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## METRONLIKE STRUCTURES

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**Abstract.** The simplest tool which man could found to interpret space is the concept of distance. The concept of distance gives a sense of duality and helps us to decide the positional difference between points. Our view is that, if the effect of the observer’s point is counted in measuring of distances between different points, then the measure may be more relevant. Keeping in mind the properties of metric space, we tried to establish new properties of distance by including the observer’s point. Based on properties of angles, Sonwane and Prasad [29] established a mathematical structure called metron as a generalization of metric space. The present paper continues the study of Metron and proposes various metronlike structures by weakening some properties of metrons. The relationships between metron and metron-like structures are discussed by providing appropriate examples and diagrams.

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## 1. INTRODUCTION AND PRELIMINARIES

Metric spaces [11] are one of the basic structures in mathematics. During the past hundred and fifteen years many mathematicians such as Kurepa [16], Azam, Fisher and Khan [6], Beg, Abbas and Nazir [7], Gähler [12], Matthews [17, 18], Huang and Zhang [13], Azam Arshad and Beg [5], Karapinar and Erhan [15], Sedghi and his coworkers [23, 24], Zand and Nezhad [31], Ahmed, Omran and Asad [2], Stoltenberg [30], Roldán and Karapinar [21], Mustafa and Sims [20], Aghajani and his coworkers [1], Mukheimer [19], Dhage [10], Ceder [8], An and his coworkers [3], Asadi, Karapinar and Salimi [4], Salimi and Vetro [22], Shah and Husain [25], Smirnov [28], Shukla [26, 27], Jleli and Samet [14], Deza and Deza [9], and others proposed various generalizations of metric spaces.

**Definition 1.1.** [11] *Let  $X$  be a non empty set and  $d : X \times X \rightarrow \mathbb{R}$  such that:*

- (D1)  $d(x, y) \geq 0, \forall x, y \in X$  , ( *Property of non negativity* ).
- (D2)  $x = y \Rightarrow d(x, y) = 0, \forall x, y \in X$  ( *Vanishing property* ).
- (D3)  $d(x, y) = 0 \Rightarrow x = y, \forall x, y \in X$  ( *Identification or separation property* ).
- (D4)  $d(x, y) = d(y, x) \forall x, y \in X$  ( *Symmetric property* ).
- (D5)  $d(x, z) \leq d(x, y) + d(y, z) \forall x, y, z \in X$  ( *Triangle inequality property* ).

Then  $d$  is called a metric on  $X$  and the pair  $(X, d)$  is called a metric space.

Metric-like structures that are assumed to possess (at least) some properties of a metric space are presented in the following table.

TABLE 1. Metric like structures with their properties

| Structure/ Properties   | $D1$ | $D2$ | $D3$ | $D4$ | $D5$ |
|-------------------------|------|------|------|------|------|
| Metric space            | ✓    | ✓    | ✓    | ✓    | ✓    |
| Semi metric space       | ✓    | ✓    |      | ✓    | ✓    |
| Quasi metric space      | ✓    | ✓    | ✓    |      | ✓    |
| Quasi semi metric space | ✓    | ✓    |      |      | ✓    |

Recently Sonwane and Prasad [29] established a new mathematical structure called Metron as a generalization of metric spaces as follows:

**Definition 1.2.** [29] *Let  $X$  be a nonempty set and  $m : X \times X \times X \rightarrow \mathbb{R}$  (where  $\mathbb{R}$  is the set of all real numbers) defined with*

- (M1)  $m\langle x, y, z \rangle \geq 0$ ,  $\forall x, y, z \in X$  (Property of non-negativity).
- (M2)  $m\langle x, y, z \rangle = 0$ , if  $x = z$  (Vanishing Property).
- (M3) if  $x \neq z$ , then  $\exists y \in X$  such that  $m\langle x, y, z \rangle > 0$  (Survival property).
- (M4) if  $m\langle u, x, v \rangle = m\langle u, y, v \rangle$ ,  $\forall u, v \in X$  then  $x = y$  (Identification property).
- (M5)  $m\langle x, y, z \rangle = m\langle z, y, x \rangle$ ,  $\forall x, y, z \in X$  (Symmetric Property).
- (M6)  $m\langle x, y, z \rangle \leq m\langle x, y, u \rangle + m\langle u, y, z \rangle$ ,  $\forall x, y, z, u \in X$  (Property of Triangle inequality).
- (M7)  $m\langle x, y, z \rangle \leq m\langle x, u, z \rangle + m\langle u, z, x \rangle + m\langle z, x, u \rangle$ ,  $\forall x, y, z, u \in X$  (property of Triangular chain inequality).

Then the function  $m$  is called a metron function on  $X$  and the set  $X$  associated with  $m$  is called a metron with metron function  $m$ , which is denoted by  $(X, m)$ . The value  $m\langle x, y, z \rangle$  is called the metron value at the triple  $\langle x, y, z \rangle$  under  $m$  on  $X$  the points  $x$  and  $z$  are called Marked points of the triple and  $y$  is called the Base point of the triple.

**Remark 1.3.** [29] *Properties M3 and M4 in the definition of metron are equivalent to the following properties respectively:*

- M3(A) If  $m\langle x, y, z \rangle = 0$ ,  $\forall y \in X \Rightarrow x = z$  (Implication of Equality Property; i.e. IE-Property).
- M4(A) Whenever  $x \neq y$ , then  $\exists u, v \in X$ , such that  $m\langle u, x, v \rangle \neq m\langle u, y, v \rangle$  (Implication of Nonequality property; i.e. IN-Property).

**Lemma 1.4.** [29] *Let  $x, y, a, b \in \mathbb{R}$  are elements of arbitrary choice, then*

$$||x-b|-|b-y|| \leq |x-y| \leq ||x-a|-|a-y||+||a-y|-|y-x||+||y-x|-|x-a||.$$

In the study of metron structures, it is very natural to study those structures which are formed on the basis of some of the properties of metron or those which require additional properties with a metron. These variations will help us to have an analytical study of metrons. The examples and theorems presented ahead lead us to study the natural development. Situations as such always help one to look the structures more deeply and the structural picture becomes more transparent. The present paper continue the study of metron and proposed metronlike structures by weakening some properties of Metron. In section 2, definitions and sufficient number of examples are presented to establish metronlike structures them rigorously. In section 3, a hierarchy of these structures is established and a lattice of these structures

is formed. The section 4 presented the equivalences and convertibilities of metriclike spaces and metronlike structures. The section 5 constructed some metrons from weak variations of metron.

## 2. METRONLIKE STRUCTURES: DEFINITIONS AND EXAMPLES

**Definition 2.1.** Let  $X$  be a nonempty set and  $m : X \times X \times X \rightarrow \mathbb{R}$ . We call  $(X, m)$  a metronlike structure if  $m$  has at least properties (M1) and (M2).

**Example 2.2.** Let  $X$  be a nonempty set and define  $f : X \rightarrow \mathbb{R}$  and

$$m_1\langle x, y, z \rangle = ||f(x) - f(y)| - |f(y) - f(z)|| \quad \forall x, y, z \in X.$$

Then  $(X, m_1)$  is a metronlike structure because  $m_1$  satisfies the properties M1 and M2.

Metronlike structures that are assumed to possess (at least) some properties of a metron are presented in the following table.

TABLE 2. Metronlike structures with their properties

| Structure/ Properties | M1 | M2 | M3 | M4 | M5 | M6 | M7 |
|-----------------------|----|----|----|----|----|----|----|
| Metron                | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  |
| Metronlike            | ✓  | ✓  |    |    |    |    |    |
| Semi-metron           | ✓  | ✓  |    |    | ✓  | ✓  | ✓  |
| Sur-semi-metron       | ✓  | ✓  | ✓  |    | ✓  | ✓  | ✓  |
| Ide-semi-metron       | ✓  | ✓  |    | ✓  | ✓  | ✓  | ✓  |
| Pre-metron            | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  |    |
| Semi-pre-metron       | ✓  | ✓  |    |    | ✓  | ✓  |    |
| Sur-semi-pre-metron   | ✓  | ✓  | ✓  |    | ✓  | ✓  |    |
| Ide-semi-pre-metron   | ✓  | ✓  |    | ✓  | ✓  | ✓  |    |

**Example 2.3.** Let  $X = ]0, \infty[ \cup \{-1\}$  and  $f(x) = |x|$ ,  $\forall x \in X$ . Define  $m_1\langle x, y, z \rangle = ||f(x) - f(y)| - |f(y) - f(z)||$ ,  $\forall x, y, z \in X$ . Then  $(X, m_1)$  is a semi-metron. For, the properties M1, M2, M5 and M6 hold obviously. We see that M3 and M4 do not hold because

$$m_1\langle -1, y, 1 \rangle = ||-1| - |y|| - ||y| - |1|| = ||1 - |y|| - ||y| - 1|| = 0,$$

for all  $y \in X$  and  $m_1\langle u, 1, v \rangle = m_1\langle u, -1, v \rangle$ ,  $\forall u, v \in X$  but  $1 \neq -1$ . Lastly, to prove M7, we note that  $m_1\langle x, y, z \rangle = m_1\langle \bar{x}, \bar{y}, \bar{z} \rangle$ , when  $|x| =$

$\bar{x}$ , etc; because  $\|x\| = |x|$  so  $|\bar{x}| = |x|$  or  $f(x) = f(\bar{x})$ .

Hence

$$\begin{aligned}
 m_1\langle x, y, z \rangle &= m_1\langle \bar{x}, \bar{y}, \bar{z} \rangle \\
 &= ||\bar{x} - \bar{y}| - |\bar{y} - \bar{z}|| \\
 &\leq ||\bar{x} - \bar{u}| - |\bar{u} - \bar{z}|| + ||\bar{u} - \bar{z}| - |\bar{z} - \bar{x}|| + ||\bar{z} - \bar{x}| - |\bar{x} - \bar{u}|| \\
 &\leq m_1\langle \bar{x}, \bar{u}, \bar{z} \rangle + m_1\langle \bar{u}, \bar{z}, \bar{x} \rangle + m_1\langle \bar{z}, \bar{x}, \bar{u} \rangle \\
 &= m_1\langle x, u, z \rangle + m_1\langle u, z, x \rangle + m_1\langle z, x, u \rangle \quad \forall x, y, z, u \in X.
 \end{aligned}$$

Thus  $M7$  property holds. Therefore, we have that  $(X, m_1)$  is a semi-metron.

**Example 2.4.** Let  $X = \mathbb{R}$  and  $f(x) = |x|$ ,  $\forall x \in \mathbb{R}$ , define  $m_1\langle x, y, z \rangle = ||f(x) - f(y)| - |f(y) - f(z)||$ ,  $\forall x, y, z \in X$ , then  $(X, m_1)$  is a semi-metron. The justification is on the lines of Example 2.3.

**Example 2.5.** Let  $X$  be a non empty set and  $f : X \rightarrow \mathbb{R}$  with  $f$  is a non-injective map. Define  $m_1\langle x, y, z \rangle = ||f(x) - f(y)| - |f(y) - f(z)||$ ,  $\forall x, y, z \in X$ , then  $(X, m_1)$  is a semi-metron. The justification are on the lines of Example 2.3.

**Example 2.6.** Let  $X = \mathbb{R}$  and  $f : X \rightarrow \mathbb{R}$  given by  $f(x) = |x|$ ,  $\forall x \in X$ , define  $m_2\langle x, y, z \rangle = ||f(x) - y| - |y - f(z)||$ ,  $\forall x, y, z \in X$ , then the properties  $M1$ ,  $M2$ ,  $M5$  and  $M6$  clearly hold. For the remaining properties, situations are as follows-

$M3$  does not hold, as  $m_2\langle -1, y, 1 \rangle = 0$ ,  $\forall y \in X$ .

$M4$  does not hold, as

$$\begin{aligned}
 m_2\langle u, -1, v \rangle &= ||f(u) - (-1)| - |(-1) - f(v)|| \\
 &= ||f(u) + 1| - |-1 - f(v)|| \\
 &= ||f(u) + 1| - |f(v) + 1|| \\
 &= |(f(u) + 1) - (f(v) + 1)|, \text{ for } f(u), f(v) \geq 0; \\
 &= |f(u) - f(v)| \\
 &= |(f(u) + 2) - (f(v) + 2)| \\
 &= ||(f(u) - (-2)) - |(-2) - f(v)|| \\
 &= m_2\langle u, -2, v \rangle \quad \forall u, v \in X.
 \end{aligned}$$

Thus we have that,  $m_2\langle u, -1, v \rangle = m_2\langle u, -2, v \rangle \quad \forall u, v \in X$ , where as  $-2 \neq -1$ . Similarly  $m_2\langle u, -x, v \rangle = m_2\langle u, -y, v \rangle \quad \forall u, v \in X$ , and

$x > 0, y > 0, x \neq y$ . For the property M7, note that

$$m_2\langle x, y, z \rangle = \begin{cases} ||f(x) - y| - |y - f(z)|| = ||f(x) - f(y)| - |f(y) - f(z)||, & \text{if } y \geq 0 \\ ||f(x) - y| - |y - f(z)|| \geq ||f(x) - f(y)| - |f(y) - f(z)||, & \text{if } y < 0 \end{cases}$$

So,

$$\begin{aligned} m_2\langle x, y, z \rangle &= ||f(x) - y| - |y - f(z)|| \\ &\geq ||f(x) - f(y)| - |f(y) - f(z)|| \\ &= m_2\langle x, f(y), z \rangle \end{aligned}$$

$\forall x, y, z \in X$ . Hence

$$\begin{aligned} m_2\langle x, y, z \rangle &= ||f(x) - y| - |y - f(z)|| \\ &\leq ||f(x) - f(u)| - |f(u) - f(z)|| \\ &\quad + ||f(u) - f(z)| - |f(z) - f(x)|| \\ &\quad + ||f(z) - f(x)| - |f(x) - f(u)|| \\ &= m_2\langle x, f(u), z \rangle + m_2\langle u, f(z), x \rangle + m_2\langle z, f(x), u \rangle \\ &\leq m_2\langle x, u, z \rangle + m_2\langle u, z, x \rangle + m_2\langle z, x, u \rangle \end{aligned}$$

$\forall x, y, z \in X$ . Hence M7 holds. This shows that  $(X, m_2)$  is a semi-metron.

