"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 34 (2024), No. 1, 47 - 56

REMARKS ON GENERALIZATIONS OF TOPOLOGICAL SPACES VIA PROPERTIES OF CLOSURE FUNCTIONS

SHYAMAPADA MODAK, TAKASHI NOIRI

Abstract. The fixed points of a closure function are known as closed sets in the corresponding generalized closure space and their complements are called open sets. We identify among the combinations of usual properties of a closure function some that are sufficient (but not necessary, as we show through counterexamples) in order to obtain that the family of open sets is a specific generalization of the notion of topology (namely, weak structure, minimal structure, generalized topology in the sense of Csaszar, supratopology, generalized topology in the sense of Lugojan M-structure). The properties of other operators associated to a closure function (interior, exterior and boundary operators) are also investigated.

1. Introduction and Preliminaries

Let Z be a nonempty set, 2^Z be the collection of all subsets of Z and $cl : 2^Z \to 2^Z$ be a set-valued function, called here closure functions, also known as generalized closure operator. We call $cl(A)$, the closure of A for each $A \in 2^Z$ and the pair (Z, cl) is called a generalized closure space.

Keywords and phrases: Generalized closure space, minimal structure, generalized topology.
(2020) Mathematics Subject Classification: 554A05.

Definition 1. *The closure function* $cl : 2^Z \to 2^Z$ *in a generalized closure space* (Z, cl) *is called:*

(a) grounded if $\emptyset = cl(\emptyset)$,
(b) isotonic if $A \subseteq B$ *implies* $cl(A) \subseteq cl(B)$,
(c) extensive if $cl(A) \supseteq A$ *for all* $A \in 2^Z$,
(d) subadditive if $cl(A) \cup cl(B) \supseteq cl(A \cup B)$,
(e) idempotent if $cl(A) = cl(cl(A))$,
(f) additive if $cl(A \cup B) = cl(A) \cup cl(B)$.

Note that closure function cl is isotonic if and only if cl is supraaditive, i.e. $cl(A \cup B) \supseteq cl(A) \cup cl(B)$. Therefore every closure function is isotonic and that every isotonic and subadditive closure function is additive.

A *Čech closure operator* (also known in General topology as preclosure operator) is defined by three properties: grounded, extensive and additive.

A Kuratowski closure operator is an idempotent *Čech closure operator*. Pervin [14] had shown that, a closure function $cl : 2^Z \to 2^Z$ is a Kuratowski closure operator if and only if it satisfies the single axiom: $A \cup cl(A) \cup (cl(B)) = cl(A \cup B) \setminus cl(\emptyset)$ for any $A, B \in 2^Z$.

A set $F \in 2^Z$ is said to be closed in the generalized closure space (Z, cl) if $F = cl(F)$ holds (similar type of closed set has been defined in [17]). F is said to be open if $Z \setminus F$ is closed i.e. $cl(Z \setminus F) = Z \setminus F$. We denote $clop(Z)$ as the collection of all open sets in a generalized closure space (Z, cl).

However, if a closure function cl satisfies the conditions (a), (c) and (f) then the space (Z, cl) is called a closure space and it was introduced by Čech [2]. The author Chattopadhyay and Thron [3] and Modak and Islam [12] have studied this type of closure spaces.

Through this paper, we shall discuss about the properties of generalized closure functions and find out the boundary points and exterior points of a set in the generalized closure space.

2. Properties of closure functions

Theorem 2. *For a generalized closure space* (Z, cl), *the closure function cl is grounded if and only if* $Z \in clop(Z)$.

Proof. Suppose $cl(\emptyset) = \emptyset$. Then $cl(Z \setminus Z) = Z \setminus Z$ implies $Z \in clop(Z)$.

Conversely suppose that $Z \in clop(Z)$. Then $cl(Z \setminus Z) = Z \setminus Z$ implies $cl(\emptyset) = \emptyset$. ∎

Theorem 3. *Let (Z, cl) be a generalized closure space. If the closure function cl is extensive, then $\emptyset \in clop(Z)$.*

Proof. Given that $Z \subseteq cl(Z)$. Then $Z \subseteq cl(Z) \subseteq Z$ implies $Z = cl(Z)$. Hence $cl(Z \setminus \emptyset) = Z \setminus \emptyset$, so $\emptyset \in clop(Z)$. ∎

The converse of Theorem 3 does not hold in general:

