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ON THE SUM OF INVERSES OF SUBGROUP ORDERS IN FINITE GROUPS

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Abstract. Let G be a finite group and $f(G) = \sum_{H \leq G} \frac{1}{|H|}$. In this note, we study the finite groups G such that $f(G) \leq 2$.

1. Introduction

In the last years there has been a growing interest in studying some arithmetical functions associated to finite groups (see, e.g., [3]). Given a finite group G, we first recall the function

$$\sigma(G) = \frac{1}{|G|} \sum_{H \in L(G)} |H|,$$

where $L(G)$ denotes the subgroup lattice of G. It has been investigated in [5, 11, 12], and generalized in [5, 9] by replacing $L(G)$ with the poset $C(G)$ of cyclic subgroups of G or with the lattice $N(G)$ of normal subgroups of G. Clearly, the inequality $\sigma(\mathbb{Z}_n) \leq 2$ characterizes the set consisting both of deficient numbers and of perfect numbers (for more details on these numbers, see [6]). In [5], this set has been extended to the class of finite groups G satisfying $\sigma(G) \leq 2$.

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Another function that inspired us, the sum of inverses of element orders

$$m(G)=\sum_{a\in G}\frac{1}{o(a)}\,,$$

was introduced in [1] and used to give characterizations for commutativity, cyclicity and nilpotency of finite groups.

In the current note, we consider the function

$$f(G)=\sum_{H\in L(G)}\frac{1}{|H|}\,.$$

Note that f is multiplicative and strictly increasing on $L(G)$. We observe that $f(\mathbb{Z}_n)=\sigma(\mathbb{Z}_n)$ and therefore the inequality $f(\mathbb{Z}_n)\le 2$ also characterizes the set of deficient numbers and perfect numbers. This leads to the following natural question

Which are the finite groups G such that $f(G)\le 2$?

In the current note, we will answer the above question by proving that these groups are particular ZM-groups. We recall that a *ZM-group* is a finite group with all Sylow subgroups cyclic. By [4], such a group is of type

$$\mathrm{ZM}(m,n,r)=\langle a,b\mid a^m=b^n=1,\ b^{-1}ab=a^r\rangle,$$

where the triple (m,n,r) satisfies the conditions

$$(m,n)=(m,r-1)=1 \text{ and } r^n\equiv 1 \pmod{m}.$$

Our main result is stated as follows.

Theorem 1.1. *Let G be a finite group with $f(G)\le 2$. Then there exists a triple (m,n,r) such that $G\cong \mathrm{ZM}(m,n,r)$. Moreover, we have*

$$f(G)=\frac{1}{mn}\sum_{m_1|m,\, n_1|n} m_1n_1\left(m_1,\frac{r^n-1}{r^{n_1}-1}\right).$$

Since $|\mathrm{ZM}(m,n,r)|=mn$ and

$$\frac{\sigma(mn)}{mn}=\frac{1}{mn}\sum_{m_1|m,\, n_1|n} m_1n_1\le f(G),$$

we get the following corollary.

Corollary 1.2. *Let G be a finite group with $f(G)\le 2$. Then $|G|$ is either a deficient number or a perfect number.*

The formula in Theorem 1.1 can be simplified if n or m is prime.

Corollary 1.3. *If n is prime we have*

$$f(G) = \frac{\sigma(m)}{m} + \frac{1}{mn} \sum_{m_1 | m} m_1^2,$$

while if m is prime we have

$$f(G) = \frac{1}{mn} \left[(m^2+1)\sigma(n) - m(m-1)d\sigma\left(\frac{n}{d}\right) \right],$$

where d is the multiplicative order of r modulo m, i.e.

$$d = o_m(r) = \min\{k \in \mathbb{N}^* \mid r^k \equiv 1 \pmod{m}\}.$$

Let p and q be two distinct primes. From Corollary 1.3 we easily infer that there is no non-cyclic group G of order pq such that $f(G) \leq 2$. An example of a non-cyclic group G of order pq^2 such that $f(G) \leq 2$ is

$$G \cong \mathrm{ZM}(11, 25, 3) = \mathrm{SmallGroup}(275, 1).$$

Note that in this case we have

$$f(G) = \frac{482}{275} < 2.$$

For the proof of Theorem 1.1, we need the following well-known result (see e.g. (4.4) of [8], II).

Theorem A. *A finite p-group has a unique subgroup of order p if and only if it is either cyclic or a generalized quaternion 2-group.*

We recall that a *generalized quaternion 2-group* is a group of order 2^n for some positive integer $n \geq 3$, defined by

$$Q_{2^n} = \langle a, b \mid a^{2^{n-2}} = b^2, a^{2^{n-1}} = 1, b^{-1}ab = a^{-1} \rangle.$$

Finally, we formulate a natural open problem related to our results.

