The best-known example of a quasi-normed function space which is not a normed space is that of Lebesgue spaces $L^p(X, \mu)$ for $0 < p < 1$, with the usual quasi-norm $\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$ having $2^{1/p-1}$ as modulus of concavity [22]. The Lorentz spaces $L^{p,q}(X, \mu)$, $1 < p < \infty$, $1 \leq q \leq \infty$ are endowed with a quasi-norm that is a norm if $1 \leq q \leq p$, but is only equivalent to a norm if $p < q \leq \infty$ [11]. For an Orlicz space $L^{\Phi}(X, \mu)$ generated by a generalized Orlicz function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, which is strictly increasing, but not necessarily convex, such that $\Phi(0) = 0$ and $\lim\limits_{u \to \infty} \Phi(u) = \infty$, the Minkowski functional of the set of measurable functions $f : X \to \overline{\mathbb{R}}$ with $I_{\Phi}(f) := \int_X |f| \, d\mu \leq 1$ is a quasi-norm if and only if there exist

$p > 0$ and $c > 0$ such that $\Phi(at) \geq ca^p \Phi(t)$ for all $a \geq 1$ and $t \geq 0$, this quasi-norm being the Luxemburg norm if Φ is convex [20].

The density of smooth functions in Sobolev spaces on open sets in $\mathbb{R}^n$ is of great importance for the theory and applications of Sobolev spaces, the celebrated $H = W$ theorem of Meyers and Serrin [27] showing that $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ whenever $\Omega \subset \mathbb{R}^n$ is an open set and $1 \leq p < \infty$.

If X is a metric measure space, Lipschitz functions are dense in the Hajłasz-Sobolev space $M^{1,p}(X)$ with $1 < p < \infty$, both in norm and in Lusin's sense [14, Theorem 5]. The density of Lipschitz functions in Newtonian spaces $N^{1,p}(X)$ with $1 \leq p < \infty$ has been proved by Shanmugalingam [33, Theorem 4.1], in doubling metric measure spaces (X, d, μ) supporting a weak $p-$Poincaré inequality. Every Newtonian function $u \in N^{1,p}(X)$ with a $p-$integrable upper gradient g is approximated in $N^{1,p}(X)$, with an error that tends to zero as $\lambda \to \infty$, by $\lambda-$Lipschitz functions u_λ that coincide with u a.e. in the complement of the set where the non-centered Hardy-Littlewood maximal function of g^p is above λ^p. Corresponding density results have been proved for Orlicz-Sobolev spaces [34], [1] and Sobolev-Lorentz spaces [11]. Under the above assumptions on (X, d, μ) plus the completness of the metric space, Durand-Cartagena and Jaramillo [13] proved that $LIP^\infty(X) = M^{1,\infty}(X) = N^{1,\infty}(X)$ with equivalent norms, where $LIP^\infty(X)$ is the space of bounded Lipschitz functions on X. Moreover, assuming that (X, d, μ) is connected complete and doubling, Durand-Cartagena, Jaramillo and Shanmugalingam have shown that $LIP^\infty(X) = N^{1,\infty}(X)$ with comparable energy seminorms if and only if X supports a weak $\infty-$Poincaré inequality [12, Theorem 4.7]. Ambrosio, Colombo and Di Marino [3] obtained the density of Lipschitz functions in $N^{1,p}(X)$ with $1 < p < \infty$ without assuming that the measure is doubling or that X supports a Poincaré inequality, provided that X is proper and endowed with a doubling metric. Ambrosio, Pinamonti and Speight [2] investigated weighted Sobolev spaces on metric measure spaces and provided sufficient conditions for the density of Lipschitz functions in these spaces. Malý [23] carried out a thorough and deep study of density of Lipschitz functions in Newtonian spaces based on quasi-Banach function lattices with absolutely continuous quasi-norm, using weak boundedness properties of a fractional maximal operator and various non-centered maximal operators of Hardy-Littlewood type. The absolute continuity of the function lattice quasi-norm is an essential assumption here, as it is shown through counterexamples. In [23] several density results that are very general are obtained and some concretizations are given, that extend many known results.

The density of Lipschitz functions in Newtonian spaces has important consequences. Recall that a function $u : X \to \mathbb{R}$ is said to be $\mathbf{E}$-quasicontinuous if for every $\varepsilon > 0$ there is an open set $U \subset X$ such that $Cap_{\mathbf{E}}(U) < \varepsilon$ and the restriction of u to $X \setminus U$ is continuous. The Sobolev capacity $Cap_{\mathbf{E}}(A)$ of a set $A \subset X$ is defined as $Cap_{\mathbf{E}}(A) = \inf \left\{ \|u\|_{N^{1,\mathbf{E}}(X)} : u \geq 1 \text{ on } \mathbf{E} \right\}$. If continuous functions are dense in $N^{1,\mathbf{E}}(X)$, then every function in $N^{1,\mathbf{E}}(X)$ has an $\mathbf{E}$-quasicontinuous representative and the quasi-continuity of functions in $N^{1,\mathbf{E}}(X)$ is equivalent to the outer regularity of a Sobolev capacity $Cap_{\mathbf{E}}$ [32], see [33] for $\mathbf{E} = L^p(X)$. Moreover, in the case $\mathbf{E} = L^p(X)$ with X proper Björn, Björn and Shanmugalingam [8] proved that every function in $N^{1,\mathbf{E}}(\Omega)$ is quasicontinuous in the open set $\Omega \subset X$, provided that continuous functions are dense in $N^{1,\mathbf{E}}(X)$. The results of Björn, Björn and Shanmugalingam in [8] have been generalized by Malý [25] to the case where $\mathbf{E}$ is a quasi-Banach function lattice; it is established that, assuming that X is locally compact and $\mathbf{E}$ satisfies a Vitali-Carathéodory property, the density of continuous functions in $N^{1,\mathbf{E}}(X)$ implies the quasicontinuity of all functions in $N^{1,\mathbf{E}}(X)$. If X is proper, $\mathbf{E}$ is a Banach function space with absolutely continuous norm and Lipschitz functions are dense in $N^{1,\mathbf{E}}(X)$, then the set $Lip_C(X)$ of compactly supported Lipschitz functions is dense in $N^{1,\mathbf{E}}(X)$ and for every open set $\Omega \subset X$ the closure of $Lip_C(\Omega)$ in $N^{1,\mathbf{E}}(X)$ is $N_0^{1,\mathbf{E}}(\Omega)$ [30], see [8] for $\mathbf{E} = L^p(X)$.

The aim of this paper is to give a simple proof for the density of Lipschitz functions in a Newtonian space $N^{1,\mathbf{E}}(X)$, in the spirit of corresponding proofs from [33] and [11], under the assumptions that the rearrangement-invariant Banach function space $\mathbf{E}$ has an absolutely continuous norm and its fundamental function satisfies a certain lower estimate in the sense of [5]. It is assumed that (X, d, μ) is a doubling metric measure space (with μ nonatomic), that supports an appropriate $(1, \mathbf{E})$ $-$Poincaré inequality. It is not assumed that (X, d) is complete. We also provide sufficient conditions for the local Hölder continuity of a representative of each function in $N^{1,\mathbf{E}}(X)$, using a quasiconcavity property of a power $p > \log_2 C_\mu$ of the fundamental function of $\mathbf{E}$, where C_μ is the doubling constant of the measure μ.

2. Preliminaries

Throughout this paper $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ is a Banach function space in the sense of Bennet and Sharpley [6], over a complete and $\sigma-$finite measure space (X, μ). We recall that $\mathbf{E} \subset \mathbf{L}^1_{loc}(X)$.

Definition 1. *A function $f \in \mathbf{E}$ is said to have absolutely continuous (AC) norm in $\mathbf{E}$ if and only if $\|f\chi_{E_k}\|_{\mathbf{E}} \to 0$ for every sequence $(A_k)_{k\geq 1}$ of measurable sets satisfying $A_k \to \emptyset$ $\mu-$a.e. (i.e. $\mu\left(\limsup_{k\to\infty} A_k\right) = 0$). The space $\mathbf{E}$ is said to have absolutely continuous norm if every $f \in \mathbf{E}$ has AC norm .*

Some examples of Banach function spaces with absolutely continuous norm are the Orlicz space $L^{\Psi}(X)$ determined by a doubling Young function Ψ, in particular Lebesgue spaces $L^p(X)$ with $1 \leq p < \infty$ and Lorentz spaces $L^{p,q}(X)$ with $1 < p < \infty$ and $1 \leq q < \infty$. In contrast, if the measure space (X, μ) is nonatomic, then the only function $f \in L^{\infty}(X)$ with AC norm is the zero function [6, Example I. 3.3.].

