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$G(m)-$INVARIANTS AND HOMOTOPY INVARIANTS OF PARTIAL PARALLELIZABLE MANIFOLDS

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Abstract. Using a partial and global frame of tangent bundle TM, we construct some $G(M)-$invariants $[M \times \mathcal{R}^k]$ and homotopy classes $c_k(M)$ corresponding to $[M \times \mathcal{R}^k]$, $k = 1, 2, ..., p_0$. Some properties of the partial parallelizable and parallelizable manifolds are obtain with the aid of partial trivial structures in tangent bundles and certain homotopy classes.

1. Introduction

The concept of a partial parallelizable manifold (p.p.m.) is fundamental in this paper. It is inspired from [1],...,[5]. For to obtain in $G(m)-$invariants and homotopy invariants of a p.p.m. we use [5] and partial trivial structure (p.t.s.) in tangent vector bundle. We can obtain some properties of p.p.m. using the theory of vector bundles and parallelizable Grassmann manifolds are presented in [6], [7], [8], [9], ..., [15], [16].

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*In the memory of my daughter, Narcisa Apreutesei-Dumitriu

2. Partial parallelizable structures on manifolds and partial trivial structures in tangent bundles

These results belongs of manifolds endowed with vector fields. Analysis of partial trivial manifolds arose in connection with the study of vector fields on manifolds. Our idea is to replace the tangent vector fields on a differentiable manifold M with the subbundle of TM spanned by these fields. We look p.p.m. as a generalization of parallelizable manifolds. The geometrical objects considered in this paper are of $C^\infty-$classes, the morphisms are of constant rank and manifolds are paracompact manifolds. We present a construction of $G(M)-$invariants $[M \times \mathcal{R}^k]$ and homotopy classes $c_k(M) \in [M, G_k]$ of a p.p.m. (M, ρ_p).

Let M be a paracompact manifold of dimension m, TM its tangent vector bundle, and p a fixed natural number, $1 \le p < m$. Let $P_p = (X_1, X_2, ..., X_p)$ be a global partial frame (g.p.f.) of TM (i.e. $X_1(x), ..., X_p(x)$ are independent vectors, $\forall x \in M$).

Definition 2.1. *The couple (M, ρ_p) is called a partial parallelizable manifold (p.p.m.).*

To justify this term we consider two arbitrary vector fields $v = \sum_{i\equiv 1}^{p} v^i X_i$, $w = \sum_{k=1}^{p} w^k X_k$. Then fields v and w are identically if and only if $v^i = w^i$, $i = 1, 2, ..., p$.

We suppose that the topology of M is induced by its differentiable structure.

Consider the group $G(M)$ of the global diffeomorphisms of M on M. An action $\varphi \in G(M)$ induce a bijection φ' on vector fields defined by:

$$\varphi'(X)(\varphi(x)) = (\varphi(x); \varphi'_x(X(x))),$$

where φ'_x denotes the differential of φ at $x \in M$. Then, the action of $G(M)$ on ρ_p is defined by the relation:

$$(\varphi, \varphi')(\rho_p(x)) = (\varphi(x); \varphi'_2(X_1(x)), ..., \varphi'_2(X_p(x))). \tag{1}$$

Now, let $E_x(\rho_p)$ be the vector subspace of TM spanned by $\rho_p(x) = (X_1(x), ..., X_p(x))$ for $x \in M$. Then $E(\rho_p) = \cup_{x\in\mathbb{N}} E_x(\rho_p)$ is the total space of a vector bundle over M, $\xi(\rho_p) = (E(\rho_p), \Pi^p, M)$ where Π^p is the projection of $\xi(\rho_p)$. Since ρ_p is a global frame of $\xi(\rho_p)$ and M a paracompact manifold, the exact sequence

$$O \longrightarrow E(\rho_p) \longrightarrow TM \longrightarrow TM/E(\rho_p) \longrightarrow 0$$

gives the vector bundle isomorphisms $TM \simeq E(\rho_p) \oplus TM/E(\rho_p) \simeq (M \times \mathcal{R}^p) \oplus TM/E(\rho_p)$, where $E(\rho_p) \simeq M \times \mathcal{R}^p$.