Example 4. *Let $Z = \{o_1, o_2, o_3\}$. Define $cl : 2^Z \to 2^Z$ by $cl(\emptyset) = \emptyset$, $cl(Z) = Z$, $cl(\{o_1\}) = Z$, $cl(\{o_2\}) = \{o_2\}$, $cl(\{o_3\}) = \{o_3\}$, $cl(\{o_1, o_2\}) = \{o_1, o_2\}$, $cl(\{o_1, o_3\}) = \{o_1, o_2\}$, $cl(\{o_2, o_3\}) = \{o_2, o_3\}$. Here $\emptyset \in clop(Z)$, but the closure function cl is not extensive, since $\{o_1, o_3\} \nsubseteq cl(\{o_1, o_3\})$.*

Theorem 5. *If the closure function cl of a generalized closure space (Z, cl) is isotonic and extensive, then for $\{V_i : i \in J\} \subseteq clop(Z)$, $\bigcup\limits_i V_i \in clop(Z)$.*

Proof. Note that $cl(Z \setminus \bigcup\limits_i V_i) \subseteq cl(Z \setminus V_i)$, for each i. Then $cl(Z \setminus \bigcup\limits_i V_i) \subseteq (Z \setminus V_i)$, for each i. This implies that $cl(Z \setminus \bigcup\limits_i V_i) \subseteq \bigcap\limits_i (Z \setminus V_i) = (Z \setminus \bigcup\limits_i V_i)$. Thus we have $cl(Z \setminus \bigcup\limits_i V_i) \subseteq (Z \setminus \bigcup\limits_i V_i) \subseteq cl(Z \setminus \bigcup\limits_i V_i)$ (due to extensive). Therefore, $\bigcup\limits_i V_i \in clop(Z)$. ∎

Corollary 6. *If the closure function cl of a generalized closure space (Z, cl) is isotonic and extensive, then for $\{V_i : i \in \mathbb{N}\} \subseteq clop(Z)$, $\bigcup\limits_{i \in \mathbb{N}} V_i \in clop(Z)$.*

For the converse of Theorem 5, we discuss following:

Example 7. *Let $Z = \{o_1, o_2, o_3\}$. Define $cl : 2^Z \to 2^Z$ by $cl(\emptyset) = \emptyset$, $cl(Z) = Z$, $cl(\{o_1\}) = Z$, $cl(\{o_2\}) = \{o_2\}$, $cl(\{o_3\}) = \{o_3\}$, $cl(\{o_1, o_2\}) = \{o_1, o_2\}$, $cl(\{o_1, o_3\}) = \{o_1, o_2\}$, $cl(\{o_2, o_3\}) = \{o_2, o_3\}$. Here $\emptyset \in clop(Z)$, but the closure function cl is not extensive, since $\{o_1, o_3\} \nsubseteq cl(\{o_1, o_3\})$. Again, the function cl is not isotonic, since for $\{o_3\} \subseteq \{o_1, o_3\}$, $cl(\{o_3\} \nsubseteq cl(\{o_1, o_3\})$.*

Theorem 8. *Let (Z, cl) be a generalized closure space. If the closure function cl is extensive and subadditive, then $\bigcap\limits_{i=1}^{k} V_i \in clop(Z)$ for every $V_1,\ V_2, ..., V_k \in clop(Z)$.*

Proof. $Z \setminus \bigcap V_i \subseteq cl(Z \setminus \bigcap V_i) = cl(\bigcup(Z \setminus V_i)) \subseteq cl(Z \setminus V_1) \cup ..... \cup cl(Z \setminus V_n) = (Z \setminus V_1) \cup .... \cup (Z \setminus V_n) = Z \setminus \bigcap V_i$. Therefore, $cl(Z \setminus \bigcap V_i) = Z \setminus \bigcap V_i$. ∎

For the converse of Theorem 8, we give the following example.