Let $f : X \to \overline{\mathbb{R}}$ be a $\mu-$measurable function. The distribution function of f is defined by $d_f(t) = \mu(\{x \in X : |f(x)| > t\})$ for $t \geq 0$. The nonincreasing rearrangement of f is

$$f^*(t) = \inf\{s \geq 0 : d_f(s) \leq t\}, \, t \geq 0.$$

Definition 2. *A Banach function space $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ is said to be rearrangement invariant if $f^* = g^*$ implies $\|f\|_{\mathbf{E}} = \|g\|_{\mathbf{E}}$.*

Lebesgue spaces and some of their generalizations, namely Orlicz spaces and Lorentz spaces are rearrangement invariant Banach function spaces.

Definition 3. *The fundamental function of a rearrangement invariant space $\mathbf{E}$ over (X, μ) is $\Phi_{\mathbf{E}} : [0, \infty) \to [0, \infty)$ defined by $\Phi_{\mathbf{E}}(t) = \|\chi_A\|_{\mathbf{E}}$, where $A \subset X$ is a $\mu-$measurable set with $\mu(A) = t$.*

Lemma 1. [6, Corollary II. 5.3]*Let $\mathbf{E}$ be a rearrangement invariant Banach function space over a resonant measure space (X, μ). The fundamental function $\Phi_{\mathbf{E}}$ satisfies: $\Phi_{\mathbf{E}}$ is increasing, vanishes only at the origin, is continuous (except perhaps at the origin) and $t \mapsto \frac{\Phi_{\mathbf{E}}(t)}{t}$ is decreasing.*

Every measure space (X, μ) with a $\sigma-$finite and nonatomic measure μ is resonant [6, Theorem II.2.7]. If (X, d) is a quasi-metric space endowed with doubling measure μ and X has no isolated points, then μ is nonatomic [21, Theorem 1].

Throughout the paper, the triple (X,d,μ) denotes a metric measure space, which is a metric space (X,d) equipped with a Borel regular measure μ, that is finite and positive on balls. Then μ is regular, i.e. inner regular and outer regular [17, p. 3]. Obviously, μ is $\sigma-$finite.

Denote by $B(x,r)=\{y\in X : d(y,x)<r\}$ and $\overline{B}(x,r)=\{y\in X : d(y,x)\leq r\}$ the open, respectively the closed ball in X, centered at $x\in X$ and with radius $r>0$.

The measure μ on the metric space (X,d) is said to be *doubling* if there is a constant $C\geq 1$ such that for every ball $B(x,r)\subset X$,

$$\mu(B(x,2r))\leq C\mu(B(x,r)). \tag{2.1}$$

The doubling constant C_μ of the doubling measure μ is the smallest constant C satisfying the above condition.

A metric space is called *proper* if all closed and bounded subsets are compact. Every proper metric space is complete and every complete metric space with a doubling measure is proper [16, Lemma 4.1.14].

If the measure μ is doubling, then Lebesgue differentiation theorem holds [17, Theorem 1.8], [16, page 77]: if $f\in \mathbf{L}^1_{loc}(X)$, for a.e. $x\in X$ we have $\lim\limits_{r\to 0}\int\limits_{B(x,r)}|f(y)-f(x)|\,d\mu(y)=0$, in particular $\lim\limits_{r\to 0}\int\limits_{B(x,r)}f(y)d\mu(y)=f(x)$.

The $\mathbf{E}-$*modulus of a family* Γ of curves in X is defined by $Mod_{\mathbf{E}}(\Gamma)=\inf\|\rho\|_{\mathbf{E}}$,where the infimum is taken over all Borel functions $\rho : X\to[0,\infty]$ satisfying $\int\limits_\gamma \rho\, ds\geq 1$ for all locally rectifiable curves γ in X.

Definition 4. *A Borel measurable function $g : X\to[0,\infty]$ is said to be an upper gradient of a function $u : X\to\mathbb{R}$ if for every rectifiable curve $\gamma : [a,b]\to X$ the following inequality holds*

$$|u(\gamma(a))-u(\gamma(b))|\leq\int\limits_\gamma gds. \tag{2.2}$$

A $\mathbf{E}-$*weak upper gradient* of a function $u : X\to\mathbb{R}$ is a Borel measurable function $g : X\to[0,\infty]$ such that (2.2) holds for all rectifiable curves $\gamma : [a,b]\to X$ except for a curve family with zero $\mathbf{E}-$modulus.

For a function $u : X \to \mathbb{R}$ denote by $G_{u,\mathbf{E}}$ the family of all $\mathbf{E}-$weak upper gradients $g \in \mathbf{E}$ of u in X. Let $\widetilde{N}^{1,\mathbf{E}}(X)$ be the set of functions $u : X \to \mathbb{R}$ with $u \in \mathbf{E}$, for which $G_{u,\mathbf{E}}$ is non-empty. For $u \in \widetilde{N}^{1,\mathbf{E}}$ define $\|u\|_{\widetilde{N}^{1,\mathbf{E}}} := \|u\|_{\mathbf{E}} + \inf\{\|g\|_{\mathbf{E}} : g \in G_{u,\mathbf{E}}\}$. The Newtonian space $N^{1,\mathbf{E}}(X)$ is defined as the quotient normed of $\widetilde{N}^{1,\mathbf{E}}(X)$ with respect to the equivalence relation defined by: $u \sim v$ if $\|u - v\|_{\widetilde{N}^{1,\mathbf{E}}} = 0$. Then $N^{1,\mathbf{E}}(X)$ is a Banach space with the norm $\|u\|_{1,\mathbf{E}} = \|u\|_{\widetilde{N}^{1,\mathbf{E}}(X)}$. [32].

For an arbitrary Banach function space $\mathbf{E}$ over a metric measure space X, the $(1,\mathbf{E})-$Poincaré inequality was introduced in [29] as a natural generalization of the Orlicz-Poincaré inequality introduced by Aïssaoui [1] and the Poincaré inequality based on Lorentz spaces, introduced by Costea and Miranda [11], that both generalize the weak $p-$Poincaré inequality introduced in the setting of metric measure spaces by Heinonen and Koskela [19]. The case $\mathbf{E} = L^{\infty}(X)$ has been studied first by Durand-Cartagena, Jaramillo and Shanmugalingam [12].

For every measurable set $A \subset X$ with $0 < \mu(A) < \infty$ and every $u \in L^1(A)$ denote the integral average of u over A by $u_A = \frac{1}{\mu(A)} \int_A u d\mu$. For every ball $B = B(x,R)$ and each constant $\tau > 0$ we denote $\tau B = B(x, \tau R)$.

Definition 5. *Let $u : X \to \mathbb{R}$ be locally integrable and $g : X \to [0,\infty]$ be Borel measurable. The pair (u,g) is said to satisfy a weak $(1,\mathbf{E})-$Poincaré inequality if there exist some constants $C_P > 0$ and $\tau \geq 1$ such that for all balls $B \subset X$*

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq C_P diam(B) \frac{\|g\chi_{\tau B}\|_{\mathbf{E}}}{\|\chi_{\tau B}\|_{\mathbf{E}}}. \tag{2.3}$$

The space (X,d,μ) is said to support a weak $(1,\mathbf{E})-$Poincaré inequality if there exist some constants $C_P > 0$ and $\tau \geq 1$ independent of u and g such that the inequality (2.3) holds whenever $u \in L^1_{loc}(X)$ and g is an upper gradient of u.

Here $\|g\chi_{\tau B}\|_{\mathbf{E}}$ stands for $N(g\chi_{\tau B})$, even in the case $N(g\chi_{\tau B}) = \infty$.

If (X,d,μ) supports a weak $(1,\mathbf{E})-$Poincaré inequality, then (2.3) holds whenever g is a $\mathbf{E}-$weak upper gradient of u, since for every $\mathbf{E}-$weak upper gradient g of a function u on X there is a sequence $(g_i)_{i\geq 1}$ of upper gradients of u, such that $\lim_{i\to\infty} \|g_i - g\|_{\mathbf{E}} = 0$ [32, Proposition 2].