**Example 2.7.** Let  $X = [a, b]$  or a finite set of reals with  $c(X) \geq 3$ , where  $c(X)$  denotes the cardinality and  $m_1\langle x, y, z \rangle = ||x - y| - |y - z||$ ,  $\forall x, y, z \in X$ . Then  $(X, m_1)$  is a sur-semi-metron. For, the properties M1, M2, M5, M6 and M7 hold obviously. The property M3 also holds, because whenever  $x \neq z, \exists y \neq \frac{x+z}{2}$  and  $y \in x$ , then  $||x - y| - |y - z|| = m_1\langle x, y, z \rangle > 0$ . Further we see that M4 does not hold, because, when  $a = \inf X$  and  $b = \sup X$ , then  $m\langle u, a, v \rangle = m_1\langle u, b, v \rangle$ ,  $\forall u, v \in X$  with  $a \neq b$ .

**Example 2.8.** Let  $X = ]0, \infty[ \cup \{-1\}$  and  $f : X \rightarrow \mathbb{R}$  given by  $f(x) = |x|$ ,  $\forall x \in X$ . Define  $m_2\langle x, y, z \rangle = ||f(x) - y| - |y - f(z)||$ ,  $\forall x, y, z \in X$ . Then following situations arise about the metron properties.

- (i) Properties M1 and M2 clearly hold.
- (ii) The property M3 does not holds because  $x = -1 \neq 1 = z$ , Then  $f(x) = 1$  and  $f(z) = 1$ , so,  $m_2\langle -1, y, 1 \rangle = ||1 - y| - |y - 1|| = 0$ ,  $\forall y \in X$ .

(iii) *Property M4(A) holds. For, when  $y \neq y'$  (assume  $y < y'$  without the loss of generality) choose  $f(u) \in ]y, y'[\cap X$  and  $f(v) > y'$ . Then*

$$\begin{aligned} m_2\langle u, y, v \rangle &= ||f(u) - y| - |y - f(v)|| \\ &= |f(u) - f(v)| \\ &> ||f(u) - y'| - |y' - f(v)|| \\ &= m_2\langle u, y', v \rangle. \end{aligned}$$

*Hence, whenever  $y \neq y', \exists u, v \in X$  such that  $m_2\langle u, y, v \rangle \neq m_2\langle u, y', v \rangle$ .*

(iv) *Property M5 holds clearly.*

(v) *Property M6 holds. As,*

$$\begin{aligned} m_2\langle x, y, z \rangle &= ||f(x) - y| - |y - f(z)|| \\ &\leq ||f(x) - y| - |y - f(u)|| + ||y - f(u)| - |y - f(z)|| \\ &= ||f(x) - y| - |y - f(u)|| + ||f(u) - y| - |y - f(z)|| \\ &= m_2\langle x, y, u \rangle + m_2\langle u, y, z \rangle, \forall x, y, z, u \in X. \end{aligned}$$

(vi) *For the property M7, note that,*

$$\begin{aligned} m_2\langle x, y, z \rangle &= ||f(x) - y| - |y - f(z)|| \\ &= ||\bar{x} - y| - |y - \bar{z}||, \text{ where } \bar{x} = f(x) = |x| \text{ etc} \\ &= \begin{cases} ||\bar{x} - \bar{y}| - |\bar{y} - \bar{z}|, & \text{if } y > 0. \\ |\bar{x} - \bar{z}| \geq ||\bar{x} - \bar{y}| - |\bar{y} - \bar{z}|| & \text{if } y = -1. \end{cases} \end{aligned}$$

*Because,  $-1 = y < \bar{x}$ ,  $-1 = y < \bar{z}$ , and  $-1 = y < \bar{y} = 1$ . Hence,  $m_2\langle x, y, z \rangle \geq m_2\langle x, f(y), z \rangle$ ,  $\forall x, y, z \in X$ . Thus,*

$$\begin{aligned} m_2\langle x, y, z \rangle &= ||f(x) - y| - |y - f(z)|| \\ &\leq ||f(x) - f(u)| - |f(u) - f(z)|| \\ &\quad + ||f(u) - f(z)| - |f(z) - f(x)|| \\ &\quad + ||f(z) - f(x)| - |f(x) - f(u)|| \\ &= m_2\langle x, f(u), z \rangle + m_2\langle u, f(z), x \rangle + m_2\langle z, f(x), u \rangle \\ &\leq m_2\langle x, u, z \rangle + m_2\langle u, z, x \rangle + m_2\langle z, x, u \rangle \end{aligned}$$

$\forall x, y, z \in X$ . Hence M7 holds. This shows that  $(X, m_2)$  is a Idesemi-metron.

**Example 2.9.** Let  $\mathbb{R}$  be the set of all reals and  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f$  is injective. Define  $m_3 : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$m_3\langle x, y, z \rangle = ||x - f(y)| - |f(y) - z|| \quad \forall x, y, z \in \mathbb{R}.$$

Then  $(\mathbb{R}, m_3)$  is necessarily a pre-metron. The justification is as follows.

- (i) Properties M1 and M2 clearly hold, by definition.
- (ii) If  $x \neq z$ , then  $|x - f(y)| \neq |z - f(y)|$ , when  $f(y) \neq \frac{x+z}{2}$ ,  
So,  $||x - f(y)| - |f(y) - z|| > 0$ , If  $f(y) \neq \frac{x+z}{2}$ .  
Hence whenever  $x \neq z$ ,  $\exists y \in \mathbb{R}$  such that  $m_3\langle x, y, z \rangle > 0$ , Thus property M3 holds.
- (iii) If  $x \neq y$ , so  $f(x) \neq f(y)$ , and therefore there is always a choice available as  $u \in ]f(x), f(y)[$  and  $v \notin ]f(x), f(y)[$  with  $v < f(x)$ .  
Then for these choice of  $x, y, u, v$ , we get

$$\begin{aligned} m_3\langle u, x, v \rangle &= ||u - f(x)| - |f(x) - v|| \\ &< |u - v| \\ &= ||u - f(y)| - |f(y) - v|| \\ &= m_3\langle u, y, v \rangle. \end{aligned}$$

Thus property M4(A) holds.

- (iv) The property M5, clearly holds by definition.
- (v) Consider,

$$\begin{aligned} m_3\langle x, y, z \rangle &= ||x - f(y)| - |f(y) - z|| \\ &\leq ||x - f(y)| - |f(y) - u| + ||f(y) - u| - |f(y) - z|| \\ &= ||x - f(y)| - |f(y) - u| + ||u - f(y)| - |f(y) - z|| \\ &= m_3\langle x, y, u \rangle + m_3\langle u, y, z \rangle \quad \forall x, y, z, u \in \mathbb{R}. \end{aligned}$$

So M6 holds.

- (vi) The property M7 not necessarily holds.  
For example, consider,

$$f(x) = \begin{cases} \frac{1}{x}, & \text{for } |x| \geq 1 \\ 0, & \text{for } x = 0 \\ x + 1, & \text{for } 0 < x < 1 \\ x - 1, & \text{for } -1 < x < 0 \end{cases}$$

Then,  $f$  is a injective map. But

$$\begin{aligned}
& m_3\langle 1/4, 8, 2 \rangle \\
&= ||1/4 - f(8)| - |f(8) - 2|| \\
&= ||1/4 - 1/8| - |1/8 - 2|| \\
&= |1/8 - 15/8| = 14/8 = 7/4. \\
&> m_3\langle 1/4, 1/4, 2 \rangle + m_3\langle 1/4, 2, 1/4 \rangle + m_3\langle 2, 1/4, 1/4 \rangle \\
&= ||1/4 - f(1/4)| - |f(1/4) - 2|| + 0 + ||2 - f(1/4)| - |f(1/4) - 1/4|| \\
&= ||1/4 - 5/4| - |5/4 - 2|| + 0 + ||2 - 5/4| - |5/4 - 1/4|| \\
&= 2|1 - 3/4| = 2 \cdot 1/4 = 1/2.
\end{aligned}$$

This means that  $m_3$  does not satisfy the M7 property. Thus we observe that the property M7 does not hold in general for all the injective maps. Hence we conclude that  $(\mathbb{R}, m_3)$  is a pre-metron necessarily.

**Example 2.10.** Let  $\mathbb{R}$  be the set of reals and  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f$  is injective and  $\sup f$  and  $\inf f$  exists. Define  $m_2 : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$m_2\langle x, y, z \rangle = ||f(x) - y| - |y - f(z)||, \quad \forall x, y, z \in \mathbb{R}.$$

Then  $(\mathbb{R}, m_2)$  is a semi-pre-metron. The observations are as follows.

- (A) The properties M1, M2, M5 and M6 hold by  $m_2$  on  $\mathbb{R}$  as given in Example 2.9
- (B) The properties M3, M4 and M7 don't hold by  $m_2$ .

- (i) The function  $f$  is non-injective, so  $\exists$  two points  $x \neq z$  such that  $f(x) = f(z)$ . Then  $|f(x) - y| = |f(z) - y|, \forall y \in \mathbb{R}$ , gives  $m_2\langle x, y, z \rangle = ||f(x) - y| - |y - f(z)|| = 0, \forall y \in \mathbb{R}$ . Thus M3 does not hold.
- (ii) The property M4 does not hold. Let  $\sup f = b$  and then choose  $b < x < y$ , then

$$\begin{aligned}
m_2\langle u, x, v \rangle &= ||f(u) - x| - |x - f(v)|| \\
&= |f(u) - f(v)| \\
&= ||f(u) - y| - |y - f(x)|| \\
&= m_2\langle u, y, x \rangle \forall u, v \in X.
\end{aligned}$$

Thus, M4 does not hold.

- (iii) For the property M7. Consider,

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ -2a, & \text{if } x = a \\ 2a, & \text{for } x = -a \\ x & \text{if } x \notin \{0, \pm a\} \end{cases} \quad \forall a > 0, y_0 > 2a.$$

Then,

$$\begin{aligned}
 L.H.S. &= m_2\langle a, y_0, -a \rangle = ||f(a) - y_0| - |y_0 - f(a)|| \\
 &= ||-2a - y_0| - |y_0 - 2a|| \\
 &= |(2a + y_0) - (y_0 - 2a)| = 4a > 0.
 \end{aligned}$$

$$\begin{aligned}
 R.H.S. &= m_2\langle a, 0, -a \rangle + m_2\langle 0, -a, a \rangle + m_2\langle -a, a, 0 \rangle \\
 &= ||f(a) - 0| - |0 - f(-a)|| \\
 &\quad + ||f(0) - (-a)| - |(-a) - f(a)|| \\
 &\quad + ||f(-a) - a| - |a - f(o)|| \\
 &= ||-2a| - |-2a|| + ||a| - |-a + 2a|| + ||2a - a| - |a - 0|| \\
 &= 0 + 0 + 0 = 0.
 \end{aligned}$$

So,  $m_2\langle a, y_0, -a \rangle \geq m_2\langle a, 0, -a \rangle + m_2\langle 0, -a, a \rangle + m_2\langle -a, a, 0 \rangle$ .  
Hence M7 does not hold.

**Example 2.11.** Let  $\mathbb{R}$  be the set of reals and  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f$  is injective. Define  $m_2 : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$m_2\langle x, y, z \rangle = ||f(x) - y| - |y - f(z)||, \quad \forall x, y, z \in \mathbb{R}.$$

Then  $(\mathbb{R}, m_2)$  is necessarily a sur-semi-pre-metron. Observations are as follows.

- (i) The properties M1, M2 and M5 clearly hold.
- (ii) If  $x \neq z$  then  $f(x) \neq f(z)$ , because  $f$  is a 1-1 function. Choose  $y \neq 1/2(f(x) + f(z))$ , then  $||f(x) - y| - |y - f(z)|| \neq 0$ , gives  $||f(x) - y| - |y - f(z)|| = m_2\langle x, y, z \rangle > 0$ . Hence M3 holds.
- (iii) We observe that the property M4 may not hold in general. For, under an assumption, if  $\mathbb{R}(f) \subset ]a, b[$  and if  $b < x < y$  be chosen then,

$$\begin{aligned}
 m_2\langle u, x, v \rangle &= ||f(u) - x| - |x - f(v)|| \\
 &= |f(u) - f(v)|, \text{ for } f(u) < x \text{ and } f(v) < x. \\
 &= ||f(u) - y| - |y - f(v)||, \text{ for } f(u) < y \text{ and } f(v) < y. \\
 &= m_2\langle u, y, v \rangle, \quad \forall u, v \in \mathbb{R}.
 \end{aligned}$$

Gives M4 does not hold.

(iv) We have

$$\begin{aligned}
 m_2\langle x, y, z \rangle &= ||f(x) - y| - |y - f(z)|| \\
 &\leq ||f(x) - y| - |y - f(u)|| + ||y - f(u)| - |y - f(z)|| \\
 &= ||f(x) - y| - |y - f(u)|| + ||f(u) - y| - |y - f(z)|| \\
 &= m_2\langle x, y, u \rangle + m_2\langle u, y, z \rangle, \quad \forall x, y, z, u \in X.
 \end{aligned}$$

Thus the property M6 holds.