Conversely: let (e_i), $i = 1, 2, ..., p$, be the canonical base of $\mathcal{R}^p$ and $r = \{(x, e_i) \mid x \in M\}$ the global frame of trivial bundle $O^p = (M \times \mathcal{R}^p \to M)$. Then, r is a global partial frame of TM. An arbitrary vector field $E(\rho_p)-$valued X has the representation $X = \sum_{i=1}^{p} \lambda^i X_i$, where λ^i are real functions. With the precedent notations, we have the following

Lemma 2.1. *Let (M, ρ) be a paracompact, partial parallelizable manifold. Then, there is a trivial vector subbundle $E(\rho_p)$ of TM. Conversely, if TM admits a trivial vector subbundle, then M is a partial parallelizable manifold. In this condition, the structure of TM is given by the isomorphism $TM \simeq (M \times \mathcal{R}^p) \oplus TM/E(\rho_p)$.*

Corollary 2.1. *The set $[M \times \mathcal{R}^p] = \{E(\rho_p)$ is the subbundle of TM generated by $\rho_p\}$ is a $G(M)-$invariant.*

Proof. If $\widetilde{\rho}_p \neq \rho_p$ and $E(\widetilde{\rho}_p)$ is the subbundle generated by $\widetilde{\rho}_p = (\varphi, \varphi')(\rho_p)$, $\varphi \in G(M)$ then $E(\rho_p) \simeq E(\widetilde{\rho}_p) \simeq M \times \mathcal{R}^p$. ∎

Definition 2.2. *The trivial subbundle $E(\rho_p) \simeq M \times \mathcal{R}^p$ is called a partial trivial structure of TM. TM is called a partial trivial vector bundle (p.t.b.).*

The partial parallelism of M is equivalent to a partial trivialization of TM. Hence, lemma 2.1 transfer some result from theory of partial parallelizable manifold (M, ρ_p) in the study of partial trivial bundle TM.

Analysis of excepted case $p = m$. In this situation, M is a parallelizable manifold. Let $\rho_m = (X_1, X_2, ..., X_m)$ be a global frame of TM. Then $E(\rho_m) = TM \simeq M \times \mathcal{R}^m$.

Let $\widehat{\rho}_m$, $\widehat{E}(\rho_m)$ be the equivalence classes of ρ_m and $E(\rho_m)$ defined by $G(M)-$actions. Finally, we have $\widehat{E}(\rho_m) = g_m(\widehat{\rho}_m)$ where g_m is a bijection given by lemma 2.1.

Remark. Similarly, we consider $G(M)-$invariants $[M \times \mathcal{R}^k] = \{E(\rho'_k)/E(\rho'_k)$ is isomorphic to $E(\rho_k)$, $(\rho, \rho')(\rho_k) = \rho'_k$, where $\rho_1 \subset \rho_2 \subset ... \subset \rho_p$, $\varphi \in G(M)\}$.

Now, we consider the set $A_k = \{\sigma_k \mid \sigma_k$ is an arbitrar g.p.f. of $TM\}$. We have two ways of viewing an element $\sigma_k \in A_k$, $1 \leq k \leq p$. It is a global partial frame of TM, and σ_k is a partial parallelism of

M. Action of $G(M)$ on A_k is given by the relation (2.1). Denote λ this equivalence relation and let $\widehat{\sigma}_k$ be the equivalence class of σ_k. Let $\widehat{E}(\sigma_k)$ be the equivalence class of $E(\sigma_k)$ relative to $G(M)-$actions. In this way the class $\widehat{E}(\rho_k)$ correspond to $\widehat{\rho}_k$. Group $G(M)$ actions on the frames of trivial vector bundle $\Theta^k = (M \times \mathcal{R}^k \to M)$ through $E(M) \simeq \Theta^k$ and the relation (2.1). Consider the fibre bundle $F(\rho_k) = E(\rho_k) \oplus \Theta^k$. Interpreting Whitney sum of two vector bundles that a fiber product [[11], p. 15], the action of $G(M)$ on frames of $F(\rho_k)$ is defined by the following relation

$$(\varphi, \varphi')(s_k) = ((\varphi, \varphi')(\rho_k), (\varphi, \varphi')(\lambda_k)),\ \varphi \in G(M), \tag{2}$$

where s_k, λ_k are global frames of $F(\rho_k)$, corresponding to action (2.2). Precedent consideration and lemma 2.1 give the following

Theorem 2.1. *Let (M, ρ_p) be a $C^\infty-$differentiable, paracompact, partial*
parallelizable manifold. Then (M,ρ_p) admit the equivalence classes $\widehat{\rho}_k, \widehat{E}(\rho_k)$, $\check{E}(\rho_k)$, $k = 1, 2, ..., p$. Each of these classes determines the remained classes.