The $(1, \mathbf{E})-$Poincaré inequality is stronger than the weak $\infty-$Poincaré inequality, since $\|g\chi_{\tau B}\|_{\mathbf{E}} \leq \|g\chi_{\tau B}\|_{L^\infty(X)} \|\chi_{\tau B}\|_{\mathbf{E}}$. We recall that every complete and doubling metric measure space (X, d, μ) that supports a weak $\infty-$Poincaré inequality is quasiconvex [12, Proposition 3.4], hence it has no isolated points, in particular μ is nonatomic.

Next we prove that the $(1, \mathbf{E})-$Poincaré inequality is weaker than the $p-$Poincaré inequality with $p = 1$.

The Hölder's inequality for a Banach function space $\mathbf{E}$ and its associate Banach function space $\mathbf{E}'$ on (X, μ) [6, Theorem I. 2.4] shows that for every $f \in \mathbf{E}$ and $g \in \mathbf{E}'$ we have

$$\int_X |fg| \, d\mu \leq \|f\|_{\mathbf{E}} \|g\|_{\mathbf{E}'} . \tag{2.4}$$

By [6, Theorem II. 5.2], if $\mathbf{E}$ is a rearrangement invariant Banach function space over a resonant measure space (X, μ) and $\mathbf{E}'$ is its associate space, than the product of the fundamental functions of these spaces is the identity function on $[0, \mu(X))$, i.e.

$$\Phi_{\mathbf{E}}(t) \, \Phi_{\mathbf{E}'}(t) = t \tag{2.5}$$

for every $t \in [0, \mu(X))$.

Proposition 1. *Assume that the metric measure space (X, d, μ) supports a weak $1-$Poincaré inequality. Let $\mathbf{E}$ be a rearrangement invariant Banach function space over the resonant measure space (X, μ). Then (X, d, μ) supports a $(1, \mathbf{E})-$Poincaré inequality, with the constants from the weak $1-$Poincaré inequality.*

Proof. There exist some constants $C_P > 0$ and $\tau \geq 1$, such that for every $u \in L^1_{loc}(X)$ and g an upper gradient of u

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq C_P r \frac{1}{\mu(\tau B)} \int_{\tau B} g d\mu.$$

By Hölder's inequality (2.4), $\int_{\tau B} g d\mu \leq \|g\chi_{\tau B}\|_{\mathbf{E}} \cdot \|\chi_{\tau B}\|_{\mathbf{E}'}$.

Using the identity (2.5) satisfied by the fundamental functions of $\mathbf{E}$ and its associate space $\mathbf{E}'$, we have

$$\|\chi_{\tau B}\|_{\mathbf{E}} \|\chi_{\tau B}\|_{\mathbf{E}'} = \mu(\tau B) .$$

Then $\frac{1}{\mu(\tau B)}\int\limits_{\tau B} g d\mu \leq \|g\chi_{\tau B}\|_{\mathbf{E}} \cdot \frac{\|\chi_{\tau B}\|_{\mathbf{E}'}}{\mu(\tau B)} = \frac{\|g\chi_{\tau B}\|_{\mathbf{E}}}{\|\chi_{\tau B}\|_{\mathbf{E}}}$, hence

$$\frac{1}{\mu(B)}\int\limits_{B} |u - u_B|\, d\mu \leq C_P r \frac{\|g\chi_{\tau B}\|_{\mathbf{E}}}{\|\chi_{\tau B}\|_{\mathbf{E}}}.$$

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Bastero, Milman and Ruiz [5] defined for a Banach function space $\mathbf{E}$ on $\mathbb{R}^n$ the maximal operator $\mathcal{M}_{\mathbf{E}}f(x) = \sup\limits_{Q\ni x}\frac{1}{\|\chi_Q\|_{\mathbf{E}}}\|f\chi_Q\|_{\mathbf{E}}$ for $f \in \mathbf{E}$, where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ which contain x and have sides parallel to the coordinate axes.

In [29] we considered an analogue of the maximal operator from [5] in the setting of metric measure spaces. Assume that f is a $\mu-$measurable function. If $f\chi_B \notin \mathbf{E}$ for some ball B, then $N(f\chi_B) = \infty$ and we write $\|f\chi_B\|_{\mathbf{E}} = \infty$.

Definition 6. *The noncentered maximal operator associated with the Banach function space* $\mathbf{E}$ *is defined by*

$$\mathcal{M}_{\mathbf{E}}f(x) = \sup_{B}\frac{\|f\chi_B\|_{\mathbf{E}}}{\|\chi_B\|_{\mathbf{E}}},$$

where the supremum is taken over all balls $B \subset X$ containing the point x. Here f is any $\mu-$measurable function.

For each $f \in L^1_{loc}(X)$, the function $\mathcal{M}_{\mathbf{E}}f$ is lower semicontinuous, as each superlevel set $G_t := \{x \in X : \mathcal{M}_{\mathbf{E}}f(x) > t\}$, $t > 0$ is open in X. Indeed, for every $x \in G_t$ there exist a ball $B_0 \subset X$ with $x \in B_0$ such that $\|f\chi_{B_0}\|_{\mathbf{E}} > t\,\|\chi_{B_0}\|_{\mathbf{E}}$ and $r > 0$ such that $B(x,r) \subset B_0$, and we see that $B(x,r) \subset G_t$.

The maximal operator $M_{\mathbf{E}}$ has been thoroughly studied by Malý [23], in the general case where $\mathbf{E}$ is a quasi-Banach function lattice.

The validity of a $(1,\mathbf{E})-$Poincaré inequality in doubling metric spaces, for a pair (u,g), implies a pointwise estimate for u in terms of the maximal function $\mathcal{M}_{\mathbf{E}}g$. In the setting of metric measure spaces this was first proved by Hajłasz and Koskela for $\mathbf{E} = L^p(X)$, $1 \leq p < \infty$ [15, Theorem 3.2], using a restricted maximal function.

Proposition 2. *Let (X,d,μ) be a doubling metric measure space and let $\mathbf{E}$ be a Banach function space over (X,μ). Assume that the pair (u,g) satisfies a weak $(1,\mathbf{E})-$Poincaré inequality with constants C_P and τ. Then there exists a set $A \subset X$ with $\mu(A) = 0$ such that*

$$|u(x) - u(y)| \leq C' d(x,y)\left(\mathcal{M}_{\mathbf{E}}g(x) + \mathcal{M}_{\mathbf{E}}g(y)\right) \tag{2.6}$$

for every x, $y \in X \setminus A$. Here C' is some constant depending only on C_P and the doubling constant C_μ.

3. Local Hölder continuity of Newtonian functions

We recall the notion of relative lower volume decay [16, page 213]. A metric measure space (X, d, μ) is said to satisfy a *relative lower volume decay of order* $Q \geq 0$ if there is a constant $C_0 \geq 1$ such that

$$\frac{\mu(B(x,s))}{\mu(B(a,r))} \geq \frac{1}{C_0}\left(\frac{s}{r}\right)^Q, \tag{3.1}$$

whenever $a \in X$, $0 < s \leq r$ and $x \in B(a,r)$.

It is well-known that every doubling metric measure space (X, d, μ), with the doubling constant $C_\mu \geq 1$, satisfies a relative lower volume decay of order $Q = \log_2 C_\mu$ with $C_0 = (C_\mu)^2$, see [16, Lemma 8.1.13].