Example 9. *In Example 4, since* $clop(Z) = \{Z, \emptyset, \{o_1\}, \{o_3\}, \{o_1, o_2\}, \{o_1, o_3\}\}$, $clop(Z)$ *is closed under finite intersection. But the closure function* cl *is not extensive.*

Example 10. *Let* $Z = \{o_1, o_2, o_3\}$, $cl(\emptyset) = \emptyset$, $cl(\{o_1\}) = \{o_1, o_2\}$, $cl(\{o_2\}) = \{o_2, o_3\}$, $cl(\{o_3\}) = \{o_2, o_3\}$, $cl(\{o_1, o_2\}) = Z$, $cl(\{o_2, o_3\}) = Z$, $cl(\{o_1, o_3\}) = \{o_1, o_3\}$, $cl(Z) = Z$. *Here the closure function* cl *is not subadditive because:* $cl(\{o_2\} \cup \{o_3\}) = cl(\{o_2, o_3\}) = Z \nsubseteq cl(\{o_2\}) \cup cl(\{o_3\}) = \{o_2, o_3\}$. *However* $clop(Z) = \{Z, \emptyset, \{o_2\}\}$, $clop(Z)$ *is closed under finite intersection.*

Theorem 11. *If the closure function* cl *of a generalized closure space* (Z, cl) *is additive, then* $\bigcap_{i=1}^{k} U_i \in clop(Z)$ *for every* $U_1, U_2, \ldots, U_k \in clop(Z)$.

Proof. For the subcollection $\{U_1, U_2, ..., U_k\}$ of $clop(Z)$, $cl(Z \setminus \bigcap_{i=1}^{k} U_i)) = cl(\bigcup_{i=1}^{k}(Z \setminus U_i) = (Z \setminus U_1) \cup .... \cup (Z \setminus U_k) = (Z \setminus \bigcap_{i=1}^{k} U_i)$. Thus $\bigcap_{i=1}^{k} U_i \in clop(Z)$. ∎

In Theorem 11, the condition is sufficient and it is followed by the following example:

Example 12. *In Example 4, the closure function* cl *is not additive but finite intersections of the members of* $clop(Z)$ *belongs to* $clop(Z)$.

Theorem 13. *Let* (Z, cl) *be a generalized closure space. If* cl *is isotonic and subadditive, then* $\bigcap V_i \in clop(Z)$ *for every* $V_1, V_2, ..., V_k \in clop(Z)$.

Proof. If cl is isotonic and subadditive, then it is additive. Thus the proof is obvious by Theorem 11. ∎

In the following example, we shall discuss the converse of Theorem 13:

Example 14. *Let* $Z = \{o_1, o_2, o_3\}$, $cl(\emptyset) = \emptyset$, $cl(\{o_1\}) = \{o_1, o_2\}$, $cl(\{o_2\}) = \{o_2, o_3\}$, $cl(\{o_3\}) = \{o_2, o_3\}$, $cl(\{o_1, o_2\}) = Z$, $cl(\{o_2, o_3\}) = Z$, $cl(\{o_1, o_3\}) = \{o_1, o_3\}$, $cl(Z) = Z$. *Then* $clop(Z) = \{\emptyset, Z, \{o_2\}\}$. *Here, the closure function* cl *is neither isotonic nor subadditive because:* $\{o_3\} \subseteq \{o_1, o_3\}$ *but* $cl(\{o_3\}) \nsubseteq cl(\{o_1, o_3\})$ *and* $cl(\{o_2\} \cup \{o_3\}) \nsubseteq cl(\{o_2\}) \cup cl(\{o_3\})$.

3. Applications of *clop*(Z)

Definition 15. *A collection $\mathcal{F}$ of subsets of a nonempty set Z is called*

(1) a weak structure [4] *(denoted by $\mathcal{W}$) if $\emptyset \in \mathcal{F}$,*

(2) a minimal structure [15] *(denoted by m_Z) if $\emptyset,\ Z \in \mathcal{F}$,*

(3) a generalized topology [5] *(denoted by μ) if $\emptyset \in \mathcal{F}$ and $\mathcal{F}$ is closed under arbitrary unions,*

(4) a supratopology [9] *(denoted by τ^*) if $Z \in \mathcal{F}$ and $\mathcal{F}$ is closed under arbitrary unions,*

(5) a generalized topology [8] *if $\emptyset,\ Z \in \mathcal{F}$ and $\mathcal{F}$ is closed under arbitrary unions,*

(6) an $\mathcal{M}$-structure [1] *if $\emptyset, Z \in \mathcal{F}$ and $\mathcal{F}$ is closed under finite intersections.*

There are some mathematical relations among the above mathematical structures (see [13])

Theorem 16. *For the closure function cl of a generalized closure space (Z, cl), the following properties hold:*

(1) If cl is extensive, then $clop(Z)$ is a weak structure,

(2) If cl is extensive and grounded, then $clop(Z)$ is a minimal structure,

(3) If cl is extensive and isotonic, then $clop(Z)$ is a generalized topology (in the sense of Csaszar),

(4) If cl is isotonic, extensive and grounded, then $clop(Z)$ is a supratopology,

(5) If cl is isotonic, extensive and grounded, then $clop(Z)$ is a generalized topology (in the sense of Lugojan),

(6) If cl is subadditive, extensive and grounded, then $clop(Z)$ is an $\mathcal{M}$-structure,

(7) If cl is isotonic, extensive, grounded and subadditive, then $clop(Z)$ is a topology.