3. $G(M)-$invariants of partial parallelizable manifold

In this section we produce certain invariants at the action of $G(M)$.

Theorem 3.1. *Let (M, ρ_p) be a $C^\infty-$differentiable, paracompact and partial parallelizable manifold. Then, there exist and are unique determined certain $G(M)-$invariants $[M \times \mathcal{R}^k]$ of (M, ρ_p), $1 \leq k \leq kp$.*

Proof. For an arbitrary fixed natural number k, $1 \leq k \leq p$ and the set $\rho_1 \subset \rho_2 \subset ... \subset \rho_k \subset ... \subset \rho_p$,, we consider the vector bundle over M, $\xi(\rho_k) = (E(\rho_k), \Pi^k, M)$, where Π^k is its projection. If $\widetilde{\rho_k} = (\varphi, \varphi')(\rho_k)$, $\varphi \in G(M)$ is a g.p.f. of TM, $\widetilde{\rho_k \neq \rho_k}$, then $\widetilde{\rho}_k$ spanned a vector subbundle of TM, $\xi(\widetilde{\rho}_k) = (E(\widetilde{\rho}_k), \widetilde{\Pi}^k, M)$. Now, we consider the equivalence class $[M \times \mathcal{R}^k]$ which is composed only of isomorphic trivial subbundles of TM: $E(\rho_k), E(\widetilde{\rho}_k)$,... Since each subbundle $E(\rho_k)$, $E(\widetilde{\rho}_k)$,... isisomorphic with $M \times \mathcal{R}^k$, we denote: $[M \times \mathcal{R}^k] = \{E(\varphi, \varphi')(\rho_k) \mid \rho_k = (X_{i_1}, .., X_{i_k}) \subset \rho_p, \varphi \in G(M)\}$. Since k is arbitrary, the couple (M, ρ_p) admits the equivalence class $[M \times \mathcal{R}^s]$, $[M \times \mathcal{R}^2]$, ..., $[M \times \mathcal{R}^p]$. These classes are determined in unique way by $\rho_1 \subset \rho_2 \subset ... \subset \rho_p$, respectively. ∎

Remark. The class $[M \times \mathcal{R}^k]$ is an invariant to action of $G(M)$ and to selection of ρ_k, $\widetilde{\rho}_k$, if $\widetilde{\rho}_k = (\varphi, \varphi')(\rho_k)$, $\varphi \in G(M)$, $k = 1, 2, ..., p$.

Definition 3.1. *The class* $[M \times \mathcal{R}^k]$ *is called a* $G(M)-$*invariant of* M, ρ_p, $k = 1, ..., p$.

Analysis of excepted case $p = m$. Since M is a parallelizable manifold there is a global frame ρ_m of TM. Then, the equivalence class $[M \times \mathcal{R}^m] = \{E((\varphi, \varphi')(\rho_m)) \mid \rho_m = (X_1, ..., X_m), \varphi \in G(M)\}$ is an $G(M)-$invariant of (M, ρ_m). Therefore (M, ρ_m) admits the invariants $[M \times \mathcal{R}^1]$, $[M \times \mathcal{R}^2], ...,$ $[M \times \mathcal{R}^m]$.

4. Homotopy invariants of partial parallelizable manifolds

Let G_s be the Grassmann variety consisting of the set of $s-$subspace of $\mathcal{R}^\infty$, $s = k, p, m-p, m$. In this section we will demonstrate some homotopy properties of p.p.m. (M, ρ_p).

Theorem 4.1. *Let* (M, ρ_p) *a* $C^\infty-$*differentiable paracompact and partial parallelizable manifold. There exist some homotopy classes* $c_k(M), c_{m-k}(M)$, $C_m(M)$ *of the partial parallelizable manifold* (M, ρ_p).