(v) By considering the following example, we conclude that the property M7 does not hold in general. For, consider,

$$f(x) = \begin{cases} 1/x, & \text{if } |x| > 1 \\ 0, & \text{if } x = 0 \\ -(x+1), & \text{if } 0 < x \leq 1 \\ -(x-1), & \text{if } -1 \leq x < 0 \end{cases}$$

Then,

$$\begin{aligned}
 m_2\langle 1, 5, -1 \rangle &= ||-2 - 5| - |5 - 2|| = |7 - 3| = 4 \\
 &> m_2\langle 1, 0, -1 \rangle + m_2\langle 0, -1, 1 \rangle + m_2\langle -1, 1, 0 \rangle \\
 &= ||-2 - 0| - |0 - 2|| \\
 &\quad + ||0 + 1| - |-1 + 2|| \\
 &\quad + ||2 - 1| - |1 - 0|| \\
 &= 0 + 0 + 0 = 0.
 \end{aligned}$$

Hence  $(\mathbb{R}, m_2)$  is necessarily a sur-semi-pre-metron.

**Example 2.12.** Let  $X = ]0, \infty[ \cup \{-1\}$  and  $f : X \rightarrow \mathbb{R}$ , given by

$$f(x) = \begin{cases} -2, & \text{if } x = 1 \\ 2, & \text{if } x = -1 \\ x, & \text{if } x \neq \pm 1 \end{cases}$$

Define  $m_2\langle x, y, z \rangle = ||f(x) - y| - |y - f(z)||$ ,  $\forall x, y, z \in \mathbb{R}$ . Then  $(X, m_2)$  is an idesemi-pre-metron. For,

- (i) The properties M1, M2, M5 and M6 clearly hold.
- (ii)  $m_2$  does not satisfy the property M3. As  $2 \neq -1$ , but  $m_2\langle 2, y, -1 \rangle = ||2 - y| - |y - 2|| = 0$ ,  $\forall y \in X$ .

(iii)  $m_2$  satisfies the property M4. For, take  $x \neq y$  (assume  $x < y$ ) and choose  $f(u) \in ]x, y[ \cap X$  and  $f(v) > y$ . Then,

$$\begin{aligned} m_2\langle u, x, v \rangle &= ||f(u) - x| - |x - f(v)||, \\ &= |f(u) - f(v)| \text{ as, } x < f(u) \text{ and } x < f(v) \\ &> ||f(u) - y| - |y - f(v)||, f(u) < y < f(v) \\ &= m_2\langle u, y, v \rangle. \end{aligned}$$

Hence, if  $x \neq y$  then  $\exists u, v \in X$  such that  $m_2\langle u, x, v \rangle \neq m_2\langle u, y, v \rangle$ .

(iv)  $m_2$  does not satisfy the property M7. For,

$$\begin{aligned} m_2\langle 1, 5, -1 \rangle &= ||-2 - 5| - |5 - 2|| = |7 - 3| = 4 \\ &> m_2\langle 1, 1/n, -1 \rangle + m_2\langle 1/n, -1, 1 \rangle + m_2\langle -1, 1, 1/n \rangle \\ &= ||-2 - 1/n| - |1/n - 2|| \\ &\quad + ||1/n + 1| - |-1 + 2|| \\ &\quad + ||2 - 1| - |1 - 1/n|| \\ &= 2/n + 1/n + 1/n = 4/n < 4, \forall n > 1. \end{aligned}$$

**Remark 2.13.** Metron like structures  $m_1, m_2, m_3$  defined in previous examples can be generalizes as:

$$\tilde{m}\langle x, y, z \rangle = ||f(x) - g(y)| - |g(y) - f(z)||.$$

Where  $f, g : X \rightarrow \mathbb{R}$ .

- (i) If  $g = f$ , then  $\tilde{m} = m_1$  (Examples 2.2, 2.3, 2.4, 2.5, and 2.7).
- (ii) If  $g = 1_{\mathbb{R}}$ , then  $\tilde{m} = m_2$  (Examples 2.6, 2.8, 2.10, 2.11, and 2.12).
- (iii) If  $f = 1_{\mathbb{R}}$  and  $g$  is redenoted by  $f$ , then  $\tilde{m} = m_3$  (Example 2.9).

**Definition 2.14.** If a metronlike structure is not assumed to satisfy the symmetry property (M5), then it is called a quasi-metronlike structure.

Quasi-metronlike structures that are assumed to possess (at least) some properties of a metron are presented in the following table.

**Example 2.15.** Let  $\mathbb{R}$  be the set of all the reals. Define  $m_4 : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$m_4\langle x, y, z \rangle = \begin{cases} \min\{1, (|x - y| - |y - z|)\}, & \text{if } (|x - y| - |y - z|) \geq 0. \\ 1, & \text{otherwise.} \end{cases}$$

Then  $(\mathbb{R}, m_4)$  is a quasimetron. For,

- (i)  $m_4\langle x, y, z \rangle \geq 0$  by definition. Hence M1 holds.

TABLE 3. Quasi-metronlike structures with their properties

| Structure/ Properties     | M1 | M2 | M3 | M4 | M5 | M6 | M7 |
|---------------------------|----|----|----|----|----|----|----|
| Quasi-metron              | ✓  | ✓  | ✓  | ✓  |    | ✓  | ✓  |
| Quasi-semi-metron         | ✓  | ✓  |    |    |    | ✓  | ✓  |
| Quasi-sur-semi-metron     | ✓  | ✓  | ✓  |    |    | ✓  | ✓  |
| Quasi-ide-semi-metron     | ✓  | ✓  |    | ✓  |    | ✓  | ✓  |
| Quasi-pre-metron          | ✓  | ✓  | ✓  | ✓  |    | ✓  |    |
| Quasi-semi-pre-metron     | ✓  | ✓  |    |    |    | ✓  |    |
| Quasi-sur-semi-pre-metron | ✓  | ✓  | ✓  |    |    | ✓  |    |
| Quasi-ide-semi-pre-metron | ✓  | ✓  |    | ✓  |    | ✓  |    |

- (ii) If  $x = z$ , then  $|x - y| = |y - z|$ ,  $\forall y \in \mathbb{R}$ ,  
or  $(|x - y| - |y - z|) = 0, \forall y \in \mathbb{R}$ ,  
or  $m_4\langle x, y, z \rangle = 0, \forall y \in \mathbb{R}$ . Hence  $m_4\langle x, y, z \rangle = 0, \forall y \in \mathbb{R}$  when  $x = z$ . Thus the property M2 holds.
- (iii) If  $x \neq z$ , then choose a  $y \neq 1/2(x + z)$ , so  $|x - y| \neq |y - z|$  and therefore  $|x - y| - |y - z| > 0$  or  $< 0$ . Hence  $m_4\langle x, y, z \rangle > 0$  or  $1$  (by definition). Thus in either of the cases whenever  $x \neq z \exists y \in \mathbb{R}$  such that  $m_4\langle x, y, z \rangle > 0$ . It shows that M3 holds.
- (iv) If  $x \neq y$ , assume  $x < y$  (without the loss of generality) and choose  $u$  and  $v$  such that  $u < x < v < y$  and  $|u - v| < 1$ . Then  $|u - x| - |x - v| < |u - v| = (|u - y| - |y - v|) < 1$ . But  $|u - x| - |x - v|$  is either negative or nonnegative less than  $|u - v|$ . So,  $m_4\langle u, x, v \rangle$  is either 1 or nonnegative and less than  $|u - v| < 1$ . Thus in either of the cases,  $m_4\langle u, x, v \rangle \neq |u - v| = |u - y| - |y - v| = m_4\langle u, y, v \rangle < 1$ . Hence, when  $x \neq y$ ,  $\exists u, v \in \mathbb{R}$  such that  $m_4\langle u, x, v \rangle \neq m_4\langle u, y, v \rangle$ . It shows that  $m_4$  satisfies the  $M4(A)$  property.
- (v) The M5 property does not hold. For, suppose  $x, y, z \in \mathbb{R}$  such that,  $|x - y| < |y - z|$ , then  $m_4\langle x, y, z \rangle = 1$  and  $|x - y| < |y - z|$  gives  $|z - y| > |y - x|$  so  $m_4\langle z, y, x \rangle > 0$ . This will not be always equal to 1. Thus  $\exists$  triple  $x, y, z \in \mathbb{R}$  such that  $m_4\langle x, y, z \rangle \neq m_4\langle z, y, x \rangle$ .
- (vi) The property M6 holds. Let  $x, y, z, u \in X$ , then

$$m_4\langle x, y, z \rangle = \begin{cases} \min\{1, (|x - y| - |y - z|)\}, & \text{if } (|x - y| \geq |y - z|). \\ 1, & \text{otherwise.} \end{cases}$$

By definition, here two situations arise. Consider the following situations.

**Situation (I)**

$$\begin{aligned}
0 \leq m_4\langle x, y, z \rangle &\leq |x - y| - |y - z| \\
&= |x - y| - |y - u| + |y - u| - |y - z| \\
&= (|x - y| - |y - u|) + (|u - y| - |y - z|)
\end{aligned}$$

Then either both the values  $(|x - y| - |y - u|)$  and  $(|u - y| - |y - z|)$  are positive or any one of these values is negative. Since,  $m_4\langle x, y, z \rangle$  bounded by 1 and for the expression of negative value, the metron value is +1, hence in either cases  $m_4\langle x, y, z \rangle$  does not exceed  $m_4\langle x, y, u \rangle + m_4\langle u, y, z \rangle$ .

**Situation (II)** The value  $m_4\langle x, y, z \rangle = 1$  when  $|x - y| - |y - z| < 0$ . Now,  $|x - y| - |y - z| = (|x - y| - |y - u|) + (|u - y| - |y - z|)$ , gives that either both the expressions are negative or any one of these is negative. Therefore in either of the cases, at least one of the values  $m_4\langle x, y, u \rangle$  or  $m_4\langle u, y, z \rangle$  is +1. Hence  $m_4\langle x, y, z \rangle$  does not exceed  $m_4\langle x, y, u \rangle + m_4\langle u, y, z \rangle$ . Therefore in both the situations,  $m_4\langle x, y, z \rangle \leq m_4\langle x, y, u \rangle + m_4\langle u, y, z \rangle \forall x, y, z, u \in \mathbb{R}$ .

(vii) The property M7 holds. Consider the identity,

$$0 = (|x - u| - |u - z|) + (|u - z| - |z - x|) + (|z - x| - |x - u|)$$

Then by the observation, it is clear that either all the expressions of this identity have zero value or at least one of these expressions have negative value.

Take the situation(I), when all the expressions have zero value. Then it implies that  $|x - y| = |u - z| = |z - x|$  But we know that  $|x - u| = |u - z| = |z - x| \Leftrightarrow x = u = z$ , whenever  $x, z, u$  are reals. So,  $x = z$ , gives  $m_4\langle x, y, z \rangle = |x - y| - |y - z| = 0$ ,  $\forall y \in \mathbb{R}$  and therefore the inequality  $m_4\langle x, y, z \rangle \leq m_4\langle x, y, z \rangle + m_4\langle u, z, x \rangle + m_4\langle z, x, u \rangle$  is satisfied. In situation(II), when at least one of the expressions is negative, then at least one of the values from  $m_4\langle x, u, z \rangle, m_4\langle u, z, x \rangle$  and  $m_4\langle z, x, u \rangle$  is 1. But by definition,  $m_4\langle x, y, z \rangle$  is bounded by 1. Hence the inequality  $m_4\langle x, y, z \rangle \leq m_4\langle x, u, z \rangle + m_4\langle u, z, x \rangle + m_4\langle z, x, u \rangle \forall x, y, z, u \in \mathbb{R}$  is satisfied. Thus M7 holds.

**Example 2.16.** Let  $X$  be a nonempty set with at least four elements and  $f : X \rightarrow \mathbb{R}$  such that  $f$  is injective map. Define  $m_5 : X \times X \times X \rightarrow$

$\mathbb{R}$  given by

$$m_5\langle x, y, z \rangle = \begin{cases} \min\{1, (|f(x) - f(y)| - |f(y) - f(z)|)\}, & \text{if } |f(x) - f(y)| \geq |f(y) - f(z)| \\ 1, & \text{otherwise.} \end{cases}$$

Then  $(X, m_5)$  is necessarily a quasi-sur-semi-metron. Because metron value of any triple is determined by reals and the  $f$  is injective, so  $f(x), f(y), f(z)$  is an independent triple as is  $x, y, z$ . Hence all the situations arises same except (iv) as in the Example 2.15

**Example 2.17.** Let  $X$  be a nonempty set with at least four elements and  $f : X \rightarrow \mathbb{R}$  such that  $f$  is non-injective but not constant. Define  $m_5 : X \times X \times X \rightarrow \mathbb{R}$  as in Example 2.16. Then  $(X, m_5)$  is a quasi-semi-metron.

**Example 2.18.** Let  $(X, d)$  be a metric space and  $m_6 : X \times X \times X \rightarrow \mathbb{R}$  given by

$$m_6\langle x, y, z \rangle = \begin{cases} \min\{1, (d(x, y) - d(y, z))\}, & \text{if } d(x, y) \geq d(y, z). \\ 1, & \text{otherwise.} \end{cases}$$

Then  $(X, m_6)$  is a quasi-sur-semi-pre-metron.