Lemma 2. *If the metric measure space (X, d, μ) satisfies the lower volume decay of order $Q \geq 0$ given by (3.1), then for $f \in L^1_{loc}(X)$ and $a \in X$, $x \in B(a,r)$ and $0 < s \leq r - d(x,a)$*

$$\left|f_{B(x,s)} - f_{B(a,r)}\right| \leq C_0\left(\frac{r}{s}\right)^Q \frac{1}{\mu(B(a,r))}\int_{B(a,r)} \left|f(x) - f_{B(a,r)}\right| d\mu.$$

Proof. $\left|f_{B(x,s)} - f_{B(a,r)}\right| = \left|\frac{1}{\mu(B(x,s))}\int_{B(x,s)} f(x)\, d\mu - f_{B(a,r)}\right|$
$= \left|\frac{1}{\mu(B(x,s))}\int_{B(x,s)} \left(f(x) - f_{B(a,r)}\right) d\mu\right| \leq \frac{1}{\mu(B(x,s))}\int_{B(x,s)} \left|f(x) - f_{B(a,r)}\right| d\mu.$

By triangle inequality, $x \in B(a,r)$ and $0 < s \leq r - d(x,a)$ imply $B(x,s) \subset B(a,r)$, hence $\int_{B(x,s)} \left|f(x) - f_{B(a,r)}\right| d\mu \leq \int_{B(a,r)} \left|f(x) - f_{B(a,r)}\right| d\mu.$

Using (3.1) it follows that

$$\begin{aligned}
\left|f_{B(x,s)} - f_{B(a,r)}\right| &\leq \frac{\mu(B(a,r))}{\mu(B(x,s))}\frac{1}{\mu(B(a,r))}\int_{B(a,r)} \left|f(x) - f_{B(a,r)}\right| d\mu \\
&\leq C_0\left(\frac{r}{s}\right)^Q \frac{1}{\mu(B(a,r))}\int_{B(a,r)} \left|f(x) - f_{B(a,r)}\right| d\mu.
\end{aligned}$$

∎

The following proposition provides sufficient conditions for the existence of a Hölder continuous representative of the first component of a pair satisfying a $(1,\mathbf{E})$ −Poincaré inequality; for $\mathbf{E} = L^p(X)$ we recover some conclusions of [4, Theorem 4.1] and [15, Theorem 5.1].

Theorem 1. *Let (X,d,μ) be a doubling metric measure space with a doubling constant C_μ. Let $\mathbf{E}$ be a rearrangement invariant Banach function space over X. Assume that the fundamental function $\Phi = \Phi_{\mathbf{E}}$ of $\mathbf{E}$ satisfies the following condition: $\Phi^p(t)/t$ is nonincreasing for $t \in (0,\delta)$ for some constants $p > Q := \log_2 C_\mu$ and $\delta > 0$.*

If the pair (f,g) satisfies a $(1,\mathbf{E})$ −Poincaré inequality, where $f : X \to \mathbb{R}$ is a locally integrable function and $g \in \mathbf{E}$, then f has a representative that is locally $\left(1-\frac{Q}{p}\right)$ −Hölder continuous.

In particular, if (X,d,μ) supports a $(1,\mathbf{E})$ −Poincaré inequality, then every $f \in N^{1,\mathbf{E}}(X)$ has a locally $\left(1-\frac{Q}{p}\right)$ −Hölder continuous representative.

Proof. **Step 1.** Consider a ball $B(a,R)$ in X and $\lambda \in (0,1)$. Let $x \in B(a,(1-\lambda)R)$ be a Lebesgue point of f. By triangle inequality, $B(x,\lambda R) \subset B(a,R)$. We use a telescoping argument. Let $B_0 := B(a,R)$ and $B_i := B(x,\lambda^i R)$ for all integers $i \geq 1$. Then $B_{i+1} \subset B_i$ for every integer $i \geq 0$.

Since x is a Lebesgue point of f , we have

$$|f(x) - f_{B_0}| = \left|\lim_{n\to\infty} f_{B_n} - f_{B_0}\right| = \left|\sum_{i=0}^{\infty}\left(f_{B_{i+1}} - f_{B_i}\right)\right| \leq \sum_{i=0}^{\infty}\left|f_{B_{i+1}} - f_{B_i}\right|.$$

By Lemma 2, $\left|f_{B_{i+1}} - f_{B_i}\right| \leq C_0\lambda^Q \frac{1}{\mu(B_i)} \int_{B_i} |f(x) - f_{B_i}|\, d\mu$, for $i \geq 0$.

Using the $(1,\mathbf{E})$ −Poincaré inequality

$$\frac{1}{\mu(B)}\int_B |f(x) - f_B|\, d\mu \leq C_P diam(B) \frac{\|g\chi_{\tau B}\|_{\mathbf{E}}}{\|\chi_{\tau B}\|_{\mathbf{E}}},$$

we get $\frac{1}{\mu(B_i)}\int_{B_i} |f(x) - f_{B_i}|\, d\mu \leq 2C_P R\lambda^i \frac{\|g\chi_{\tau B_i}\|_{\mathbf{E}}}{\|\chi_{\tau B_i}\|_{\mathbf{E}}}$. Then

$$|f(x) - f_{B_0}| \leq CR\lambda^Q \sum_{i=0}^{\infty} \lambda^i \frac{\|g\chi_{\tau B_i}\|_{\mathbf{E}}}{\|\chi_{\tau B_i}\|_{\mathbf{E}}}, \tag{3.2}$$

where $C = 2C_P C_0$.

Using the monotonicity of the $\mathbf{E}-$norm induced by (P2) and the definition of the fundamental function $\Phi = \Phi_{\mathbf{E}}$ we get

(3.3)

$$\frac{\|g\chi_{\tau B_i}\|_{\mathbf{E}}}{\|\chi_{\tau B_i}\|_{\mathbf{E}}} \leq \frac{\|g\chi_{\tau B_0}\|_{\mathbf{E}}}{\|\chi_{\tau B_i}\|_{\mathbf{E}}} = \frac{\|\chi_{\tau B_0}\|_{\mathbf{E}}}{\|\chi_{\tau B_i}\|_{\mathbf{E}}} \frac{\|g\chi_{\tau B_0}\|_{\mathbf{E}}}{\|\chi_{\tau B_0}\|_{\mathbf{E}}} = \frac{\Phi\left(\mu\left(\tau B_0\right)\right)}{\Phi\left(\mu\left(\tau B_i\right)\right)} \frac{\|g\chi_{\tau B_0}\|_{\mathbf{E}}}{\|\chi_{\tau B_0}\|_{\mathbf{E}}}.$$

Now we will use the assumption that $\Phi^p\left(t\right)/t$ is nonincreasing for $t \in (0, \delta)$, hence $0 < t_1 \leq t_2 < \delta$ implies $\frac{\Phi(t_2)}{\Phi(t_1)} \leq \left(\frac{t_2}{t_1}\right)^{\frac{1}{p}}$.

We choose R, as we may, such that $\mu\left(\tau B_0\right) < \delta$. The above inequality and (3.1) imply

$$\frac{\Phi\left(\mu\left(\tau B_0\right)\right)}{\Phi\left(\mu\left(\tau B_i\right)\right)} \leq \left(\frac{\mu\left(\tau B_0\right)}{\mu\left(\tau B_i\right)}\right)^{\frac{1}{p}} \leq \left(C_0\right)^{\frac{1}{p}} \lambda^{-i\frac{Q}{p}}, \tag{3.4}$$

for all $i \geq 0$, where $Q = \log_2 C_\mu$ and $C_0 = \left(C_\mu\right)^2$.

By (3.2), (3.3) and (3.4), using $p > Q$ we get

$$\left|f\left(x\right) - f_{B_0}\right| \leq C\left(C_0\right)^{\frac{1}{p}} \frac{\|g\chi_{\tau B_0}\|_{\mathbf{E}}}{\|\chi_{\tau B_0}\|_{\mathbf{E}}} \sum_{i=0}^{\infty} \left(\lambda^{1-\frac{Q}{p}}\right)^i.$$

We proved that for every ball $B(a, R) \subset X$, if $x \in B(a, (1-\lambda)\, R)$ is a Lebesgue point of f, then the following estimate holds

$$\left|f\left(x\right) - f_{B(a,R)}\right| \leq C' R \frac{\left\|g\chi_{B(a,\tau R)}\right\|_{\mathbf{E}}}{\left\|\chi_{B(a,\tau R)}\right\|_{\mathbf{E}}}. \tag{3.5}$$

Here $C' = C\left(C_0\right)^{\frac{1}{p}} \frac{\lambda^Q}{1-\lambda^{1-\frac{Q}{p}}} = 2C_P\left(C_0\right)^{\frac{p+1}{p}} \frac{\lambda^Q}{1-\lambda^{1-\frac{Q}{p}}}$.