Proof. We obtain the classes $c_x(M)$ by $[M \times \mathcal{R}^x]$ and some bijections. Denote $\mathrm{Vect}_k(M)$ the set of isomorphism classes of vector bundles over M and of fiber $\mathcal{R}^k$.

Let $[M, G_k]$ be the set of homotopy classes of maps $f : M \to G_k$. There is a bijection f_k :$\mathrm{Vect}_k(M) \to [M, G_k]$ for each $k = 1, 2, ..., p$ [8]. For the sake of simplicity, we denote this theorem by the symbol $(H.C.)$ (=homotopy calssification).

Let f_k be a fixed bijection, f_k :$\mathrm{Vect}_k(M) \to [M, G_k]$. Since f_k is a bijection, the invariant $[M \times \mathcal{R}^k]$ has associated with it a unique homotopy class $c_k(M) \in [M, G_k]$ defined by the relation $c_k(M) = f_k([M \times \mathcal{R}^k])$, $k = 1, 2, .., p$.

Similarly, we obtain $c_{m-k}(M) = f_{m-k}([M \times \mathcal{R}^{m-k}])$, $c_m(M) = f_m([M \times \mathcal{R}^k])$ which are the homotopy classes corresponding to $[M \times \mathcal{R}^{m-k}]$, $[M \times \mathcal{R}^m]$ through the bijection f_{m-k} :$\mathrm{Vect}_{m-k}(M) \to [M, G_{m-k}]$, f_m :$\mathrm{Vect}_m(M) \to [M, G_m]$, respectively. It is clear that these classes are well-defined for $f_k : f_{m-k}$, f_m fixed bijections. ∎

Remark. Since f_s :$\mathrm{Vect}_s(m) \to [M, G_s]$, $s = k, m-k, m$ are bijections, precedent relations also give $G(M)-$invariants $[M \times \mathcal{R}^s]$ applying to $c_s(M)$.

We describe the transformations of classes $c_*(m) \to \widetilde{c}_*(m)$ relative to transformations $\rho_* \to \widetilde{\rho}_*$ and $f_* \to \widetilde{f}_*$, where $\rho_*, \widetilde{\rho}_*$ are g.p.f. and $f_*, \widetilde{f}_*$ are bijections given by theorem (H.C.).

We can formulate the following

Theorem 4.2. *Let (M, ρ_p) be a partial parallelizable manifold and $1 \leq k \leq p$. Then:*

1) The classes $c_k(M)$, $c_{m-k}(M)$, $c_m(M)$ are $G(M)-$invariants;

2) These classes satisfy the relations (4.1), (4.2);

3) All these classes are homotopy invariants for fixed maps f, g.

Proof. 1) Let $f, g \in c_k(M)$, $F : M \times [0,1] \to G_k$ a homotopy of maps f, g and $\varphi \in G(M)$. Then, the map $H : M \times [0,1] \to G_k$, $H(x,t) = H_t(x) = F_t(\varphi(x)) = F(\varphi(x), t)$, $x \in M$, is a homotopy of $f \circ \varphi$, $g \circ \varphi$. Consequently, $f \circ \varphi$, $g \circ \varphi \in c_k(m)$. Similarly, results that $c_{m-k}(M)$ and $c_m(M)$ are $G(M)-$invariants.

2) Since f_k, g_k are bijections, $f_k \neq g_k$, by relations $c'_k(m) = g_k(\{E(\rho_k)\})$, $c_k(M) = f_k(\{E(\rho_k)\})$ we obtain

$$c_k(M) = f_k g_k^{-1}(c'_k(m)). \tag{3}$$

This method can be applied for the classes $c_{m-k}(M), c_m(M)$ too. Then result the relations

$$c_{m-k}(M) = f_{m-k}(c'_{m-k}(M), c_m(M) = f_m g_m^{-1}(c'_m(M))), \tag{4}$$

where f_* and g_* are bijections given by the theorem (H.C.).