**Example 2.19.** Let  $(X, d)$  be a semimetric space and  $m_6 : X \times X \times X \rightarrow \mathbb{R}$  given by

$$m_6\langle x, y, z \rangle = \begin{cases} \min\{1, (d(x, y) - d(y, z))\}, & \text{if } d(x, y) \geq d(y, z). \\ 1, & \text{otherwise.} \end{cases}$$

Then  $(X, m_6)$  is a quasi-semi-pre-metron.

**Example 2.20.** Let  $(X, d)$  be a quasisemimetric space and  $m : X \times X \times X \rightarrow \mathbb{R}$  defined by  $m\langle x, y, z \rangle = |d(x, y) - d(y, z)|$ . Then  $(X, m)$  is a quasi-sur-semi-pre-metron.

Observations about Examples 2.17 to 2.20 can be seen on the lines of Example 2.15.

**Remark 2.21.** The examples of quasiide-semi-metron, quasisemi-metron, quasipre-metron, quasi-sur-semi-pre-metron, quasiidesemi-pre-metron, quasi-semi-pre-metron can be made by varying the corresponding metronlike functions for symmetry.

**Theorem 2.22.** Let  $(X, m)$  be a quasimetron. If we define  $m_c : X \times X \times X \rightarrow \mathbb{R}$  by  $m_c\langle x, y, z \rangle = m\langle z, y, x \rangle$ , then  $(X, m_c)$  is also a quasimetron.

*Proof.* Let  $(X, m)$  be a quasimetron and  $m_c$  is a metronlike function given by  $m_c\langle x, y, z \rangle = m\langle z, y, x \rangle \forall x, y, z \in X$ . Then observations are as follows.

- (i)  $m_c\langle x, y, z \rangle = m\langle z, y, x \rangle \geq 0, \forall x, y, z \in X$ .
- (ii) If  $x = z$  so,  $z = x$  and then  $m\langle z, y, x \rangle = 0, \forall y \in X$  Thus,  $m_c\langle x, y, z \rangle = 0, \forall y \in X$ .
- (iii) If  $x \neq z$  so  $z \neq x$  then  $\exists a y \in X$  such that  $m\langle z, y, x \rangle > 0$  or  $m_c\langle x, y, z \rangle > 0$ .
- (iv) If  $x \neq y$  then  $\exists u, v \in X$  such that  $m\langle v, x, u \rangle \neq m\langle v, y, u \rangle$  or  $m_c\langle u, x, v \rangle \neq m_c\langle u, y, v \rangle$ .
- (v) When  $m\langle x, y, z \rangle \neq m\langle z, y, x \rangle$  then obviously,  $m_c\langle z, y, x \rangle \neq m_c\langle x, y, z \rangle$  (by definition). This gives that  $M5$  property is not possible.
- (vi) The property  $M6$  follows on note that

$$\begin{aligned}
 m_c\langle x, y, z \rangle &= m\langle z, y, x \rangle \\
 &\leq m\langle z, y, u \rangle + m\langle u, y, x \rangle \\
 &= m_c\langle u, y, z \rangle + m_c\langle x, y, u \rangle \\
 &= m_c\langle x, y, u \rangle + m_c\langle u, y, z \rangle \forall x, y, z, u \in X
 \end{aligned}$$

- (vii) The property  $M7$  follows on note that

$$\begin{aligned}
 m_c\langle x, y, z \rangle &= m\langle z, y, x \rangle \\
 &\leq m\langle z, u, x \rangle + m\langle u, x, z \rangle + m\langle x, z, u \rangle \\
 &= m_c\langle x, u, z \rangle + m_c\langle z, x, u \rangle + m_c\langle u, z, x \rangle \\
 &= m_c\langle x, u, z \rangle + m_c\langle u, z, x \rangle + m_c\langle z, x, u \rangle \forall x, y, z, u \in \mathbb{R}.
 \end{aligned}$$

Thus  $m_c$  satisfies all the properties  $M1$  to  $M7$  except  $M5$ . Hence  $(X, m_c)$  is a quasimetron.

□

**Definition 2.23.** Let  $(X, m)$  be a quasimetronlike structure and define  $m_c : X \times X \times X \rightarrow \mathbb{R}$  by  $m_c\langle x, y, z \rangle = m\langle z, y, x \rangle, \forall x, y, z \in X$ , then  $(X, m_c)$  is called conjugate of  $(X, m)$ .

In the previous definitions and examples we have presented weak variations of metron structure, in which the property itself is to be excluded or included in the structure itself. Some times the properties  $M2$ ,  $M3$  and  $M4$  are not the properties of structure itself but these properties seem in the family of structures. These weak structures are defined and presented in the following definitions and examples.

**Definition 2.24. (Weak vanishing property etc.)** Let  $X$  be a nonempty set and  $\{m_\alpha : \alpha \in \Lambda\}$  is a family of trivariate functions on  $X$ . Then we say that the family  $\{m_\alpha : \alpha \in \Lambda\}$  satisfies-

- (i) Weak vanishing property on  $X$  if, for each  $x \in X$ ,  $\exists \alpha_x \in \Lambda$  such that  $m_{\alpha_x}\langle x, y, x \rangle = 0$ ,  $\forall y \in X$ .
- (ii) Weak survival property on  $X$  if, for each pair of distinct points  $x \neq z \in X$ ,  $\exists$  an  $\alpha_{(x,z)} \in \Lambda$  and a point  $y \in X$  such that  $m_{\alpha_{(x,z)}}\langle x, y, z \rangle > 0$ .
- (iii) Weak identification property on  $X$  if,  $m_\alpha\langle u, x, v \rangle = m_\alpha\langle u, y, v \rangle$ ,  $\forall u, v \in X$  and  $\forall \alpha \in \Lambda \Rightarrow x = y$ .

**Definition 2.25. (Weak V-metron etc):** Let  $X$ , be a nonempty set and  $\{m_\alpha : \alpha \in \Lambda\}$  is a family of trivariate functions on  $X$ . Then we say that:

- (A)  $(X, m_\alpha)$  is a Weak Vanishing Metron (Weak V-metron) with respect to the family  $\{m_\alpha : \alpha \in \Lambda\}$  or  $\{(X, m_\alpha) : \alpha \in \Lambda\}$  is a family of weak V- metrons, if
  - (i)  $(X, m_\alpha)$  satisfy all the metron properties except possibly vanishing property M2 for each  $\alpha \in \Lambda$ .
  - (ii) The family  $\{m_\alpha\}$  satisfies weak vanishing property on  $X$ .
- (B)  $(X, m_\alpha)$  is a weak survival metron (weak S-metron) with respect to the family  $\{m_\alpha\}$  or  $\{(X, m_\alpha) : \alpha \in \Lambda\}$  is a family of weak S-metrons, if
  - (i)  $(X, m_\alpha)$  is an ide-semi-metron,  $\forall \alpha \in \Lambda$ .
  - (ii) The family  $\{m_\alpha\}$  satisfies weak survival property on  $X$ .
- (C)  $(X, m_\alpha)$  is a weak identification metron (weak I-metron) with respect to the family  $\{m_\alpha\}$  or  $\{(X, m_\alpha) : \alpha \in \Lambda\}$  is a family of weak I-metrons, if
  - (i)  $(X, m_\alpha)$  is a sur-semi-metron,  $\forall \alpha \in \Lambda$ .
  - (ii) The family  $\{m_\alpha\}$  satisfies weak identification property on  $X$ .

**Example 2.26. (Family of weak vanishing metrons)** Let  $E = [0, n]$ ,  $n \in \mathbb{N}$  and  $e_p : E \times E \times E \rightarrow \mathbb{R}$  for all  $p \in E$ , defined by,

$$e_p\langle x, y, z \rangle = \begin{cases} 0, & \text{if } x = z \neq p \\ 1, & \text{if } x = z = p \\ 1 + (y/n) & \text{if } x \neq z. \end{cases}$$

Then  $(E, e_p)$  is a weak vanishing metron w.r.t the family  $F \subset \{e_p : p \in E\}$  or  $\{(E, e_p) : e_p \in F\}$  is a family of weak vanishing metrons if  $\{e_p, e_q\}_{p \neq q} \subset F$ . For,

- (i)  $e_p\langle x, y, z \rangle \geq 0$ ,  $\forall x, y, z \in E$ .

- (ii) if  $x \in E$  and  $x \neq p$  then  $e_p\langle x, y, x \rangle = 0$  and  $e_q\langle p, y, p \rangle = 0$ ,  $q \in E$  with  $p \neq q$ .
- (iii) if  $x \neq z$ , then  $e_p\langle x, y, z \rangle = 1 + y/n > 0$ ,  $\forall y \in X$ .
- (iv) When  $x \neq y$ , then  $1 + x/n \neq 1 + y/n$ , so,  $e_p\langle u, x, v \rangle \neq e_p\langle u, y, v \rangle$ ,  $\forall u \neq v$ .
- (v)  $e_p\langle x, y, z \rangle = e_p\langle z, y, x \rangle$ ,  $\forall x, y, z \in X$ .
- (vi) The triangle inequality

$$e_p\langle x, y, z \rangle \leq e_p\langle x, y, u \rangle + e_p\langle u, y, z \rangle$$

is vacuously true when  $x = z \neq p$  and it is also true when  $x = z = p$ , as  $e_p\langle p, y, p \rangle = 1$  where as  $e_p\langle p, y, u \rangle = 1$  or  $1 + y/n$  etc. Further, when  $x \neq z$ , the inequality is true, for  $u$  surely differs with at least one point from  $x$  and  $z$ , so,  $e_p\langle x, y, z \rangle = 1 + y/n$  and one of  $e_p\langle x, y, u \rangle$  and  $e_p\langle u, y, z \rangle$  is  $1 + y/n$ .

- (vii) The triangular chain inequality

$$e_p\langle x, y, z \rangle \leq e_p\langle x, u, z \rangle + e_p\langle u, z, x \rangle + e_p\langle z, x, u \rangle$$

is vacuously true then  $x = z \neq p$  and  $x = z = p$ . And for  $x \neq z$ ,  $u$  surely differs with at least one point from  $x$  and  $z$  and therefore,  $e_p\langle x, y, z \rangle = 1 + y/n$ ,  $e_p\langle x, y, z \rangle \geq 1$  and at least one of  $e_p\langle u, z, x \rangle$  and  $e_p\langle z, x, u \rangle$  is greater than or equal to 1. Thus,  $e_p\langle x, y, z \rangle + e_p\langle u, z, x \rangle + e_p\langle z, x, u \rangle \geq 2$ , whereas  $e_p\langle x, y, z \rangle \leq 2$ . so inequality holds.

Hence in all cases, the triangular chain inequality is true.

**Example 2.27. (Family of weak survival metrons)** Let  $X = ]0, \infty[ \cup \{-1\}$  and  $f : X \rightarrow \mathbb{R}$ , given by

$$f_\mu(x) = \begin{cases} x, & \text{if } x \neq -1 \\ \mu, & \text{if } x = -1, \text{ where } \mu \in ]0, 1]. \end{cases}$$

Define  $m_\mu\langle x, y, z \rangle = ||f_\mu(x) - y| - |y - f_\mu(z)||$ ,  $\forall x, y, z \in X$ . Then  $(X, m_\mu)$  is an ide-semi-metron for all  $\mu \in ]0, 1]$ .

In particular,  $(X, m_\mu)$  does not satisfy the survival property for the pair  $-1, \mu$ , as  $m_\mu\langle -1, y, \mu \rangle = ||\mu - y| - |y - \mu|| = 0$ ,  $\forall y \in X$ . But  $m_v\langle -1, y, \mu \rangle = ||v - y| - |y - \mu|| > 0$ , when  $v \neq \mu$ .

Therefore the family  $\{(X, m_\mu) : \mu \in ]0, 1]\}$  satisfies the weak survival property. Hence the family  $\{(X, m_\mu) : \mu \in ]0, 1]\}$  is a family of weak  $S$ -metrons. In particular  $\{(X, m_\mu), (X, m_v)\}_{v \neq \mu}$  is a family of weak  $S$ -metrons.

**Example 2.28. (Family of weak identification metrons)** Let  $J_n = \{1, 2, \dots, n\}$  and  $f_i : J_n \rightarrow J_n$  for  $i = 1, 2$ , given by

$$f_1(x) = \begin{cases} x, & \text{if } x \neq 1, n \\ 1, & \text{if } x = 1, n \end{cases}$$

$$f_2(x) = \begin{cases} x, & \text{if } x \neq 1, n \\ n, & \text{if } x = 1, n \end{cases}$$

and  $m_i\langle x, y, z \rangle = ||f_i(x) - f_i(y)| - |f_i(y) - f_i(z)||$ ,  $i = 1, 2$ .

Then  $(J_n, m_1)$  and  $(J_n, m_2)$  are sur-semi-metrons and do not follow the identification property for pair,  $\{1, (n-1)\}$  in  $(J_n, m_1)$  and  $\{2, n\}$  in  $(J_n, m_2)$ . Because  $m_1\langle x, 1, y \rangle = m_1\langle x, n-1, y \rangle$ ,  $\forall x, y \in J_n$  and  $m_2\langle x, 2, y \rangle = m_2\langle x, n, y \rangle$ ,  $\forall x, y \in J_n$ . But it obeys the weak identification property, i.e.  $m_i\langle u, x, v \rangle = m_i\langle u, y, v \rangle$ ,  $\forall u, v \in J_n$  and  $i = 1, 2 \Rightarrow x = y$ . It shows that  $\{(J_n, m_i) : i = 1, 2\}$  is a family of weak identification metrons.