Proof. (1) Obvious from Theorem 3.

(2) Obvious from Theorem 2 and Theorem 3.

(3) Obvious from Theorem 3 and Theorem 5.

(4) Obvious from Theorem 2 and Theorem 5.

(5) Obvious from Theorem 2, Theorem 3 and Theorem 5.

(6) Obvious from Theorem 2, Theorem 3 and Theorem 11.

(7) Obvious from Theorem 2, Theorem 3, Theorem 5 and Theorem 11. ∎

Note that the closure operator associated to a minimal structure m_Z is the intersection of all supersets of a given set that are m_Z closed, i.e. are complements of sets in m_Z. This closure has been discussed in [10, 11, 15] and it is induced by the grounded, isotonic and extensive closure function. More closure functions have been considered in [16, 7, 6, 17].

Interrelations between various mathematical structures induced from closure function have been shown in Diagram 1.

$$\begin{tikzcd}
\boxed{\text{Extensive, Grounded, Isotonic, Subadditive}} \arrow[d] & \boxed{\text{Isotonic, Extensive, Grounded}} \arrow[d] & \boxed{\text{Extensive, Isotonic}} \arrow[d] \\
\text{Topology} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Generalized Topology(Lugojan)} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Generalized Topology(Csazar)} \arrow[d, Rightarrow] \\
\mathcal{M}\text{-Structure} \arrow[r, Rightarrow] & \text{Minimal Structure} \arrow[r, Rightarrow] & \text{Weak Structure} \\
\boxed{\text{Subadditive, Extensive, Grounded}} \arrow[u] & \boxed{\text{Extensive, Grounded}} \arrow[u] & \boxed{\text{Extensive}} \arrow[u]
\end{tikzcd}$$

FIGURE 1. Diagram 1

Boundary points:

The conjugate of the closure function $cl : 2^Z \to 2^Z$ is called the interior function $int : 2^Z \to 2^Z$ and defined as:

$$int(A) = Z \setminus cl(Z \setminus A).$$

Lemma 17. *Let A and B be two subsets of a generalized closure space (Z, cl). Then:*

(1) $int(A) \subseteq A$, if the closure function cl is extensive.

(2) for $A \subseteq B$, $int(A) \subseteq int(B)$, if the closure function cl is isotonic.

(3) $int(Z \setminus int(Z)) = \emptyset$, if the closure function is grounded.

Definition 18. *Let A be a subset of a generalized closure space (Z, cl), $bd(A) = cl(A) \cap cl(Z \setminus A)$ is said to be boundary of A.*

Theorem 19. *Let A be a subset of a generalized closure space (Z, cl), the following statements hold:*

(1) $cl(A) = int(A) \cup bd(A)$.

(2) $bd(A) = bd(Z \setminus A)$.

(3) $Z \setminus bd(A) = int(A) \cup int(Z \setminus A)$.

(4) $bd(A) = cl(A) \setminus int(A) = cl(Z \setminus A) \setminus int(Z \setminus A)$.

(5) $bd(A)$ is the set of all $x \in Z$ such that $x \notin int(A)$ and $x \notin int(Z \setminus A)$.

(6) $A \cup bd(A) \subseteq cl(A)$, if the closure function cl is extensive.

(7) $int(A) \subseteq A \setminus bd(A) \subseteq cl(A)$ if the closure function cl is expanding.

(8) for closed set A, $A \cup bd(A) \subseteq A$, when the closure function cl is extensive.

(9) A is open if and only if $A \cap bd(A) = \emptyset$, when the closure function is extensive.

Proof. (1) $bd(A) \cup int(A) = [cl(A) \cap cl(Z \setminus A)] \cup (int(A)) = [cl(A) \cap (Z \setminus int(A))] \cup int(A) = cl(A)$.