3) Using the definitions of $c_k(M)$, $c_{m-k}(M)$, $c_m(M)$ for fixed bijections f_k, f_{m-k}, f_m, the result follows. ∎

Definition 4.1. *For f_k, f_{m-k}, f_m fixed, the classes $c_k(M)$, $c_{m-k}(M)$, $c_m(M)$ are called homotopy invariants of (M, ρ_p), corresponding to $G(M)-$ invariants $[M \times \mathcal{R}^k]$, $[m \times \mathcal{R}^{m-k}]$, $[m \times \mathcal{R}^m]$, respectively; $k = 1, 2, ..., p$.*

Remark. Since we use only g.p.f. $(\varphi, \varphi')(\rho_k)$, $\varphi \in G(M)$, $k = 1, 2, ..., p$, and classes $\{E(\rho_k)\}$, the precedent result is natural. The set $\Gamma_k(M) = \{c'_k(M) c'_k(M) = g_k f_k^{-1}(c_k(M)), \forall g_k, f_k\}$ is independent of g_k, f_k.

Our present goal is to establish a relation between homotopy invariants $c_k(M)$, $c_{m-k}(M)$ and $c_m(M)$.

We have two ways to interpret an element $E(\rho_k) \in [M \times \mathcal{R}^k]$:

1) It is a vector subbundle of TM, and

2) $E(\rho_k)$ is a vector bundle over M, $k = 1, 2, ..., k$.

Convention: If E is a vector bundle over M, $[E]$ denotes the set of trivial subbundles of TM isomorphic to E, and $\{E\}$ denotes the class of vector bundles over M isomorphic to E. (It is clear that $[E] \neq \{E\}$). Hence, $[E]$ and $\{E\}$ are distinct classes.

We will identify the direct sum of two vector bundles with its couple of term, i.e. $E(\rho_k) \oplus TM/E(\rho_k) \leftrightarrow (E(\rho_k), TM/E(\rho_k))$. In the following result the subbundle $E(\rho_k)$ is considered only that a vector bundle, $k = 1, 2, ..., p$.

Theorem 4.3. *Let (M, ρ_p) be a $C^\infty-$differentiable, paracompact partial parallelizable manifold, TM its tangent vector bundle, and $c_k(M) = [hk]$ where $h_k : M \to G_k$ is a constant map. Then, between the homotopy classes $c_m(M)$, $c_k(M)$, $c_{m-k}(M)$ there are the relations $c_m(M) = (c_k(M), c_{m-k}(M))$. Conversely: if between these homotopy classes there exist the relations $c_m(M) = (c_k(M), c_{m-k}(M))$, then $\{TM\} = (\{E(\rho_k)\}, \{TM/E(\rho_k)\})$, $k = 1, 2, ..., p$.*

Proof. Step I. One consider the Whitney sum that a fiber product of two vector bundle [[11], p. 16] and the relation between bijections $f_m = f_k \oplus f_{m-k} = (f_k, f_{m-k})$. Then we have $c_m(M) = f_m(\{TM\}) = (f_k \oplus f_{m-k})(\{E(\rho_k) \oplus TM(E(\rho_k))\}) = (f_k(\{E(\rho_k)\}), f_{m-k}(\{TM/E(\rho_k)\})) = (c_k(M), c_{m-k}(M))$.

Step II. Let $\rho_k = (X_{i_1}, X_{i_2}, ..., X_{i_k})$, $\rho_p = (X_1, X_2, ..., X_p)$ be a partial and global frame of TM. Now, for an arbitrary fixed k, using a similar argument from the step I, we have $c_m(M) = (c_m(M), c_{m-k}(M))$. Reciprocal affirmation results by the identification of maps: $f_k^{-1} \oplus f_{m-k}^{-1} = (f_k^{-1}, f_{m-k}^{-1})$ and a similar precedent calculation.

Except case: parallelizable manifold, $p = m$. Now, $\{TM\} = \{M \times \mathcal{R}^k\}$. Therefore, $c_s(M) = f_s(\{M \times \mathcal{R}^s\})$, $c_{m-s}(M) = f_{m-s}(\{M \times \mathcal{R}^{m-s}\})$ and $c_m(M) = (c_s(M), c_{m-s}(M))$, $s = 1, 2, ..., m-1$. ∎

Remark. Therefore, some results of the p.p.m. are obtained with the aid of the p.t.s. in TM and bijections $\mathrm{Vec}_s(M) \to [M, G_s]$, $s = 1, 2, ..., m-1$.

4.1. **A characterization of partial parallelizable manifolds.** In this sections we obtain a fundamental relation between partial parallelism of (M, ρ_p) and homotopy class $c_p(M)$.

For a vector bundle E_p over M, we denote p =dimension of fiber of E_p.