Step 2. Fix $b \in X$ and consider a ball $B(b, r)$. Consider $\lambda \in (0, 1)$ from Step 1. Assume that $x, y \in B(b, kr)$ are Lebesgue points of f, where $k \in (0, 1)$. The constant k and the radius r will be chosen later.

We may use the final estimate (3.5) from Step 1 with $a = x$ and $R = \frac{2}{1-\lambda} d\left(x, y\right)$, since $x, y \in B(a, (1-\lambda)\, R)$. We denote for short $B\left(x, \frac{2\tau}{1-\lambda} d(x, y)\right) = B\left(a, \tau R\right)$. Then

$$\left|f\left(x\right) - f(y)\right| \leq \frac{2C'}{1-\lambda} d(x, y) \frac{\left\|g\chi_{B(a,\tau R)}\right\|_{\mathbf{E}}}{\left\|\chi_{B(a,\tau R)}\right\|_{\mathbf{E}}}. \tag{3.6}$$

In order to have $B\left(a, \tau R\right) \subset B(b, \tau r)$, we require $\frac{2\tau}{1-\lambda} d(x, y) + kr \leq \tau r$, but $d(x, y) < 2kr$, therefore it suffices to choose $0 < k \leq \frac{(1-\lambda)\tau}{(1-\lambda)+4\tau}$.

We estimate the right hand side of (3.6) using again the assumption that $\Phi^p\left(t\right)/t$ is nonincreasing for $t \in (0, \delta)$, taking into account that

$\mu\left(B(p,\tau r)\right) < \delta$ for $r > 0$ small enough:
$$\frac{\left\|g\chi_{B(a,\tau R)}\right\|_{\mathbf{E}}}{\left\|\chi_{B(a,\tau R)}\right\|_{\mathbf{E}}} \leq \frac{\left\|\chi_{B(b,\tau r)}\right\|_{\mathbf{E}}}{\left\|\chi_{B(a,\tau R)}\right\|_{\mathbf{E}}} \frac{\left\|g\chi_{B(b,\tau r)}\right\|_{\mathbf{E}}}{\left\|\chi_{B(b,\tau r)}\right\|_{\mathbf{E}}} \leq \left(\frac{\mu(B(b,\tau r))}{\mu(B(a,\tau R))}\right)^{\frac{1}{p}} \frac{\left\|g\chi_{B(b,\tau r)}\right\|_{\mathbf{E}}}{\left\|\chi_{B(b,\tau r)}\right\|_{\mathbf{E}}}.$$
By (3.1) it follows that

$$\frac{\left\|g\chi_{B(a,\tau R)}\right\|_{\mathbf{E}}}{\left\|\chi_{B(a,\tau R)}\right\|_{\mathbf{E}}} \leq (C_0)^{\frac{1}{p}} \left(\frac{(1-\lambda)\tau}{2}\right)^{\frac{Q}{p}} d\,(x,y)^{-\frac{Q}{p}} \frac{\left\|g\chi_{B(b,\tau r)}\right\|_{\mathbf{E}}}{\left\|\chi_{B(b,\tau r)}\right\|_{\mathbf{E}}}. \tag{3.7}$$

Finally, from (3.6) and (3.7) we obtain

$$|f\,(x) - f(y)| \leq C'' d(x,y)^{1-\frac{Q}{p}} \frac{\left\|g\chi_{B(b,\tau r))}\right\|_{\mathbf{E}}}{\left\|\chi_{B(b,\tau r)}\right\|_{\mathbf{E}}}, \tag{3.8}$$

whenever $x, y \in B(b, \frac{(1-\lambda)\tau}{(1-\lambda)+4\tau} r)$ are Lebesgue points of f.

Here $C'' = \frac{2C'}{1-\lambda}\,(C_0)^{\frac{1}{p}} \left(\frac{(1-\lambda)\tau}{2}\right) = 4C_P\,(C_0)^{\frac{p+2}{p}} \left(\frac{\tau}{2}\right)^{\frac{Q}{p}} \frac{\lambda^Q}{(1-\lambda)^{1-\frac{Q}{p}}\left(1-\lambda^{1-\frac{Q}{P}}\right)}$.

Step 3. Inequality (3.8) shows that for every point $b \in X$ there exists a radius $\rho_b > 0$ depending on b and the constants $\tau, \lambda, \delta > 0$ and a constant $C_b > 0$ depending on $b, \tau, \delta, \lambda, C_\mu, C_P$ such that $|f\,(x) - f(y)| \leq C_b d(x,y)^{1-\frac{Q}{p}}$ whenever $x, y \in B(b, \rho_b)$ are Lebesgue points of f.

Since f is locally integrable and the measure μ is doubling, the complement $X \setminus L_f$ of the set L_f of Lebesgue points of f has measure zero, by Lebesgue differentiation theorem [17, Theorem 1.8], [16, page 77].

The restriction of f to L_f, that is $\left(1 - \frac{Q}{p}\right)$ −locally Hölder continuous, can be extended by continuity to a function $\widetilde{f}$ that is also $\left(1 - \frac{Q}{p}\right)$ −locally Hölder continuous. Since L_f is dense in X, for every $z \in X$ we find a sequence $(z_n)_{n\geq 1}$ that is convergent to z, with $z_n \in L_f \cap B(z, \rho_z)$ for $n \geq 1$. Using the local Hölder continuity of $f|_{L_f}$ it follows that the sequence $(f\,(z_n))_{n\geq 1}$ is Cauchy in $\mathbb{R}$, therefore it has a limit, that is independent of the sequence $(z_n)_{n\geq 1}$ converging to z and will be denoted by $\widetilde{f}\,(z)$. If $z \in L_f$, then $|f\,(z_n) - f(z)| \leq C_b d(z_n, z)^{1-\frac{Q}{p}}$ for all $n \geq 1$, hence $\widetilde{f}\,(z) = f(z)$. In particular, $\widetilde{f} = f$ a.e. in X.

Let $b \in X$. For every $v, w \in B(b, \rho_b)$ we find $(v_n)_{n\geq 1}$ convergent to v and $(w_n)_{n\geq 1}$ convergent to w, such that $v_n, w_n \in L_f \cap B(b, \rho_b)$ for $n \geq 1$. Then passing to limit as $n \to \infty$ in $|f(v_n) - f(w_n)| \leq C_b d(v_n, w_n)^{1-\frac{Q}{p}}$ we get $\left|\widetilde{f}(v) - \widetilde{f}(w)\right| \leq C_b d(v, w)^{1-\frac{Q}{p}}$, q. e.d. ∎

4. A density result

Let $\mathbf{E}$ be a Banach function space. We recall some lemmas from [31], implying that every function from $N^{1,\mathbf{E}}(X)$ in this space can be approximated by bounded functions whenever the measure μ is nonatomic, (X, d) has no isolated points and $\mathbf{E}$ is a rearrangement-invariant Banach function space with AC norm. The main motivation behind these approximation results was the preparation of the tools needed for the proof of the density result from this section.

The following lattice property of $N^{1,\mathbf{E}}(X)$ is well-known in the cases where $\mathbf{E}$ is an Orlicz space [34, Lemma 6.14] or a Lorentz space [11, Lemma 3.15, Lemma 3.16].

Lemma 3. *If $g_i \in \mathbf{E}$ is a $\mathbf{E}$-weak upper gradient of $u_i : X \to \mathbb{R}$, for $i = 1, 2$, then $u := \max\{u_1, u_2\}$ and $v := \min\{u_1, u_2\}$ have the $\mathbf{E}-$weak upper gradient $g = \max\{g_1, g_2\}$ and $g \in \mathbf{E}$.*

For $k \geq 0$ the truncation of a function $u : X \to \mathbb{R}$ at levels $\pm k$ is defined as $u_k = \max\{\min\{u, k\}, -k\}$. If $u \geq 0$, then $u_k := \min\{u, k\}$.

Lemma 4. *If $g \in \mathbf{E}$ is an $\mathbf{E}$-weak upper gradient of $u : X \to [0, \infty)$, then for every $k \in [0, \infty)$ the function g is an $\mathbf{E}$-weak upper gradient of $u_k := \min\{u, k\}$. Moreover, if $u \in \mathbf{E}$, then $u_k \in \mathbf{E}$, with $\|u_k\|_{\mathbf{E}} \leq \|u\|_{\mathbf{E}}$, for every $k \in [0, \infty)$.*

We recall that every function $u : X \to \mathbb{R}$ having an $\mathbf{E}-$weak upper gradient in $\mathbf{E}$ has a representative that belongs to $ACC_{\mathbf{E}}(X)$, i.e. is absolutely continuous along all rectifiable curves in X except for a curve family with zero $\mathbf{E}-$modulus [32, Proposition 3].