(2) $bd(Z \setminus A) = cl(Z \setminus A) \cap cl(A) = bd(A)$.

(3) $Z \setminus bd(A) = Z \setminus [cl(A) \cap cl(Z \setminus A)] = [Z \setminus cl(A)] \cup [Z \setminus cl(Z \setminus A)] = int(Z \setminus A) \cup int(A)$.

(4) The first equation: $bd(A) = cl(A) \cap cl(Z \setminus A) = cl(A) \cap (Z \setminus int(A)) = cl(A) \setminus int(A)$.

Second part: We know that $bd(A) = bd(Z \setminus A)$. Then we replaced A by $Z \setminus A$ in the above relation and we get $bd(A) = bd(Z \setminus A) = cl(Z \setminus A) \setminus int(Z \setminus A)$.

(5) $bd(A) = cl(A) \cap cl(Z \setminus A) = cl(A) \cap (Z \setminus int(A)) = [Z \setminus int(X \setminus A)] \cap [Z \setminus int(A)]$. Then for $x \in bd(A)$, $x \in [Z \setminus int(X \setminus A)]$ and $x \in [Z \setminus int(A)]$. Thus $x \notin int(Z \setminus A)$ and $x \notin int(A)$. The converse is true. Therefore, $bd(A)$ is the set of all $x \in X$ such that $x \notin int(A)$ and $x \notin int(Z \setminus A)$.

(6) Since $bd(A) \subseteq cl(A)$, then $A \cup bd(A) \subseteq cl(A)$ (since the closure function is extensive).

(7) By (6) $cl(A) \supseteq A \cup bd(A)$. Then $cl(Z \setminus A) \supseteq (Z \setminus A) \cup bd(Z \setminus A)$. Thus $Z \setminus int(A) \supseteq (Z \setminus A) \cup bd(A)$, and $int(A) \subseteq [Z \setminus (Z \setminus A)] \cap (Z \setminus bd(A)) = A \cap (Z \setminus bd(A)) = A \setminus bd(A)$.

(8) We have $bd(A) \cup A \subseteq cl(A) = A$ as A is closed. Thus, $bd(A) \subseteq A$.

(9) Suppose that $A \cap bd(A) = \emptyset$. Then $A \cap [cl(A) \cap cl(Z \setminus A)] = \emptyset$, and hence $A \cap cl(Z \setminus A) = \emptyset$. This implies that $A \cap [Z \setminus int(A)] = \emptyset$, and hence $A \setminus int(A) = \emptyset$. So $A \subseteq int(A)$, and $A = int(A)$ (from Lemma 17). Therefore, $Z \setminus A = cl(Z \setminus A)$ and $Z \setminus A$ is closed. Therefore, A is open.

Conversely, suppose A is open in (Z, cl). By (4), $A \cap bd(A) = A \cap [cl(A) \setminus int(A)] = A \setminus int(A) = \emptyset$ because A is open. ∎

Exterior points

Definition 20. *Let E be a subset of a generalized closure space (Z, cl). The $ext(E) = int(Z \setminus E)$ is said to be exterior of E.*

Theorem 21. *For subsets E and F of a generalized closure space (Z, cl), the following properties hold:*

(1) $ext(E)$ is open.

(2) $ext(E) = int(Z \setminus E) = Z \setminus cl(E)$.

(3) $ext(ext(E)) = int(cl(E))$.

(4) if $E \subseteq F$, then $ext(E) \supseteq ext(F)$, when the closure function cl is isotonic.

(5) $ext(E \cup F) \subseteq ext(E) \cup ext(F)$, when the closure function cl is isotonic.

(6) $ext(E \cap F) \supseteq ext(E) \cap ext(F)$, when the closure function cl is isotonic.

(7) $ext(\emptyset) = Z$, when the closure function cl is grounded.

(8) $ext(Z) = \emptyset$, when the closure function cl is extensive.

(9) $ext[Z \setminus ext(E)] \subseteq ext(E)$, when the closure function cl is extensive.

(10) $int(E) \subseteq ext(ext(E))$, when the closure function cl is extensive and isotonic.

(11) $Z = int(E) \cup ext(E) \cup bd(E)$.

Proof. (1) Obvious and omitted.

(2) Obvious and hence omitted.

(3) $ext(ext(E)) = ext[Z \setminus cl(E)] = int[Z \setminus (Z \setminus cl(E))] = int(cl(E))$.