Theorem 4.4. *Let M be a $C^\infty-$ differentiable, paracompact manifold and $f^*\gamma_p$ the trivial induced bundle of γ_p under a map $f : M \to G_p$.*

1) Then M is a partial parallelizable manifold if and only if there exists a subbundle E_p of TM, where E_p and $f^*\gamma_p$ are isomorphic.

In these conditions, the structure of TM is given by isomorphisms $TM \simeq f^*\gamma_p \oplus TM/E_p \simeq (M \times \mathcal{R}^p) \oplus TM/E_p$.

2) If the subbundle E_p defines a partial trivial structure in TM, then E_p and $f^*\gamma_p$ define same homotopy class $c_p(M)$.

Proof. 1) Because M is a paracompact manifold, the vector bundle E_p is isomorphic to $f^*\gamma_p$ for a map $f : M \to G_p$ [[11], p. 31]. We remark that two isomorphic vector bundle are simultaneous trivial. It is well known that a vector bundle is trivial if and only if this admit a global frame. Let σ_p be a global frame of E_p. Then the couple (M, σ_p) denotes a partial parallelizable manifold.

Conversely: Let (M, σ_p) be a partial parallelizable manifold. Using the partial global frame σ_p, we can construct a trivial bundle, $E(sigma_p)$ of TM (lemma 2.1). Therefore, $E(\sigma_p)$ is a trivial bundle, $E(\sigma_p) \simeq M \times \mathcal{R}^p$. Since (M) is a paracompact manifold, we use the result [11]: Vector bundle $E(\sigma_p)$ is isomorphic to induced vector bundle $h^*\sigma_p$ for some map $h : M \to G_p$. Thus, $E(\sigma_p) \simeq M \times \mathcal{R}^p \simeq h^*\sigma_p$. Via the lemma 2.1 and precedent results, we have $TM \simeq f^*\gamma_p \oplus TM/E_p \simeq (M \times \mathcal{R}^p) \oplus TM/E_p$.

2) Now, we use the definition of the class $c_p(M) = f([M \times \mathcal{R}^p])$ where $f_p : M \to G_p$ is a bijection (theorem (H.C.)). If the subbundle E_p defines a partial structure in TM, then $E_p \simeq M \times \mathcal{R}^p$. Evidently, $c_p(M) = f_p(\{f^*\gamma_p\}) = f_p(\{E_p\})$. ∎

Remark. There exists a similar result for each couple $(E_k, f^*\gamma_k)$ where k is a fixed index of set $(1, 2, ..., p)$.

Corollary 4.2. *If $c_k(M) = [h_k]$ where $h_k : M \to G_k$ is a constant map and $h_k^*\gamma_k$ is a subbundle of TM, then M is a partial parallelizable manifold, $k = 1, 3, ..., p$.*

Proof. For $c_k(M) = [h_k]$, where $h_k : M \to G_k$ is a constant map. it is obvious that $E_k = h_k^*\gamma_k \simeq M \times \mathcal{R}^k$. Statement follows. ∎

5. Associated equivalence classes to a partial trivial structure

In this section $E(\rho_k)$ is viewed as a vector bundle. Let $\{E(\rho_k)\}$ be the isomorphism class of vector bundle $E(\rho_k)$, $1 \leq k \leq p$. Since ρ_k is a global frame of $E(\rho_k)$, it follows that $E(\rho_k)$ is isomorphic to $M \times \mathcal{R}^k$ and $\{E(\rho_k)\} = \{M \times \mathcal{R}^k\}$. By theorem (H.C.), there is a bijection f_k :$\mathrm{Vect}_k(M) \to [M, G_k]$. Then $f_k(\{E(\rho_k)\}) = c_k(M)$. Using the bijection g_k given by lemma 2.1, we get

Lemma 5.1. *Let (M,ρ_p) be a $C^\infty-$differentiable, paracompatc partial parallelizable manifold. There exist following logical equivalences*

$$\widehat{\rho}_k \xleftrightarrow{g_k} \{E(\rho_k)\} = \{M\times\mathcal{R}^k\} \xleftrightarrow{f_k} c_k(M),\ k=1,2,...,p.$$