Lemma 5. *Let $\mathbf{E}$ be a Banach function space on X and $u \in ACC_{\mathbf{E}}(X)$, which is constant $\mu-$a.e. on $X \setminus \Omega$, where $\Omega \subset X$ is open. If g is an upper gradient of u or $g \in \mathbf{E}$ is a $\mathbf{E}-$weak upper gradient of u, then $g\chi_\Omega$ is also a $\mathbf{E}-$weak upper gradient of u.*

We say that $\mathbf{E}$ has property (C) if $\lim\limits_{k\to\infty} \mu(A_k) = 0$ for every sequence of measurable sets $A_k \subset X$, $k \geq 1$ with $\lim\limits_{k\to\infty} \|\chi_{A_k}\|_{\mathbf{E}} = 0$. Every rearrangement invariant Banach function space over a resonant measure space has property (C) [31].

The latter two lemmas imply the following

Lemma 6. *Assume that (X, d, μ) is a metric measure space, with μ non-atomic. Let $\mathbf{E}$ be a Banach function space over X that has absolutely continuous norm and has property (C). Let $u \in N^{1,\mathbf{E}}(X)$ be nonnegative. For each integer $k \geq 0$ we define $u_k := \min\{u, k\}$. Then $u_k \in \mathbf{E}$ for each $k \geq 0$ and the sequence $(u_k)_{k\geq 0}$ converges to u in the norm of $N^{1,\mathbf{E}}(X)$. Consequently, for each $u \in N^{1,\mathbf{E}}(X)$ and every $\varepsilon > 0$ there is a bounded function $v \in N^{1,\mathbf{E}}(X)$ such that $\|u - v\|_{N^{1,\mathbf{E}}(X)} < \varepsilon$.*

The above approximation result generalizes Proposition 6.5 from [11] and Proposition 6.16 from [34], and has in turn been generalized to Newtonian spaces based on quasi-Banach lattices [23, Corollary 3.6].

Definition 7. [5] *Let $\Phi : [0, \infty) \to [0, \infty)$ be an increasing bijection. The rearrangement invariant Banach function space $\mathbf{E}$ is said to satisfy a lower $\Phi-$estimate if there exists a positive constant $M < \infty$ such that for every finite family $\{f_i : i = 1, ..., n\} \subset \mathbf{E}$ of functions with disjoint supports,*

$$\left\| \sum_{i=1}^{n} f_i \right\|_{\mathbf{E}} \geq M\Phi\left(\sum_{i=1}^{n} \Phi^{-1}\left(\|f_i\|_{\mathbf{E}}\right) \right). \tag{4.1}$$

Under the assumption that $\mathbf{E}$ is a rearrangement invariant space satisfying a lower $\Phi-$ estimate, where $\Phi = \Phi_{\mathbf{E}}$ is the fundamental function of $\mathbf{E}$, Bastero, Milman and Ruiz [5] proved that there exists $C > 0$ such that for every $f \in \mathbf{E}$

$$\Phi_{\mathbf{E}}\left(\mu\left(\{x \in X : \mathcal{M}_{\mathbf{E}} f\,(x) > \lambda\}\right)\right) \leq \frac{C}{\lambda}\,\|f\|_{\mathbf{E}} \text{ for all } \lambda > 0.$$

The following result, proved in [28], generalizes the well-known properties of weak boundedness of the Hardy-Littlewood maximal operator of a locally integrable function on a doubling metric space (see [17, Theorem 2.2]) and extends [5, Theorem 1] from the Euclidean setting to the setting of metric measure spaces. Note that instead assuming that (X, d, μ) has no isolated points we may assume that μ is nonatomic, since we only need (X, μ) to be resonant.

Lemma 7. *Let $\mathbf{E}$ be a rearrangement invariant Banach function space on a doubling metric measure space (X, d, μ) without isolated points. Let $\Phi_{\mathbf{E}}$ and $\mathcal{M}_{\mathbf{E}}$ the corresponding fundamental function and maximal operator, respectively. If $\mathbf{E}$ satisfies a lower $\Phi_{\mathbf{E}}-$estimate, then there* *exists a positive constant C such that for every $f \in \mathbf{E}$ and all $\lambda > 0$ we have*

$$\Phi_{\mathbf{E}}\left(\mu\left(\left\{x \in X : \mathcal{M}_{\mathbf{E}} f(x) > \lambda\right\}\right)\right) \leq \frac{C}{\lambda}\|f\|_{\mathbf{E}}. \tag{4.2}$$

Moreover, if $\mathbf{E}$ *has absolutely continuous norm, then*

$$\lim_{\lambda \to \infty} \lambda \Phi_{\mathbf{E}}\left(\mu\left(\left\{x \in X : \mathcal{M}_{\mathbf{E}} f(x) > \lambda\right\}\right)\right) = 0. \tag{4.3}$$

For an extended real-valued function u on X, denote the superlevel set $\mathcal{L}_u(\lambda) = \{x \in X : u(x) > \lambda\}$. By the definition of the fundamental function, we may rewrite the above conditions (4.2) and (4.3) as

$$\begin{aligned} \left\|\lambda \chi_{\mathcal{L}_{(\mathcal{M}_{\mathbf{E}} f)}(\lambda)}\right\|_{\mathbf{E}} &\leq C\|f\|_{\mathbf{E}} \text{ and} \\ \lim_{\lambda \to \infty}\left\|\lambda \chi_{\mathcal{L}_{(\mathcal{M}_{\mathbf{E}} f)}(\lambda)}\right\|_{\mathbf{E}} &= 0, \text{ respectively.} \end{aligned}$$

Next we prove our density result, following the proof of [33, Theorem 4.1], based on an idea due to Semmes, that was extended to Orlicz-Sobolev spaces $N^{1,\Psi}(X)$ determined by a doubling Young function Ψ [34, Theorem 6.17], to Newtonian Lorentz spaces $N^{1,L^{p,q}}(X,\mu)$ with $1 \leq q \leq p < \infty$ [11, Theorem 6.9] and to Newtonian spaces $N^{1,p}(X : V)$ with $1 \leq p < \infty$ and V a Banach space [16, Theorem 8.2.1].

Theorem 2. *Let (X,d,μ) be a doubling metric measure space, with μ nonatomic. Let $\mathbf{E}$ be a rearrangement invariant Banach function space, that has absolutely continuous norm and satisfies a lower $\Phi-$esti-mate, where $\Phi = \Phi_{\mathbf{E}}$ is the fundamental function of $\mathbf{E}$. Assume that X supports a weak $(1,\mathbf{E})-$ Poincaré inequality.*

Then Lipschitz functions are dense in $N^{1,\mathbf{E}}(X)$ both in norm and in Lusin's sense: for every $u \in N^{1,\mathbf{E}}(X)$ and each $\varepsilon > 0$ there exists a Lipschitz function $v \in N^{1,\mathbf{E}}(X)$ such that $\mu(\{x \in X : u(x) \neq v(x)\}) < \varepsilon$ and $\|u - v\|_{1,\mathbf{E}} < \varepsilon$.

Proof. Since μ is $\sigma-$finite and nonatomic, (X,μ) is resonant. Then the fundamental function $\Phi = \Phi_{\mathbf{E}}$ of the rearrangement invariant Banach function space $\mathbf{E}$ has the properties from Lemma 1 and $\mathbf{E}$ has the so-called property (C).

Let $u \in N^{1,\mathbf{E}}(X)$ and $g \in \mathbf{E}$ be an upper gradient of u.

Step 1. (Reduction to the case of bounded functions)

It suffices to consider $u \geq 0$, using the decomposition into positive and negative parts. We may assume that $u \leq M$ for some constant M, using Lemma 6.

Step 2. (Bound on the mean oscillation on a ball with center in the "good" set)

Consider the superlevel set $G_\lambda = \{x \in X : \mathcal{M}_{\mathbf{E}} g(x) > \lambda\}$ for $\lambda > 0$. Note that the "bad" set G_λ is open.