### 3. HIERARCHY BETWEEN METRON AND METRONLIKE STRUCTURES

In the weak variations of metron, we found many metronlike structures called pre-metron, quasimetron, sur-semi-metron and ide-semi-metron by excluding the properties  $M7$ ,  $M5$ ,  $M4$  and  $M3$  respectively. Further, weak variations of structures have been found by excluding more than one properties from the metron. All these weak structures can be arranged in a lattice diagram given in Figure 3.1.

In this diagram five levels are given. These levels represent low pitched demarcations. When no property is excluded from the definition of a metron, we call the structure to be a zero level structure, which is the metron itself. Out of the properties  $M7$ ,  $M5$ ,  $M4$  and  $M3$  of a metron, if any one of them is deleted, we call the corresponding structure to be a first level structure. Similarly if any two of the properties mentioned above are deleted, the corresponding structures are called second level structures. Further, if any three of the above properties are deleted from the definition of a metron the structure obtained are called third leveled. Finally if all the

four properties mentioned above are deleted. we call the structure to be of fourth leveled. In the diagram the implications indicate the weak structures. The weakest of all is the quasi-semi-pre-metron. We note that those structures which are not connected by a flow of implications are noncomparable.

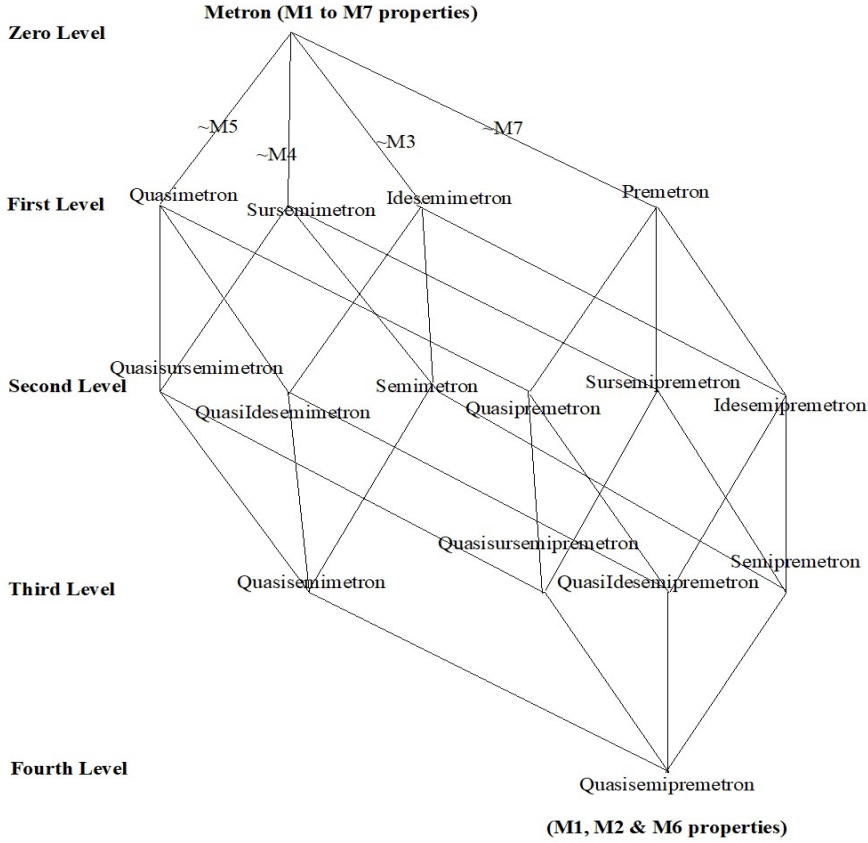


Figure 3.1. **Lattice Diagram of Metronlike Structures**

#### 4. METRIC SPACES AND METRONLIKE STRUCTURES

In this section we investigate the possibilities of constructing metronlike structures from metric spaces and vice versa.

**Theorem 4.1.** *Each metric space is a sur-semi-metron.*

*Proof.* Let  $(X, d)$  be a metric space. Define  $m : X \times X \times X \rightarrow \mathbb{R}$ , by  $m\langle x, y, z \rangle = d(x, z)$ ,  $\forall x, y, z \in X$ . Then:

- (i) The properties,  $M1$ ,  $M2$ ,  $M3$ ,  $M5$  and  $M6$  clearly hold.
- (ii) The property  $M4$  is not satisfied, because,  $m\langle u, x, v \rangle = m\langle u, y, v \rangle$ ,  $\forall x, y, u, v \in X$ .
- (iii) the property  $M7$  holds, as,

$$\begin{aligned}
 m\langle x, y, z \rangle &= d(x, z) \\
 &\leq 2d(x, z) \\
 &\leq d(x, z) + d(x, u) + d(u, z) \\
 &= d(x, z) + d(u, x) + d(z, u) \\
 &= m\langle x, u, z \rangle + m\langle u, z, x \rangle + m\langle z, x, u \rangle, \quad \forall x, y, z, u \in X.
 \end{aligned}$$

Hence  $(X, m)$  is a sur-semi-metron.  $\square$

The function  $m\langle x, y, z \rangle$  defined in the Theorem 4.1 is an invariant function with respect to  $y$ . Hence it assures that  $m\langle x, y, z \rangle$  will not satisfy the identification property.

Hence it is a sur-semi-metron function. In the case of invariant metronlike functions, the triangular chain inequality is the byproduct of the triangle inequality.

**Definition 4.2.** Let  $X$  be a nonempty set and let  $m\langle x, y, z \rangle$  be a metronlike function. Then it is said to be an invariant metronlike function, if  $m$  is invariant with respect to  $y$ .

The function  $m\langle x, y, z \rangle = d(x, z)$ , where  $(X, d)$  is a metric space, is an invariant sur-semi-metron function. A set associated with an invariant sur-semi-metron function is called an Invariable sur-semi-metron.

**Theorem 4.3.** A metric space and an invariable sur-semi-metron are equivalent structures.

*Proof.* By the Theorem 4.1, and Definition 4.2, it is clear that each metric space can be treated as an invariable sur-semi-metron.

Conversely, let  $(X, m)$  be an invariable sur-semi-metron. Define  $d(x, z) = m\langle x, y, z \rangle \quad \forall x, y, z \in X$ . Since  $m$  is an invariable function with respect to  $y$ , so clearly  $d(x, z)$  is a well defined function. Further  $d$  has following properties on  $X$ .

- (i)  $d(x, z) \geq 0$ ,  $\forall x, z \in X$  because  $m\langle x, y, z \rangle \geq 0$ ,  $\forall x, y, z \in X$ .
- (ii)  $d(x, z) = 0$ , if  $x = z$  because  $m\langle x, y, z \rangle = 0$  whenever  $x = z$ .
- (iii) Suppose,  $d(x, z) = 0$ , so,  $m\langle x, y, z \rangle = 0$ ,  $\forall y \in X$ , then  $x = z$ ; by the  $M3(A)$  property.
- (iv)  $d(x, z) = m\langle x, y, z \rangle = m\langle z, y, x \rangle = d(z, x)$ ,  $\forall x, z \in X$ .

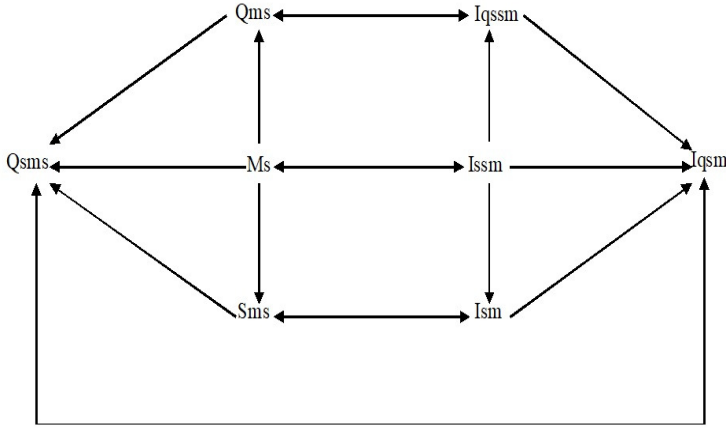
(v) Clearly, we have

$$\begin{aligned}
 d(x, z) &= m\langle x, y, z \rangle \\
 &\leq m\langle x, y, u \rangle + m\langle u, y, z \rangle \\
 &= d(x, u) + d(u, z) \quad \forall x, u, z \in X.
 \end{aligned}$$

Hence  $(X, d)$  is a metric space.

Therefore we have that, metric space and invariable sur-semi-metron are equivalent structures.  $\square$

**Remark 4.4.** *The Theorem 4.3 shows that metric space and invariable sur-semi-metron are equivalent. Similarly it can be seen that, semi-metric space, quasimetric space and quasisemimetric spaces are equivalent corresponding to invariable semi-metron, invariable quasisur-semi-metron and quasisemi-metron. Hence we get the following implications sketch of Figure 4.1.*



**Abbreviations:**

|                              |                                       |
|------------------------------|---------------------------------------|
| Ms → Metric space            | Issm → Invariable sursemimetron       |
| Sms → Semimetric space       | Ism → Invariable semimetron           |
| Qms → Quasimetric space      | Iqssm → Invariable quasisursemimetron |
| Qsms → Quasisemimetric space | Iqsm → Invariable quasisemimetron     |

*Figure 4.1. Equivalence of Metronlike structures with Metriclike structures*

**Theorem 4.5.** *Let  $(X, d)$  be a metric space, and let  $m$  be a function given by  $m\langle x, y, z \rangle = |d(x, y) - d(y, z)|$ ,  $\forall x, y, z \in X$ , then  $(X, m)$  is necessarily a sur-semi-pre-metron.*

*Proof.* Let  $(X, d)$  be a metric space and let  $m\langle x, y, z \rangle = |d(x, y) - d(y, z)|, \forall x, y, z \in X$ . Then the property observations are as follows.

- (i)  $m\langle x, y, z \rangle \geq 0, \forall x, y, z \in X$ .
- (ii) If  $x = z$  then,  $m\langle x, y, z \rangle = |d(x, y) - d(y, z)| = |d(x, y) - d(y, x)| = 0, \forall y \in X$ .
- (iii) Take  $m\langle x, y, z \rangle = 0, \forall y \in X$ , then  $|d(x, y) - d(y, z)| = 0, \forall y \in X$ ; or  $d(x, y) = d(y, z), \forall y \in X$  implies that  $x = z$ . Otherwise suppose that  $x \neq z$ , then for  $y = z, d(x, z) = d(z, z) = 0$ , gives  $x = z$ , a contradiction to our assumption  $x \neq z$ .
- (iv) The following example shows that the identification property is not satisfied.  
For, take  $J_k = \{1, 2, \dots, k\}$  as a metric space with usual metric  $d(x, y) = |x - y|$ . Let  $m_1\langle x, y, z \rangle = ||x - y| - |y - z||$  then in  $(J_k, m_1), m_1\langle x, 1, z \rangle = m_1\langle x, k, z \rangle, \forall x, z \in J_k$  whereas  $1 \neq k$ .
- (v) The symmetric property is obvious.
- (vi) The triangle inequality holds. For,

$$\begin{aligned} m\langle x, y, z \rangle &= |d(x, y) - d(y, z)| \\ &= |d(x, y) - d(y, u) + d(y, u) - d(y, z)| \\ &\leq |d(x, y) - d(y, u)| + |d(y, u) - d(y, z)| \\ &= m\langle x, y, u \rangle + m\langle u, y, z \rangle, \forall x, y, z \in X. \end{aligned}$$

- (vii) The triangular chain inequality does not hold. This is shown by the following example. Take  $(R^2, d)$  as a metric space with  $d$ , the usual metric on  $R^2$ .

Define  $m_2\langle x, y, z \rangle = |d(x, y) - d(y, z)|, \forall x, y, z \in R^2$ .

Choose,  $x = (0, 0), y = (3, 0), z = (2, 0)$  and  $u = (1, \sqrt{3})$ , then  $d(x, y) = 3, d(x, z) = 2, d(y, z) = 1, d(x, u) = 2$  and  $d(z, u) = 2$ .

So,

$$\begin{aligned} m_2\langle x, y, z \rangle &= |d(x, y) - d(y, z)| = |3 - 1| = 2 \\ &> m_2\langle x, u, z \rangle + m_2\langle u, z, x \rangle + m_2\langle z, x, u \rangle \\ &= |d(x, u) - d(u, z)| + |d(u, z) - d(z, x)| + |d(z, x) - d(x, u)| \\ &= |2 - 2| + |2 - 2| + |2 - 2| = 0. \end{aligned}$$

Thus,  $M1, M2, M3, M5$  and  $M6$  properties hold necessarily. Hence  $(X, m)$  is necessarily a sur-semi-pre-metron.