Now, we suppose that $c_k(M)=[h_k]$, where $h_k: M\to G_k$ is a constant map. Then, the induced bundle of γ_k under $h_k, h_k^*\gamma_k$, is a trivial bundle, $h^*\gamma_k\simeq M\times\mathcal{R}^k$. This result is independent from $h_k\in c_k(M)$. Indeed, let $s_k: M\to G_k$ be a map from $c_k(M)$ where $s_k\neq$constant map. Since M is a paracompact manifold and h_k, $s_k\in c_k(M)$, we have the isomorphisms of vector bundle $h_k^*\gamma_k\simeq s_k^*\gamma_k\simeq M\times\mathcal{R}^k$ [[11], p. 29]. Then, for isomorphism classes of vector bundles overs M there is the result

Lemma 5.2. *Let (M,ρ_p) be a $C^\infty-$differentiable, paracompact and partial parallelizable manifold. Suppose that $c_k(M)=[h_k]$ where $h_k: M\to G_k$ is a constant map. Then $\{h_k^*\gamma_k\}=\{s_k^*\gamma_k\}=\{M\times\mathcal{R}^k\}=f_k^{-1}(c_k(M))$ where f_k^{-1} is the inverse of f_k.*

Following result is an immediat corollary of the lemmas 5.1 and 5.2.

Theorem 5.1. *Let (M,ρ_p) be a $C^\infty-$differentiable, paracompact and partial parallelizable manifold. There are the logical equivalences*

$$\widehat{\rho}_k{}^{g_k} \longleftrightarrow \{E(\rho_k)\} \xleftrightarrow{f_k} c_k(M) \xleftrightarrow{f_k^{-1}} \{h_k^*\gamma_k\}$$

where $c_k(M)=[h_k]$, $h_k: M\to G_k$ is constant map, $1\leq k\leq p$.

Remark. Lemma 2.1 and precedent results give a homotopy description for partial trivial structure in tangent bundle TM using the class $c_k(M)$.

In the following examples ρ_k can be interpreted as a p.t.s. in tangent vector bundle or as a parallelism on manifold.

1) Let N be a parallelizable manifold, $p=$dim$N$ and M a differentiable manifold. If $f_p=(X_1,...,X_p)$ is a global frame of tangent bundle TN, then ρ_p defines a integrable partial trivial structure in $T(M\times N)$. Also, $\rho_k=(X_{i_1},...,X_{i_k})\subseteq\rho_p$ is a partial trivial structure in $T(M\times N)$, $k=1,2,...,p$. In the case $k=p$ maximal integral manifold is N.

Also, we deduce certain particular cases.

2) $M\times S^l$ admits a integrable partial structure and S^l is the maximal integral manifold, $k=1,3,7$. For $k=$odd, $k\neq 1,3,7$, $M\times S^k$ is a p.p.m.

3) Let $G_p(\mathcal{R}^n)$ be the Grassmann manifold of all linear $p-$subspaces of $\mathcal{R}^n$. The only parallelizable Grassmann manifolds are $G_1(R^2)$,

$G_1(R^4)$, $G_4(R^3)$, $G_1(R^8)$, $G_7(R^8)$ ([6], [16]). If M is differentiable manifold, then $M \times G_1(R^2), M \times G_1(R^4), M \times G_4(R^3), M \times G_1(R^8), M \times G_7(R^8)$ are endowed with integrable partial trivial structures.

4) Consider a Lie group G_r, r =dimG_r, and M a differentiable manifold. The manifold $T(M \times G\alpha)$ is endowed with an integrable partial trivial structure defined by a partial and global frame $\rho\omega = (X_1, ..., X_r)$ of TG_r. The integral maximal manifold is G_r. There exist partial trivial structures defined by partial and global frame $\rho_k \subset \rho\omega$, $1 \leq k < r$.

5) Evidently, an analogous result is given for the manifolds $M \times T^k$ and $M \times R^k$.

6) If there exists a regular vector field X on a manifold M, then $\rho_s = \{X\}$ defines an integrable partial structure in TM.

7) Let G be a Lie transformation group on a $C^\infty-$manifold M, dim$M = m$ and dim$G = n$. Consider the space $\chi(M)$ of $C^\infty-$vector fields on M and $\overline{G}$ =Lie algebra of G. Group G is a parallelizable manifold and therefore there exist partial trivial structure in the vector bundle TM.

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