Using the weak $(1, \mathbf{E})$ −Poincaré inequality and the definition of the maximal function $\mathcal{M}_{\mathbf{E}} g$ we get

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} \left|u - u_{B(x,r)}\right| d\mu \le 2C_P r \frac{\left\|g\chi_{\tau B(x,r)}\right\|_{\mathbf{E}}}{\left\|\chi_{\tau B(x,r)}\right\|_{\mathbf{E}}} \le 2C_P r \mathcal{M}_{\mathbf{E}} g(x),$$

for all $x \in X$ and $r > 0$.

For $x \in X \setminus G_\lambda$, the above inequalities imply

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} \left|u - u_{B(x,r)}\right| d\mu \le 2C_P r\lambda, \tag{4.4}$$

for every $r > 0$.

Step 3. (Distance between means on concentric balls, with center in the "good" set)

For $x \in X \setminus G_\lambda$, one finds a bound of $\left|u_{B(x,s)} - u_{B(x,r)}\right|$, where $0 < s \le r$.

Using Lemma 2, (4.4) implies $\left|u_{B(x,s)} - u_{B(x,r)}\right| \le 2C_P C_\mu r\lambda$ for $\frac{r}{2} \le s \le r$.

In the general case a similar estimate is obtained by iterating the above inequality. If $0 < s < \frac{r}{2}$, taking $k = -\left\lfloor \log_2 \frac{s}{r} \right\rfloor - 1$ we have $2^{-k-1} r \le s < 2^{-k} r$ and denoting $r_i = 2^{-i} r$ it follows that

$\left|u_{B(x,s)} - u_{B(x,r)}\right| \le \left|u_{B(x,s)} - u_{B(x,r_k)}\right| + \sum_{i=0}^{k-1} \left|u_{B(x,r_i)} - u_{B(x,r_{i+1})}\right| \le$

$2C_P C_\mu r\lambda \sum_{i=0}^{k} 2^{-i} < 4C_P C_\mu C r\lambda.$

In conclusion,

$$\left|u_{B(x,s)} - u_{B(x,r)}\right| \le 4C_P C_\mu r\lambda \text{ for } 0 < s \le r. \tag{4.5}$$

Step 4. (Lipschitz truncation defined on the "good" set)

Let $x \in X \setminus G_\lambda$.The Cauchy-type estimate $\left|u_{B(x,s)} - u_{B(x,r)}\right| \le 2Cr\lambda$ for $0 < s \le r$ implies the existence of the $\lim_{r \searrow 0} u_{B(x,r)} =: u_\lambda(x)$.

Note that $0 \le u_\lambda \le M$ everywhere on $X \setminus G_\lambda$.

If $x \in X \setminus G_\lambda$ is a Lebesgue point of u, then $u_\lambda(x) = u(x)$. Using Lebesgue differentiation theorem for $u \in \mathbf{E} \subset L^1_{loc}(X)$, we conclude that $u_\lambda = u$ μ−a.e. on $X \setminus G_\lambda$.

One proves that u_λ is $C\lambda-$Lipschitz on $X \setminus G_\lambda$, using a telescoping argument: there exists $C > 0$ such that for all $x, y \in X \setminus G_\lambda$

$$|u_\lambda(x) - u_\lambda(y)| \leq C\lambda \cdot d(x, y). \tag{4.6}$$

Let $x, y \in X \setminus G_\lambda$. Consider two chains of balls, centered at x and y, respectively: for each integer $i \geq 0$ take $B_{-i} := B\left(x, 2^{1-i} d(x, y)\right)$ and $B_{i+1} = B\left(y, 2^{-i} d(x, y)\right)$. Then $u_\lambda(x) = \lim_{i \to \infty} u_{B_{-i}}$ and $u_\lambda(y) = \lim_{i \to \infty} u_{B_{i+1}}$, hence $|u_\lambda(x) - u_\lambda(y)| \leq \sum_{k=-\infty}^{\infty} \left|u_{B_k} - u_{B_{k+1}}\right|$.

By (4.5) we get $\left|u_{B_k} - u_{B_{k+1}}\right| \leq C_P \left(C_\mu\right)^3 2^{2-k} d(x, y)\lambda$ for all integers $k \geq 1$, respectively $\left|u_{B_k} - u_{B_{k+1}}\right| \leq C_P \left(C_\mu\right)^3 2^{3+k} d(x, y)\lambda$ for all integers $k \leq -1$. For $k = 0$, using Lemma 2 and (4.4) with $r = 2d(x, y)$ we get $|u_{B_0} - u_{B_1}| \leq 4C_P \left(C_\mu\right)^3 d(x, y)\lambda$. Finally, (4.6) is obtained with $C = 16C_P \left(C_\mu\right)^3$.

Step 5. (McShane extension of the Lipschitz truncation)

By McShane extension theorem, the $C\lambda-$Lipschitz function $u_\lambda : X \setminus G_\lambda \to \mathbb{R}$ can be extended to a $C\lambda-$Lipschitz function $v_\lambda : X \to \mathbb{R}$, defined by $v_\lambda(x) = \inf\left\{u_\lambda(y) + C\lambda d(x, y) : y \in X \setminus G_\lambda\right\}$.

This McShane extension is truncated in order to maintain boundedness. Define $w_\lambda = \min(v_\lambda, M)$, $\lambda > 0$. Note that w_λ is $C\lambda-$Lipschitz on X, $0 \leq w_\lambda \leq M$ and $w_\lambda = v_\lambda = u_\lambda$ on $X \setminus G_\lambda$.

In the next two steps it is proved that $w_\lambda - u \in N^{1,\mathbf{E}}(X)$ for λ large enough and $\lim_{\lambda \to \infty} \|w_\lambda - u\|_{1,\mathbf{E}} = 0$. Moreover, w_λ solves the problem of aproximation in Lusin's sense for each $\varepsilon > 0$ there exists $\lambda_0(\varepsilon) > 0$ such that $\mu\left(\{x \in X : u(x) \neq w_\lambda(x)\}\right) < \varepsilon$ and $w_\lambda \in N^{1,\mathbf{E}}(X)$ for all $\lambda > \lambda_0(\varepsilon)$.

Step 6. (Convergence in E$-$norm of truncated McShane extensions)

We check that $w_\lambda - u \in \mathbf{E}$ for $\lambda \geq M$ and $\lim_{\lambda \to \infty} \|w_\lambda - u\|_{\mathbf{E}} = 0$

Let $\lambda \geq M$. Since $w_\lambda - u = 0$ $\mu-$a.e. on $X \setminus G_\lambda$ and $|w_\lambda - u| \leq M \leq \lambda$ on G_λ, we have $|w_\lambda - u| \leq \lambda\chi_{G_\lambda}$ $\mu-$a.e. on X.

But $\|\lambda\chi_{G_\lambda}\|_{\mathbf{E}} = \lambda\Phi\left(\mu\left(G_\lambda\right)\right) \to 0$ as $\lambda \to \infty$, according to the asymptotic estimate (4.3) for $\Phi = \Phi_{\mathbf{E}}$. The claim follows.

Step 7. (Convergence in $N^{1,\mathbf{E}}-$norm of truncated McShane extensions)

i) For every $\lambda > 0$ the function w_λ is $C\lambda$-Lipschitz, therefore the constant $C\lambda$ is an upper gradient of w_λ and w_λ is absolutely continuous on each rectifiable curve.

ii) By subadditivity of the modulus, $(C\lambda + g)$ is an upper gradient of $w_\lambda - u$ and $w_\lambda - u \in ACC_{\mathbf{E}}(X)$.

Since $w_\lambda - u = 0$ $\mu-$a.e. on $X \setminus G_\lambda$ and G_λ is open, applying Lemma 5 it follows that $(C\lambda + g)\chi_{G_\lambda}$ is a $\mathbf{E}-$weak upper gradient of $w_\lambda - u$.

As in Step 6, note that $\lambda\chi_{G_\lambda} \in \mathbf{E}$ for λ, and $\|\lambda\chi_{G_\lambda}\|_{\mathbf{E}} = \lambda\Phi(\mu(G_\lambda)) \to 0$ as $\lambda \to \infty$.