(4) Since $(Z \setminus E) \supseteq (Z \setminus F)$ as $E \subseteq F$. Then $int(Z \setminus E) \supseteq int(Z \setminus F)$ (by Lemma 17).

(5) Obvious from (4).

(6) Obvious from (4).

(7) $ext(\emptyset) = int(Z \setminus \emptyset) = Z \setminus cl(\emptyset) = Z$, since the closure function cl is grounded.

(8) By (2), $ext(Z) = Z \setminus cl(Z) = \emptyset$.

(9) $ext[Z \setminus ext(E)] = ext[Z \setminus int(Z \setminus E)] = int[Z \setminus (Z \setminus int(Z \setminus E))] = int(int(Z \setminus E)) \subseteq int(Z \setminus E)$ (Lemma 17)$= ext(E)$.

(10) $int(E) \subseteq int(cl(E))$ (from Lemma 17) $= int[Z \setminus int(Z \setminus E)] = int(Z \setminus ext(E)) = ext(ext(E))$.

(11) By Theorem 19(1), we have $int(E) \cup ext(E) \cup bd(E) = cl(E) \cup ext(E) = cl(E) \cup (Z \setminus cl(E)) = Z$. ∎

Acknowledgement: The authors are acknowledging the referees for their suggestions to improve the paper.

References

[1] A. Al-Omari and T. Noiri, **Generalized closed sets in ideal $\mathcal{M}$-spaces**, Jordan J. Math. Stat., **4**(3), (2011), 171–183.

[2] E. Čech, **Topological spaces**, rev. ed. (Publ. House Czech. Acad. Sc. Prague, English transi. Wiley), New York, 1966.

[3] K. Chattopadhyay and W. J. Thron, **Extension of closure spaces**, Can. J. Math., **29**(6), (1976), 1277–1286.
[4] A. Csaszar, **Modification of generalized topologies via hereditary classes**, Acta Math. Hungar., **115**(1-2), (2007), 29–35.
[5] A. Csaszar, **Generalized topology, generalized continuity**, Acta Math. Hungar., **96**, (2002), 351–357.
[6] S. R. Ghosh and H. Dasgupta, **Connectedness in monotone spaces**, Bull. Malays. Math. Sci. Soc., 27 (2), (2004), 129–148.
[7] Y. Lei and J. Zhang, **Closure system and its semantics**, Axioms, 10, 198, (2021), 1–30.
[8] S. Lugojan, **Generalized topology**, Stud. Cerc. Mat., **34**, (1982), 348–360.
[9] A. S. Mashhour, A. A. Allan, F. S. Mahmoud and F. H. Khedr, **On supratopological spaces**, Indian J. Pure Appl. Math., **14**(4), (1983), 502–510.
[10] S. Modak, **Dense sets in weak structure and minimal structure**, Commun. Korean Math. Soc. 28 (3), (2013), 589–596.
[11] S. Modak, **Minimal spaces with a mathematical structure**, J. Ass. Arab Univ Basic Appl. Sci., 22, (2017), 98–101.
[12] S. Modak and M. M. Islam, **New operators in ideal topological spaces and their closure spaces**, Aksaray J. Sci. Eng., **3**(2), (2019), 112–128.
[13] S. Modak and T. Noiri, **A Note on Mathematical Structures**, Bol. Soc. Paran. Mat., **37** (1), (2019), 63–69.
[14] W. J. Pervin, **Foundation of General Topology**, Academic Press, New York, 1984
[15] V. Popa and T. Noiri, **On M-continuous functions**, Anal. Univ. "Dunărea de Jos" Galaţi, Ser. Mat. Fiz. Mec. Teor., Fasc. II 18 (23), (2000), 31–41.
[16] E. Przemska, **A unified treatment of generalized closed sets in topological spaces**, Tatra Mt. Math. Publ. 85, (2023), 121–154.
[17] A. Rieser, *Čech* **closure spaces: A unified framework for discrete and continuous homotopy**, Topology and its Applications 296, (2021), 1–41.

University of Gour Banga,
Department of Mathematics ,
P.O. Mokdumpure, Malda 732 103, INDIA
e-mail: spmodak2000@yahoo.co.in

2949-1 Shiokita-cho, Hinagu
Yatsushiro-shi Kumomoto-ken, 869-5142,
JAPAN
e-mail: t.noiri@nifty.com