□

**Remark 4.6.** *Theorem 4.5 infers that the metriclike structures- metric space, semimetric space, quasimetric space, quasisemimetric space can respectively be converted into the metronlike structures- sur-semi-pre-metron, semi-pre-metron, quasi-sur-semi-pre-metron and quasi-semi-pre-metron.*

**Remark 4.7.** *The triangular chain inequality holds in  $(R, d)$  but it does not hold in  $(R^k, d)$ ,  $k \geq 2$  with  $m_k\langle x, y, z \rangle = |d(x, y) - d(y, z)|$ , where  $d$  is the usual distance function on related sets. Thus, we are able to show the basic difference, between  $(R, d)$  and  $(R^k, d)$ , when  $k \geq 2$ , with their usual metric.*

**Definition 4.8.** *Let  $(X, m)$  be a metronlike structure, then*

- (a) *it satisfies the Existence of base supremum property (EBS-Property) if  $\sup\{m\langle x, u, y \rangle : u \in X\}$  exists for each choice of  $x, y \in X$  or equivalently  $\{m\langle x, u, y \rangle : u \in X\}$  is bounded for each choice of  $x, y \in X$ .*
- (b) *it satisfies the Mark binding property (MB-Property) if  $\sup\{m\langle u, x, v \rangle : u, v \in X\}$  exists for each choice of  $x \in X$  or equivalently  $\{m\langle u, x, v \rangle : u, v \in X\}$  is bounded for each choice of  $x \in X$ .*
- (c) *it satisfies the Existence of base difference supremum property (EBDS-Property) if  $\sup\{|m\langle x, u, y \rangle - m\langle x, v, y \rangle| : u, v \in X\}$  exists for each choice of  $x, y \in X$  or equivalently  $\{|m\langle x, u, y \rangle - m\langle x, v, y \rangle| : u, v \in X\}$  is bounded for each choice of  $x, y \in X$ .*
- (d) *satisfies the Existence of mark supremum property (EMS-Property) if  $\sup\{|m\langle u, x, v \rangle - m\langle u, y, v \rangle| : u, v \in X\}$  exists for each choice of  $x, y \in X$  or equivalently,  $\{|m\langle u, x, v \rangle - m\langle u, y, v \rangle| : u, v \in X\}$  is bounded for each choice of  $x, y \in X$ .*

**Remark 4.9.** *A sur-semi-pre-metron  $(X, m)$  obtained from a metric space  $(X, d)$  according to Theorem 4.5 then  $(X, m)$  possesses the EBS-Property. As  $d(x, y) \leq d(x, z) + d(z, y)$  or  $|d(x, y) - d(y, z)| \leq d(x, z)$ ,  $\forall x, y, z \in X$  or  $\sup\{m\langle x, y, z \rangle : y \in X\}$  always exists for  $x, z \in X$ . Similarly all the conversions of Remark 4.6 have EBS-Property.*

**Theorem 4.10.** *Let  $(X, m)$  be a sur-semi-pre-metron with the EBS-Property and  $d(x, z) = \sup\{m\langle x, y, z \rangle : y \in X\}$  for all  $x, z \in X$ , then  $(X, d)$  is a metric space.*

*Proof.* Let  $(X, m)$  be a sur-semi-pre-metron with the EBS-Property and let  $d(x, z) = \sup\{m\langle x, y, z \rangle : y \in X\}$ ,  $\forall x, y, z \in X$ . Then  $d(x, z)$

always exists and so  $d$  is a well defined function on  $X \times X$ . We observe that  $d$  satisfies the following properties.

- (i)  $d(x, z) \geq 0$ ,  $\forall x, z \in X$ , which is implied by  $M1$  property of  $(X, m)$ .
- (ii) If  $x = z$ , then  $d(x, z) = 0$ , which is implied by  $M2$  property.
- (iii) If  $d(x, z) = 0$  then  $x = z$ , is implied by  $M3(A)$  property.
- (iv)  $d(x, z) = d(z, x)$ ,  $\forall x, z \in X$ , is implied by  $M5$  property.
- (v)  $d(x, z) \leq d(x, u) + d(u, z)$ ,  $\forall x, u, z \in X$ , is implied by  $M6$  property.

Thus  $d$  defined as above is a metric function on  $X$ . Hence  $(X, d)$  is a metric space.  $\square$

**Remark 4.11.** *In the statement of Theorem 4.10 if we replace sur-semi-pre-metron by some other metronlike structure viz semi-pre-metron, quasi-sur-semi-pre-metron and quasi-semi-pre-metron, then correspondingly we get the metriclike structures semimetric space, quasimetric space and quasisemimetric space.*

**Remark 4.12.** *We note from Remarks 4.6, 4.9 and 4.11, that the metronlike structures with EBS-property and metriclike structures are mutually convertible; as below:*

*Metric space  $\longleftrightarrow$  Sursemi-pre-metron with EBS-property.*

*Semimetric space  $\longleftrightarrow$  Semipre-metron with EBS-property.*

*Quasimetric space  $\longleftrightarrow$  Quasisur-semi-pre-metron with EBS-property.*

*Quasisemimetric space  $\longleftrightarrow$  Quasisemi-pre-metron with EBS-property.*

**Theorem 4.13.** *Let  $(X, m)$  be a metronlike structure with the triangular chain inequality, then  $(X, m)$  obeys the EBS-property and so there always exists a metriclike function on  $X$ .*

*Proof.* Let  $(X, m)$  be a metronlike structure with  $M7$  property. Now,  $m\langle x, y, z \rangle \leq m\langle x, u, z \rangle + m\langle u, z, x \rangle + m\langle z, x, u \rangle \forall x, y, z, u \in X$ . By varying  $y$  on  $X$  and by fixing  $u$ , it is clear that  $\{m\langle x, y, z \rangle : y \in X\}$  is a bounded set and therefore it obeys the EBS-Property.

Thus, by the result of Theorem 4.10 there always exists a metriclike function on  $X$ .  $\square$

**Remark 4.14.** *From the above findings about metriclike structures and metronlike structures of Remark 4.12 it is very clear that the identification property has no role in these metric spaces or metriclike spaces. Hence the identification property is totally independent from metric properties in these conversions.*

**Theorem 4.15.** *Let  $(X, m)$  be an idesemi-pre-metron with EMS-property and  $d_i(x, y) = \sup\{|m\langle u, x, v \rangle - m\langle u, y, v \rangle| : u, v \in X\}$  Then  $(X, d_i)$  is a metric space.*

*Proof.* Let  $(X, m)$  be an idesemi-pre-metron with EMS-property and  $d_i : X \times X \rightarrow \mathbb{R}$  is defined by,

$$d_i(x, y) = \sup\{|m\langle u, x, v \rangle - m\langle u, y, v \rangle| : u, v \in X\}, \quad \forall x, y \in X.$$

Since  $(X, m)$  possesses the EMS-property so,  $d_i(x, y)$  always exists and it is a well defined function. The observations about metric properties are following,

- (i)  $d_i(x, y) \geq 0$ ,  $\forall x, y \in X$ , by definition.
- (ii) we have,

$$\begin{aligned} d_i(x, y) = 0 &\Rightarrow \sup\{|m\langle u, x, v \rangle - m\langle u, y, v \rangle| : u, v \in X\} = 0 \\ &\Rightarrow m\langle u, x, v \rangle = m\langle u, y, v \rangle, \quad \forall u, v \in X \\ &\Rightarrow x = y, \text{ by the M4 property.} \end{aligned}$$

- (iii) if  $x = y$ , then

$$\begin{aligned} m\langle u, x, v \rangle &= m\langle u, y, v \rangle \quad \forall u, v \in X \\ &\Rightarrow |m\langle u, x, v \rangle - m\langle u, y, v \rangle| = 0, \quad \forall u, v \in X. \\ &\Rightarrow \sup\{|m\langle u, x, v \rangle - m\langle u, y, v \rangle| : u, v \in X\} = 0. \\ &\Rightarrow d_i(x, y) = 0. \end{aligned}$$

- (iv)  $d_i(x, y) = d_i(y, x)$ ,  $\forall x, y \in X$  by definition.

- (v) Consider,

$$\begin{aligned} |m\langle u, x, v \rangle - m\langle u, y, v \rangle| &= |m\langle u, x, v \rangle - m\langle u, z, v \rangle \\ &\quad + m\langle u, z, v \rangle - m\langle u, y, v \rangle| \\ &\leq |m\langle u, x, v \rangle - m\langle u, z, v \rangle| \\ &\quad + |m\langle u, z, v \rangle - m\langle u, y, v \rangle| \end{aligned}$$

By taking supremum on right hand side and then on left hand side,  $\forall x, y, z \in X$  we get,

$$\begin{aligned} \sup_{u, v \in X} \{|m\langle u, x, v \rangle - m\langle u, y, v \rangle|\} &\leq \sup_{u, v \in X} \{|m\langle u, x, v \rangle - m\langle u, z, v \rangle|\} \\ &\quad + \sup_{u, v \in X} \{|m\langle u, z, v \rangle - m\langle u, y, v \rangle|\} \end{aligned}$$

Which implies that  $d_i(x, y) \leq d_i(x, z) + d_i(z, y)$ .

Hence  $(X, d_i)$  is a metric space.

□

**Remark 4.16.** In the statement of Theorem 4.15 if we replace *idesemi-pre-metron* by *semi-pre-metron* then correspondingly we get a *semimetric space*.

**Remark 4.17.** In the Theorem 4.15, if  $(X, m)$  is a *metronlike structure* with *M4* and *EMS-property*. Then  $(X, d_i)$  is a *metric space*.

**Theorem 4.18.** Let  $(X, m)$  be a *semi-pre-metron* which possesses the *I-survival property*, the *triangle inequality of difference* and *EBDS-property*. Suppose  $d_{is}$  is a function given by,  $d_{is}(x, y) = \sup\{|m\langle x, u, y \rangle - m\langle x, v, y \rangle| : u, v \in X\}$ . Then  $(X, d_{is})$  is a *metric space*.

*Proof.* Let  $(X, m)$  be a *semi-pre-metron* with *I-survival property*, the *triangle inequality of difference* and *EBDS-property*. Define  $d_{is}(x, y) = \sup\{|m\langle x, u, y \rangle - m\langle x, v, y \rangle| : u, v \in X\}$ , then it always exists and therefore  $d_{is}(x, y)$  is a well defined function. The observations about metric properties are on the lines of Theorem 4.9 [29], shows that  $(X, d_{is})$  is a *metric space*  $\square$

**Remark 4.19.** In the statement of Theorem 4.18,

- (i) if we replace *semi-pre-metron* by *quasi-semi-pre-metron* then correspondingly we get *quasimetric space*.
- (ii) if we exclude the *I-survival property*, then correspondingly we get *semimetric space*.
- (iii) If we replace *semi-pre-metron* by *quasi-semi-pre-metron* and exclude the *I-survival property*, then correspondingly we get *quasisemimetric space*.

**Remark 4.20.** In the Theorem 4.18,  $(X, m)$  be a *semi-pre-metron*, can be replaced by  $(X, m)$  be a *metronlike structure* (with *M2 property*).

**Theorem 4.21.** Let  $(X, m)$  be a *metronlike structure* with *M7 property* then the *EBS-property* holds and therefore the *EBDS-property* also holds.

*Proof.* Let  $(X, m)$  be a *metronlike structure* with *M7 property*. So,  $m\langle x, u, y \rangle \leq \langle x, x, y \rangle + m\langle x, y, x \rangle + m\langle y, x, x \rangle = 2m\langle x, x, y \rangle, \forall x, y \in X$ . Hence  $\sup\{|m\langle x, u, y \rangle - m\langle x, v, y \rangle| : u, v \in X\} \leq 2m\langle x, x, y \rangle, \forall x, y \in X$ . This shows that *EBS-property* holds and hence *EBDS-property* also holds in  $(X, m)$ .  $\square$

**Remark 4.22.** As in the Theorem 4.18 if  $(X, m)$  is a given *metronlike structure* with *M7 property* and  $d_{is}(x, y) = \sup\{|m\langle x, u, y \rangle -$

$m\langle x, v, y \rangle| : u, v \in X\}$ , then conversions into metriclike spaces are as follows.

- (i)  $(X, d_{is})$  is a metric space if  $(X, m)$  is a semi-metron with  $I$ -survival and the triangle inequality of difference property.
- (ii)  $(X, d_{is})$  is a quasimetric space if  $(X, m)$  is a quasi-semi-metron with  $I$ -survival and the triangle inequality of difference property.
- (iii)  $(X, d_{is})$  is a semimetric space if  $(X, m)$  is a semi-metron with the triangle inequality of difference property.
- (iv)  $(X, d_{is})$  is a quasisemimetric space if  $(X, m)$  is a quasi-semi-metron with the triangle inequality of difference property.

**Definition 4.23.** Let  $(X, m)$  be a metronlike structure, then we say that it possesses the Existence of Nonsurvival Property (EN-property) if for each choice of  $x, y \in X$ ,  $\exists$  a point  $u \in X$  (depending on  $x$  and  $y$ ) such that  $m\langle x, u, y \rangle = 0$ . This property holds in  $(\mathbb{R}, m_1)$ , as,  $x, y \in \mathbb{R}$  then for  $u = 1/2(x + y)$ ,  $m_1\langle x, u, y \rangle = 0$ .

**Theorem 4.24.** Let  $(X, m)$  be a metronlike structure with EBDS-property and the existence of nonsurvival property. Suppose,  $d_s(x, y) = \sup\{m\langle x, u, y \rangle : u \in X\}$  and  $d_{is}(x, y) = \sup\{|m\langle x, u, y \rangle - m\langle x, v, y \rangle| : u, v \in X\}$ , then  $(X, m)$  possesses the EBS-property and  $d_s(x, y) = d_{is}(x, y)$ ,  $\forall x, y \in X$ .