From the properties of the fundamental function of a r.i. space, $\lim\limits_{\lambda\to\infty}\Phi(\mu(G_\lambda)) = 0$ implies $\lim\limits_{\lambda\to\infty}\mu(G_\lambda) = 0$, hence $G_\lambda \to \emptyset$ $\mu-$a.e as $\lambda \to \infty$.

Since $g \in \mathbf{E}$ and $\mathbf{E}$ has absolutely continuous norm, $\|g\chi_{G_\lambda}\|_{\mathbf{E}} \to 0$ as $\lambda \to \infty$.

Then $\|(C\lambda + g)\chi_{G_\lambda}\|_{\mathbf{E}} \to 0$ as $\lambda \to \infty$, in particular $(C\lambda + g)\chi_{G_\lambda} \in \mathbf{E}$ for λ large enough.

Consequently, we get $w_\lambda - u \in N^{1,\mathbf{E}}(X)$ for λ large enough. But $\|w_\lambda - u\|_{1,\mathbf{E}} \le \|w_\lambda - u\|_{\mathbf{E}} + \|(C\lambda + g)\chi_{G_\lambda}\|_{\mathbf{E}}$ and the first claim follows, using the conclusion of Step 6.

Finally, having $w_\lambda - u = 0$ $\mu-$a.e. on $X \setminus G_\lambda$, it follows that $\mu(\{x \in X : u(x) \ne w_\lambda(x)\}) = \mu(G_\lambda)$. Since $\mu(G_\lambda) \to 0$ as $\lambda \to \infty$, this completes the proof. ∎

The proof of the above theorem can also be obtained by Theorem 4.2 in [23], since by Proposition 2 the maximal function $\mathcal{M}_{\mathbf{E}}g$ of an upper gradient g of the approximated function u is a Hajłasz gradient of u, satisfying the weak eastimate (4.3) from Lemma 7.

References

[1] N. Aïssaoui, **Another extension of Orlicz-Sobolev spaces to metric spaces**, Abstr. Appl. Anal. 2004, no.1, 1-26.

[2] L. Ambrosio, A. Pinamonti and G. Speight, **Weighted Sobolev spaces on metric measure spaces**, Journal für die reine und angewandte Mathematik (Crelle's Journal) 2019.746 (2019), 39-65.

[3] L. Ambrosio, M. Colombo and S. Di Marino, **Sobolev spaces in metric measure spaces: reflexivity and lower semicontinuity of slope**, (in) Variational methods for evolving objects (Sapporo, 2012), 1–58. Adv. Stud. Pure Math. 67, Mathematical Soc. of Japan, Tokyo, 2015.

[4] Z. M. Balogh, K. Rogovin and T. Zürcher, **The Stepanov differentiability theorem in metric measure spaces**, J. Geom. Anal. 14 (2004), no. 3, 405–422.

[5] J. Bastero, M. Milman and F. J. Ruiz, **Rearrangements of Hardy-Littlewood maximal functions in Lorentz spaces**, Proc. Amer. Math. Soc. 128 (2000), 65-74.

[6] C. Bennett and R. Sharpley, **Interpolation of Operators**, Pure and Applied Mathematics, vol. 129, Academic Press, London, 1988.

[7] A. Björn and J. Björn, **Nonlinear potential theory on metric spaces.** Vol. 17. European Mathematical Society, 2011.

[8] A. Björn, J. Björn and N. Shanmugalingam, **Quasicontinuity of Newton-Sobolev functions and density of Lipschitz functions in metric measure spaces**, Houston J. Math. 34 (2008), no. 4, 1197-1211.

[9] J. Cheeger, **Differentiability of Lipschitz functions on metric measure spaces**, Geom. Funct. Anal. 9 (1999), 428-517.

[10] Ş. Cobzaş, R. Miculescu and A. Nicolae, **Lipschitz functions**, Lecture Notes in Mathematics, vol. 2241, Springer, 2019.

[11] Ş. Costea and M. Miranda, **Newtonian Lorentz metric spaces**, Illinois Journal of Mathematics 56.2 (2012), 579-616.

[12] E. Durand-Cartagena, J. Jaramillo and N. Shanmugalingam, **The $\infty-$Poincaré inequality in metric measure spaces**, Michigan Math. J., 61 (2012), 63-85.

[13] E. Durand-Cartagena and J. A. Jaramillo, **Pointwise Lipschitz functions on metric spaces**, J. Math. Anal. Appl. 363 (2010), 525–548.

[14] P. Hajłasz, **Sobolev spaces on an arbitrary metric space**, Potential Anal. 5 (1996), 403-415.

[15] P. Hajłasz and P. Koskela, **Sobolev met Poincaré**, Mem. Amer. Math. Soc. 145 (2000), no. 688, x+101.

[16] J. Heinonen, P. Koskela, N. Shanmugalingam and J. Tyson, **Sobolev spaces on metric measure spaces: An approach based on upper gradients,** New Mathematical Monographs, Series Number 27, Cambridge University Press, Cambridge 2015.

[17] J. Heinonen, **Lectures on Analysis on Metric Spaces**, Springer-Verlag, New York, 2001.

[18] J. Heinonen and P. Koskela, **Quasiconformal maps in metric spaces with controlled geometry**, Acta Math., 181 (1998), 1-61.

[19] J. Heinonen and P. Koskela, **From local to global in quasiconformal structures**, Proc. Nat. Acad. Sci. U.S.A. 93 (1996), 554–556.

[20] A. Kamińska and M. Żyluk, **Local geometric properties in quasi-normed Orlicz space**s, Function spaces XII, Banach Center Publications, vol. 119 (2019), Institute of Mathematics Polish Academy of Sciences, Warszawa, 197-221.

[21] R. A. Macías and C. Segovia, **Lipschitz functions on spaces of homogeneous type**, Advances in Mathematics, Vol. 33 (1979), 257-270.

[22] L. Maligranda, **Type, cotype and convexity properties of quasi-Banach space**s, in Proceedings of the International Symposium on Banach and Function Spaces (Kitakyushu, 2003), Yokohama Publ., Yokohama, 2004, pp. 83–120.

[23] L. Malý, **Regularization of Newtonian functions on metric spaces via weak boundedness of maximal operators**, Journal d'Analyse Mathématique, 134, 1–54 (2018). https://doi.org/10.1007/s11854-018-0001-7

[24] L. Malý, **Newtonian spaces based on quasi-Banach function lattices**, Mathematica Scandinavica (2016), 133-160.

[25] L. Malý, **Fine properties of Newtonian functions and the Sobolev capacity on metric measure spaces**, Rev. Mat. Iberoam. 32 (2016), no. 1, pp. 219–255.

[26] L. Malý, **Minimal weak upper gradients in Newtonian spaces based on quasi-Banach function lattices**, Ann. Acad. Sci. Fenn., 78 (2013), 727-745.

[27] N. G. Meyers and J. Serrin, $\mathbf{H} = \mathbf{W}$, Proc. Nat. Acad. Sci USA, vol. 51, 1964, 1055-1056.

[28] M. Mocanu, **On a maximal operator in rearrangement invariant Banach function spaces on metric spaces**, Sci. Stud. Res. Ser. Math. Inform. 27 (2017), 49–59.

[29] M. Mocanu, **Poincaré inequalities based on Banach function spaces on metric measure spaces**, Sci. Stud. Res., Ser. Math. Inf., vol. 23, no. 1, (2013), 63-76.

[30] M. Mocanu, **Approximation by Lipschitz functions in abstract Sobolev spaces on metric spaces**, Math. Rep. (Bucur.) 15.65 (2013), 459-475.

[31] M. Mocanu, **Dense subclasses in abstract Sobolev spaces on metric measure spaces**, Sci. Stud. Res., Ser. Math. Inform., 22 (2) (2012), 71-84.

[32] M. Mocanu, **A generalization of Orlicz-Sobolev spaces on metric measure spaces via Banach function spaces**, Complex Var. Elliptic Equ., 55 (1-3)(2010), 253-267.

[33] N. Shanmugalingam, **Newtonian spaces: an extension of Sobolev spaces to metric measure spaces**, Rev. Mat. Iberoamericana 16 (2000), no.2, 243-279.

[34] H. Tuominen, **Orlicz-Sobolev spaces on metric measure spaces**, Ann. Acad. Sci. Fenn., Diss.135 (2004), 86 pp.

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