*Proof.* Let  $(X, m)$  be a metronlike structure with EBDS-property and the existence of nonsurvival property. Since  $m\langle x, u, y \rangle \geq 0$ , so clearly  $|m\langle x, u, y \rangle - m\langle x, v, y \rangle| \leq \max\{m\langle x, u, y \rangle, m\langle x, v, y \rangle\}$ . So, by taking supremum, we get,

$$\begin{aligned}
 & \sup\{|m\langle x, u, y \rangle - m\langle x, v, y \rangle| : u, v \in X\} \\
 & \leq \sup\{\max\{m\langle x, u, y \rangle, m\langle x, v, y \rangle\} : u, v \in X\} \\
 & = \sup\{m\langle x, u, y \rangle : u \in X\}, \text{ if } m \text{ attains max at } u. \\
 & = \sup\{m\langle x, u, y \rangle : u \in X\} \text{ for supremum taking on entire } X.
 \end{aligned}$$

Thus, we obtain that

$$\sup\{|m\langle x, u, y \rangle - m\langle x, v, y \rangle| : u, v \in X\} \leq \sup\{m\langle x, u, y \rangle : u \in X\} \dots (A)$$

Conversely,  $\forall v_0 \in X$ , we have,  $\sup\{|m\langle x, u, y \rangle - m\langle x, v, y \rangle| : u, v \in X\} \geq \sup\{|m\langle x, u, y \rangle - m\langle x, v_0, y \rangle| : u, v_0 \in X\}$ . But by the existence of nonsurvival property,  $\exists$  a point  $v_0 \in X$  such that  $m\langle x, v_0, y \rangle = 0$ . It follows that,

$$\sup\{|m\langle x, u, y \rangle - m\langle x, v, y \rangle| : u, v \in X\} \geq \sup\{m\langle x, u, y \rangle : u \in X\} \dots (B).$$

Thus, by (A) and (B) we get,

$$\sup\{|m\langle x, u, y \rangle - m\langle x, v, y \rangle| : u, v \in X\} = \sup\{m\langle x, u, y \rangle : u \in X\} \dots (C)$$

Hence the *EBDS*-property with the existence of nonsurvival property implies the *EBS*-property. Further by, the definitions of  $d_s(x, y)$  and  $d_{is}(x, y)$  putting in (C) we get  $d_s(x, y) = d_{is}(x, y)$ ,  $\forall x, y \in X$ .  $\square$

**Note:** In the Theorem 4.24, if  $(X, m)$  be a metronlike structure with *EBS*-property and *EN*-property; then  $(X, m)$  possesses the *EBDS*-property and  $d_{is}(x, y) = d_s(x, y)$ ,  $\forall x, y \in X$ .

## 5. CONSTRUCTION OF METRON FROM WEAK VARIATIONS OF METRON

Every society has some social problems. These problems are created by a unit, or units or group of units of the society. The problem creators don't obey the principles set forth for the society. Then there exists a basic job for the thinkers as how to mould the units, so that they also start to follow the norms of the society. Similarly in mathematics also, we have model structures and structures which are weak in the model sense. In particular, if a metron is taken as a model structure then the structures like sur-semi-metron, ide-semi-metron and semi-metron are weak in the above model sense. In this section we make efforts to convert these weak structures into the near model structures or into model structure.

**Theorem 5.1.** *Let  $(X, m)$  be an ide-semi-metron. Then there is always possible to construct a metron on some quotient set of  $X$ .*

*Proof.* Let  $(X, m)$  be an ide-semi-metron, then it is a metronlike structure with all the metron properties excluding *M3* property. To impose *M3* on some quotient set of  $X$ , define a relation  $S$  on the set  $X$ , by

$$xSy \Leftrightarrow m\langle x, a, y \rangle = 0, \forall a \in X.$$

Then clearly  $S$  is an equivalence relation on  $X$ . Let  $E = X/S = \{E_x : x \in X\}$ , where,  $E_x = \{y \in X : m\langle x, a, y \rangle = 0, \forall a \in X\}$ . Suppose,  $\alpha : E \rightarrow X$  such that  $\alpha(E_x) \in E_x$  and then define  $\bar{m}_\alpha : E \times E \times E \rightarrow \mathbb{R}$  given by  $\bar{m}_\alpha\langle E_x, E_y, E_z \rangle = m\langle x, \alpha(E_y), z \rangle$ . This function  $\bar{m}_\alpha$  is well defined, because,

- (i)  $\alpha(E_x) \in E_x$ ,  $\forall x \in X$ , so  $\alpha(E_x) \neq \alpha(E_y)$ , if  $E_x \neq E_y$ . Hence  $\alpha$  is a injective map.

(ii)

$$\begin{aligned}
m\langle x', \alpha(E_y), z' \rangle &\leq m\langle x', \alpha(E_y), x \rangle + m\langle x, \alpha(E_y), z \rangle + m\langle z, \alpha(E_y), z' \rangle \\
&= 0 + m\langle x, \alpha(E_y), z \rangle + 0 \\
&= m\langle x, \alpha(E_y), z \rangle.
\end{aligned}$$

Similarly, by interchanging the role of  $x$  and  $x'$  etc.  $m\langle x, \alpha(E_y), z \rangle \leq m\langle x', \alpha(E_y), z' \rangle$ . Hence,  $m\langle x, \alpha(E_y), z \rangle = m\langle x', \alpha(E_y), z' \rangle \forall x' \in E_x$  and  $z' \in E_z$ . Further, the function  $\bar{m}_\alpha$  obeys the following properties.

M1  $\bar{m}_\alpha\langle E_x, E_y, E_z \rangle \geq 0$ ,  $\forall E_x, E_y, E_z \in E$ . (by definition)

M2 Suppose  $E_x = E_z$ , then  $x' \in E_x = E_z$ , so,  $\bar{m}_\alpha\langle E_x, E_y, E_z \rangle = m\langle x, \alpha(E_y), x' \rangle = 0$ ,  $\forall E_y \in E$ .

M3 Suppose,  $E_x \in E_z$ , then  $x \notin E_z$ ; and so  $\bar{m}_\alpha\langle E_x, E_x, E_z \rangle > 0$ . Otherwise, let

$$\begin{aligned}
\bar{m}_\alpha\langle E_x, E_x, E_z \rangle &= 0. \\
\Rightarrow m\langle x, \alpha(E_z), z' \rangle &= 0, \forall z' \in E_z. \\
\Rightarrow m\langle x, \alpha(E_z), \alpha(E_z) \rangle &= 0.
\end{aligned}$$

So  $\forall y \in X$ , we have,

$$\begin{aligned}
m\langle x, y, z \rangle &= m\langle x, y, \alpha(E_z) \rangle \\
&\leq m\langle x, \alpha(E_z), \alpha(E_z) \rangle \\
&\quad + m\langle \alpha(E_z), \alpha(E_z), x \rangle \\
&\quad + m\langle \alpha(E_z), x, \alpha(E_z) \rangle \\
&= 0 + 0 + 0 = 0
\end{aligned}$$

Thus,  $m\langle x, y, z \rangle = 0$ ,  $\forall y \in X$  or equivalently  $zSx$ , so  $x \in E_z$ , a contradiction.

M4 It follows on noting that

$$\begin{aligned}
\bar{m}_\alpha\langle E_u, E_x, E_v \rangle &= \bar{m}_\alpha\langle E_u, E_y, E_v \rangle, \forall E_u, E_v \in E. \\
\Rightarrow m\langle u', \alpha(E_x), v' \rangle &= m\langle u', \alpha(E_y), v' \rangle, \forall u' \in E_u, v' \in E_v. \\
\Rightarrow \alpha(E_x) &= \alpha(E_y) \\
\Rightarrow E_x &= E_y, \text{ for } f \text{ is a injective function.}
\end{aligned}$$

M5 It clear , because  $\forall E_x, E_y, E_z \in E$ , we have,

$$\begin{aligned}
\bar{m}_\alpha\langle E_x, E_y, E_z \rangle &= m\langle x, \alpha(E_y), z \rangle \\
&= m\langle z, \alpha(E_y), x \rangle \\
&= \bar{m}_\alpha\langle E_z, E_y, E_x \rangle.
\end{aligned}$$

M6 It follows because  $\forall E_x, E_y, E_z, E_u \in E$ , we have,

$$\begin{aligned}\bar{m}_\alpha \langle E_x, E_y, E_z \rangle &= m \langle x, \alpha(E_y), z \rangle \\ &\leq m \langle x, \alpha(E_y), u \rangle + m \langle u, \alpha(E_y), z \rangle \\ &= \bar{m}_\alpha \langle E_x, E_y, E_z \rangle + \bar{m}_\alpha \langle E_u, E_y, E_z \rangle.\end{aligned}$$

M7 Clearly,  $\forall E_x, E_y, E_z, E_u \in E$ , we have,

$$\begin{aligned}\bar{m}_\alpha \langle E_x, E_y, E_z \rangle &= m \langle x, \alpha(E_y), z \rangle \\ &= m \langle \alpha(E_x), \alpha(E_y), \alpha(E_z) \rangle \\ &\leq m \langle \alpha(E_x), \alpha(E_u), \alpha(E_z) \rangle \\ &\quad + m \langle \alpha(E_u), \alpha(E_z), \alpha(E_x) \rangle \\ &\quad + m \langle \alpha(E_z), \alpha(E_x), \alpha(E_u) \rangle \\ &= \bar{m}_\alpha \langle E_x, E_u, E_z \rangle \\ &\quad + \bar{m}_\alpha \langle E_u, E_z, E_x \rangle \\ &\quad + \bar{m}_\alpha \langle E_z, E_x, E_u \rangle\end{aligned}$$

Thus  $\bar{m}_\alpha$  obeys all the metron properties M1 to M7 on  $E$ . This shows that  $(E, \bar{m}_\alpha)$  is a metron, called a quotient metron on ide-semi-metron  $(X, m)$ .  $\square$

**Remark 5.2.** Since  $\alpha : E \rightarrow X$ , such that  $\alpha(E_x) \in E_x$  and for each  $\alpha$ ,  $\bar{m}_\alpha$  is a metron function on  $E$ . Hence under the machinery of Theorem 5.1 it generates a family of quotient metrons,  $\{(E, \bar{m}_\alpha) : \alpha \in \Lambda\}$  where  $\Lambda$  is the family of all such  $\alpha$  functions on  $E$ .

**Theorem 5.3.** Let  $(X, m)$  be a sur-semi-metron. Then there is always possible to construct a family of sur-semi-metrons on the quotient set of  $X$ , which possesses the weak identification property. Thus we get a family of weak identification metrons.

*Proof.* Let  $(X, m)$  be a sur-semi-metron, then it is a metronlike structure with all the metron properties except the identification property. To impose identification, we define a relation  $I$  on  $X$ , by

$$xIy \Leftrightarrow m \langle u, x, v \rangle = m \langle u, y, v \rangle, \quad \forall u, v \in X.$$

Then clearly reflexivity, symmetry and transitivity hold for  $I$ , so  $I$  is an equivalence relation on  $X$ . Suppose,  $E$  is the set of all the equivalence classes generated by  $I$  on  $X$  i.e.  $E = \{E_x : x \in X\}$  where  $E_x = \{y \in X : xIy\}$  and  $x \in X$ .

The set  $E$  is the quotient set on  $X$  with respect to  $I$ . Now, define  $\bar{m}_\alpha : E \times E \times E \rightarrow \mathbb{R}$ , by  $\bar{m}_\alpha \langle E_x, E_y, E_z \rangle = m \langle \alpha(E_x), y, \alpha(E_z) \rangle$ , where

$\alpha : E \rightarrow X$  such that  $\alpha(E_x) \in E_x$ . The function  $\bar{m}_\alpha$  is a well-defined function, because,

- (i)  $\alpha$  is a 1 – 1 map for  $\alpha(E_x) \in E_x$ .
- (ii) By the definition of relation I on X, we have,

$$\begin{aligned}\bar{m}_\alpha \langle E_x, E_y, E_z \rangle &= m \langle \alpha(E_x), y, \alpha(E_z) \rangle, \\ &= m \langle \alpha(E_x), y', \alpha(E_z) \rangle \quad \forall y' \in E_y.\end{aligned}$$

The properties of  $\bar{m}_\alpha$  on  $E$  are observed as below. □

*M1 Clearly  $\bar{m}_\alpha$  is a nonnegative function.*

*M2 We note that  $\bar{m}_\alpha$  has vanishing property, because  $\alpha(E_x)$  is unique for each  $E_x$ , and  $m$  is a function with vanishing property.*

*M3 If  $E_x \neq E_z$ , then  $\alpha(E_x) \neq \alpha(E_z)$ , so  $\exists$  a  $y \in X$  such that  $m \langle \alpha(E_x), y, \alpha(E_z) \rangle > 0$ , which gives that  $\exists E_y \in E$  such that  $\bar{m}_\alpha \langle E_x, E_y, E_z \rangle > 0$ .*

*M4 Suppose*

$$\bar{m}_\alpha \langle E_u, E_x, E_v \rangle = \bar{m}_\alpha \langle E_u, E_y, E_v \rangle \quad \forall E_u, E_v \in E.$$

$$\begin{aligned}\Rightarrow m \langle \alpha(E_u), x', \alpha(E_v) \rangle &= m \langle \alpha(E_u), y', \alpha(E_v) \rangle, \quad \forall x' \in E_x, y' \in E_y. \\ \Rightarrow x' I y'.\end{aligned}$$

*So, we conclude that  $E_x$  and  $E_y$  are not necessarily equal.*

*M4' If we suppose,  $\bar{m}_\alpha \langle E_u, E_x, E_v \rangle = \bar{m}_\alpha \langle E_u, E_y, E_v \rangle \quad \forall E_u, E_v \in E$ , and for all  $\alpha \in \Lambda$  where  $\Lambda$  is the family of all such  $\alpha$  functions defined on  $E$  to  $X$ .*

$$\text{Then } m \langle \alpha(E_u), x', \alpha(E_v) \rangle = m \langle \alpha(E_u), y', \alpha(E_v) \rangle, \quad \forall \alpha \in \Lambda, \quad \forall E_u, E_v \in E \& x' \in E_x, y' \in E_y.$$

*Thus,*

$$\begin{aligned}m \langle u, x', v \rangle &= m \langle u, y', v \rangle, \quad \forall u, v \in X. \\ \Rightarrow x' I y' \\ \Rightarrow E_x &= E_y.\end{aligned}$$

*It follows that the family  $\{\bar{m}_\alpha : \alpha \in \Lambda\}$  satisfies the weak identification property.*

*M5 It follows on noting that,*

$$\begin{aligned}\bar{m}_\alpha \langle E_x, E_y, E_z \rangle &= m \langle \alpha(E_x), y', \alpha(E_z) \rangle, \quad \forall y' \in E_y. \\ &= m \langle \alpha(E_z), y', \alpha(E_x) \rangle, \quad \forall y' \in E_y. \\ &= \bar{m}_\alpha \langle E_z, E_y, E_x \rangle \quad \forall E_x, E_y, E_z \in E.\end{aligned}$$

M6 It follows on noting that,

$$\begin{aligned}\bar{m}_\alpha\langle E_x, E_y, E_z\rangle &= m\langle \alpha(E_x), y', \alpha(E_z)\rangle, \quad \forall y' \in E_y. \\ &\leq m\langle \alpha(E_x), y', \alpha(E_u)\rangle + m\langle \alpha(E_u), y', \alpha(E_z)\rangle, \\ &= \bar{m}_\alpha\langle E_x, E_y, E_z\rangle + \bar{m}_\alpha\langle E_u, E_y, E_z\rangle \quad \forall E_x, E_y, E_z, E_u \in E.\end{aligned}$$

M7 Clearly,  $\forall E_x, E_y, E_z, E_u \in E$ , we have

$$\begin{aligned}\bar{m}_\alpha\langle E_x, E_y, E_z\rangle &= m\langle \alpha(E_x), y', \alpha(E_z)\rangle, \quad \forall y' \in E_y. \\ &\leq m\langle \alpha(E_x), \alpha(E_u), \alpha(E_z)\rangle \\ &\quad + m\langle \alpha(E_u), \alpha(E_z), \alpha(E_x)\rangle \\ &\quad + m\langle \alpha(E_z), \alpha(E_x), \alpha(E_u)\rangle \\ &= \bar{m}_\alpha\langle E_x, E_u, E_z\rangle \\ &\quad + \bar{m}_\alpha\langle E_u, E_z, E_x\rangle \\ &\quad + \bar{m}_\alpha\langle E_z, E_x, E_u\rangle.\end{aligned}$$

It shows that  $(E, \bar{m}_\alpha)$  is not necessarily a metron, but  $(E, \bar{m}_\alpha)$  is a weak  $I$ -metron with respect to the family  $\{(E, \bar{m}_\alpha) : \alpha \in \Lambda\}$ . Hence  $\{(E, \bar{m}_\alpha) : \alpha \in \Lambda\}$  is a family of weak  $I$ -metrons.

**Remark 5.4.** The constructed structure  $(E, \bar{m}_\alpha)$  in the above Theorem 5.3 may not be a metron. We note that the structure approaches to a metron or to a weak  $I$ -metron family only. The following examples illustrate the situations.

**Example 5.5.** Let  $J_k = \{1, 2, \dots, k\}$  and  $m : J_k \times J_k \times J_k \rightarrow \mathbb{R}$  defined by  $m\langle x, y, z\rangle = ||x-y| - |y-z||$ . Then  $(J_k, m)$  is a sur-semi-metron and does not satisfy the identification property for the pair  $\{1, k\}$ . Define a relation  $I$  on  $J_k$  by

$$xIy \Leftrightarrow m\langle u, x, v\rangle = m\langle u, y, v\rangle, \quad \forall u, v \in J_k.$$

Then  $I$  is an equivalence relation on  $J_k$ . Suppose  $E$  is the set of all equivalence classes on  $J_k$  with respect to  $I$ , i.e.  $E = \{E_x : x \in J_k\}$ , where  $E_x$  indicates the equivalence class of  $x$ . Then  $E_x = \{x\}$ ,  $\forall x \neq 1, k$  and  $x \in J_k$ ; and  $E_1 = E_k = \{1, k\}$ . So,  $E = \{\{1, k\}, \{2\}, \dots, \{k-1\}\}$ .

Now, define  $\alpha_i : E \rightarrow J_k$  with  $\alpha_i(E_x) \in E_x$ . Observe that all the equivalence classes except  $\{1, k\}$  contain singleton element. So there exist exactly two  $\alpha_i$  functions. Those two functions say  $\alpha_1$  and  $\alpha_2$  are as follows:

$$\alpha_1(E_x) = \begin{cases} x, & \text{if } E_x \neq \{1, k\} \\ 1, & \text{if } E_x = \{1, k\} (= E_1) \end{cases}$$

$$\alpha_2(E_x) = \begin{cases} x, & \text{if } E_x \neq \{1, k\} \\ k, & \text{if } E_x = \{1, k\} (= E_k) \end{cases}$$

Again define,  $\bar{m}_{\alpha i} : E \times E \times E \rightarrow \mathbb{R}$ , for  $i = 1, 2$  by  $\bar{m}_{\alpha i}\langle E_x, E_y, E_z \rangle = m\langle \alpha_i(E_x), \alpha_i(E_y), \alpha_i(E_z) \rangle$ . By the Theorem 5.3, it is clear that  $(E, \bar{m}_{\alpha i})$  and  $(E, \bar{m}_{\alpha 2})$  both are sur-semi-metrons.

Further, put  $\{1, k\} = E * (= E_1 = E_k)$ ; then

$$\begin{aligned} \bar{m}_{\alpha 1}\langle E_u, E*, E_v \rangle &= \bar{m}_{\alpha 1}\langle E_u, E_1, E_v \rangle \\ &= m\langle \alpha_1(E_u), \alpha_1(E_1), \alpha_1(E_v) \rangle \\ &= ||u - 1| - |1 - v|| \\ &= |u - 1| - (v - 1)|, \text{ for } 1 \leq u, 1 \leq v \\ &= |u - v| \\ &= |v - u| \\ &= |((k - 1) - u) - ((k - 1) - v)|, \text{ for } k - 1 \geq u, v. \\ &= ||u - (k - 1)| - |(k - 1) - v|| \\ &= m\langle \alpha_1(u), \alpha_1(E_{k-1}), \alpha_1(E_v) \rangle \\ &= m_{\alpha 1}\langle E_u, E_{k-1}, E_v \rangle, \forall E_u, E_v \in E. \end{aligned}$$

And,  $E * \{1, k\} = E_k$ , so

$$\begin{aligned} \bar{m}_{\alpha 2}\langle E_u, E_2, E_v \rangle &= m\langle \alpha_2(E_u), \alpha_2(E_2), \alpha_2(E_v) \rangle \\ &= ||u - 2| - |2 - v|| \\ &= |u - 2| - (v - 2)|, \text{ for } 2 \leq u, 2 \leq v \\ &= |u - v| = |v - u| \\ &= |(k - u) - (k - v)| \\ &= ||u - k| - |k - v|| \\ &= m\langle \alpha_2(E_u), \alpha_2(E_k), \alpha_2(E_v) \rangle \\ &= \bar{m}_{\alpha 2}\langle E_u, E_k, E_v \rangle, \forall E_u, E_v \in E. \end{aligned}$$

Thus,  $(E, \bar{m}_{\alpha 1})$  and  $(E, \bar{m}_{\alpha 2})$  are structures without identification property. Hence  $(E, \bar{m}_{\alpha 1})$  and  $(E, \bar{m}_{\alpha 2})$  both are not metrons.

**Example 5.6.** Let  $X = [a, b]$  and  $m : X \times X \times X \rightarrow \mathbb{R}$ , defined by

$$m\langle x, y, z \rangle = ||x - y| - |y - z||.$$

Then  $(X, m)$  is a sur-semi-metron and does not satisfy the identification property for the pair  $\{a, b\}$ .

Define a relation  $I$  on  $X$  by

$$xIy \Leftrightarrow m\langle u, x, v \rangle = m\langle u, y, v \rangle, \quad \forall u, v \in X.$$

Then  $I$  is an equivalence relation on  $X$  and the equivalence classes on  $X$  are,  $E_x = \{x\}$  if  $x \neq a, b$  and  $E_a = E_b = \{a, b\} = E^*$ . So,  $E = \{E_x : x \in ]a, b]\}$  or  $E = \{E_x : x \in [a, b[ \}$ . Now, define,  $\alpha_i : E \rightarrow X$  with  $\alpha_i(E_x) \in E_x$ .

Observe that all the equivalence classes except  $\{a, b\}$  contain singleton element. So there exist exactly two such functions. Those two functions are as follows:

$$\alpha_a(E_x) = \begin{cases} x, & \text{if } E_x \neq E^* \\ a, & \text{if } E_x = E^* (= E_a) \end{cases}$$

$$\alpha_b(E_x) = \begin{cases} x, & \text{if } E_x \neq E^* \\ b, & \text{if } E_x = E^* (= E_b) \end{cases}$$

Define,  $\bar{m}_{\alpha_i} : E \times E \times E \rightarrow \mathbb{R}$ , by,  $\bar{m}_{\alpha_i}\langle E_x, E_y, E_z \rangle = m\langle \alpha_i(E_x), \alpha_i(E_y), \alpha_i(E_z) \rangle$  then  $(E, \bar{m}_{\alpha_i})$  is a metron, for both the values  $i = a, b$ . Because whenever we choose,  $x, y \in [a, b[$  ( $x, y \in ]a, b]$ ) with  $x \neq y$  i.e.  $E_x \neq E_y$ , then  $\exists, u, v \in [a, b[$  ( $u, v \in ]a, b]$ ) with  $\bar{m}_{\alpha_i}\langle E_u, E_x, E_v \rangle \neq \bar{m}_{\alpha_i}\langle E_u, E_y, E_v \rangle$  for  $i = a$  ( $i = b$ ) respectively.

**Theorem 5.7.** Let  $(X, m)$  be a semi-metron. Then it is always possible to construct a weak  $I$ -metron on some quotient set of  $X$ .

*Proof.* Let  $(X, m)$  be a semi-metron, then it is a metronlike structure with all metron properties except  $M3$  and  $M4$ . We impose  $M3$  and  $M4$  properties including the other properties on the quotient set of  $X$  to make out the quotient set as a metron. This quotient set is a second generation quotient set on  $X$  found by the process given below.

From the set  $X$ , we form a quotient set by the use of ' $S$ ' the survival property relation. Again we make out a quotient set from the quotient set formed earlier by using ' $I$ ' the identification property relation.

At first we impose a binary relation  $S$  on  $X$  related with survival property. Define the relation  $S$  on  $X$  by,

$$xSy \Leftrightarrow m\langle x, a, y \rangle = 0, \quad \forall a \in X.$$

Then  $S$  is an equivalence relation on  $X$ . Suppose  $E = X/S = \{e_x : x \in X\}$ , where  $e_x = \{y \in X : xSy\}$ .

Define  $\bar{m}_\alpha : E \times E \times E \rightarrow \mathbb{R}$ , by,  $\bar{m}_\alpha\langle e_x, e_y, e_z \rangle = m\langle x, \alpha(e_y), z \rangle$  where,  $\alpha : E \rightarrow X$  such that  $\alpha(e_x) \in e_x$ . Then we have seen in the Theorem 5.1 that  $(E, \bar{m}_\alpha)$  is a sur-semi-metron for each  $\alpha \in \Lambda$  where

$\Lambda$  is the set of all such  $\alpha$  functions. At the second stage, we impose a binary relation  $I$  on  $E$  related with the identification property. For, define the relation  $I$  on  $E$  by,

$$e_x I e_y \Leftrightarrow \bar{m}_\alpha \langle e_u, e_x, e_v \rangle = \bar{m}_\alpha \langle e_u, e_y, e_v \rangle \quad \forall e_u, e_v \in E.$$

Then  $I$  is again an equivalence relation on  $E$ . Suppose  $\bar{E}$  is the set of all the equivalence classes generated by  $I$  on  $E$ . So,  $\bar{E} = \{\bar{E}_{e_x} : e_x \in E\}$ , where  $\bar{E}_{e_x} = \{e_y \in E : e_x I e_y\}$ . The set  $\bar{E}$  is the quotient set on  $E$  with respect to  $I$ . Now define,  $\bar{m}_{\alpha\beta} : \bar{E} \times \bar{E} \times \bar{E} \rightarrow \mathbb{R}$ , by  $\bar{m}_{\alpha\beta} \langle \bar{E}_{e_x}, \bar{E}_{e_y}, \bar{E}_{e_z} \rangle = \bar{m}_\alpha \langle \beta(\bar{E}_{e_x}), e_y, \beta(\bar{E}_{e_z}) \rangle$  where,  $\beta : \bar{E} \rightarrow E$  such that  $\beta(\bar{E}_{e_x}) \in \bar{E}_{e_x}$ . Then  $\{(E, \bar{m}_{\alpha\beta}) : \alpha \in \Lambda \text{ and } \beta \in \Gamma\}$  is a family of weak  $I$ -metrons, as we have seen in the Theorem 5.3. Here  $\Gamma$  is the family of all such  $\beta$  function.  $\square$

**Remark 5.8.** *In the above theorem if we impose the relation related with identification properly at first stage and the survival related relation at second stage, then we arrive at the same metronlike structure called weak  $I$ -metron.*

## 6. CONCLUSION

In this paper, we have generalized the structure of metron by weakening the restrictions of properties. This study gives a possible sketch of the theory of metronlike structures. Further studies could address in the setting of metron and metronlike structures generalizations of the notions of boundedness, of convergence and of other topological concepts : separation axioms, connectedness, compactness, continuity, homeomorphisms. The study of metronlike structures can shed a new light on the theory of metric spaces, even in the case of Euclidean spaces.

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