Open problem. Determine the finite groups G such that

$$f'(G) = \sum_{H \in C(G)} \frac{1}{|H|} \leq 2 \text{ or } f''(G) = \sum_{H \in N(G)} \frac{1}{|H|} \leq 2.$$

Most of our notation is standard and will not be repeated here. Basic definitions and results on groups can be found in [4, 8]. For subgroup lattice concepts we refer the reader to [7].

2. Proofs of the main results

We start with the following lemma.

Lemma 2.1. *Let G be a finite p-group such that $f(G) \leq 2$. Then G is cyclic.*

Proof. Let $|G| = p^n$ with $n \geq 2$. First of all, we observe that G has a unique subgroup of order p. Indeed, if this does not hold, then G has at least $p+1$ subgroups of order p and so

$$f(G) \geq 1 + \frac{p+1}{p} + \frac{1}{p^n} > 2,$$

a contradiction. Thus G is either cyclic or a generalized quaternion 2-group by Theorem A. Assume that $G \cong Q_{2^n}$, where $n \geq 3$. Then G has a subgroup isomorphic to Q_8, implying that

$$f(G) \geq f(Q_8) = \frac{19}{8} > 2,$$

a contradiction.

Consequently, G is cyclic, as desired. □

Lemma 2.1 can be easily extended to nilpotent groups.

Corollary 2.2. *Let G be a finite nilpotent group such that $f(G) \leq 2$. Then G is cyclic.*

Proof. Let $G = G_1 \times \cdots \times G_k$ be the decomposition of G as a direct product of its Sylow subgroups. For every $i = 1, ..., k$, we have

$$f(G_i) \leq \prod_{j=1}^{k} f(G_j) = f(G) \leq 2,$$

and so G_i is cyclic by Lemma 2.1.

Clearly, this implies that G is also cyclic, as desired. □

We are now able to prove our main result.

Proof of Theorem 1.1. Since $f(G) \leq 2$, it follows that $f(S) \leq 2$, for any Sylow subgroup S of G. Then all Sylow subgroups of G are cyclic by Lemma 2.1 and so G is a ZM-group, say $G \cong \mathrm{ZM}(m,n,r)$. The subgroups of $\mathrm{ZM}(m,n,r)$ have been completely described in [2] (see also [10]). Set

$$L=\left\{(m_1,n_1,s)\in\mathbb{N}^3 \,\middle|\, m_1|m,\ n_1|n,\ s<m_1,\ m_1|s\frac{r^n-1}{r^{n_1}-1}\right\}.$$

Then there is a bijection between L and the subgroup lattice of $\mathrm{ZM}(m,n,r)$, namely the function that maps a triple $(m_1,n_1,s)\in L$ into the subgroup $H_{(m_1,n_1,s)}$ defined by

$$H_{(m_1,n_1,s)}=\bigcup_{k=1}^{\frac{n}{n_1}}\alpha(n_1,s)^k\langle a^{m_1}\rangle=\langle a^{m_1},\alpha(n_1,s)\rangle,$$

where $\alpha(x,y)=b^xa^y$, for all $0\le x<n$ and $0\le y<m$. Note that

$$|H_{(m_1,n_1,s)}|=\frac{mn}{m_1n_1}\,,$$

for any s satisfying $(m_1,n_1,s)\in L$. Also, for fixed m_1 and n_1, there are $\left(m_1,\frac{r^n-1}{r^{n_1}-1}\right)$ such s. We get

$$f(G)=\frac{1}{mn}\sum_{(m_1,n_1,s)\in L}m_1n_1=\frac{1}{mn}\sum_{m_1|m,\,n_1|n}m_1n_1\left(m_1,\frac{r^n-1}{r^{n_1}-1}\right),$$

completing the proof. □

3. A conjecture on the function f

Inspired by certain calculations performed with the help of the formula in Theorem 1.1, we came up with the following conjecture.

Conjecture 3.1. *There is no constant $c>1$ such that $f(G)>c$, for all finite non-cyclic groups G.*

Note that, in number theory, a prime p is called a *Sophie Germain prime* if $2p+1$ is also prime. It has been conjectured that there are infinitely many Sophie Germain primes $(p_\alpha)_{\alpha\in\mathbb{N}}$, but this remains unproven. Assume that this is true and take $m_\alpha=2p_\alpha+1$, $\alpha\in\mathbb{N}$. Also, choose a positive integer r_α such that $o_{m_\alpha}(r_\alpha)=p_\alpha$ (such a positive integer exists by Cauchy's Theorem for the group $\mathbb{Z}_{m_\alpha}^\times$) and construct the groups

$$G_\alpha=\mathrm{ZM}(m_\alpha,p_\alpha^2,r_\alpha),\alpha\in\mathbb{N}.$$

Then the second formula in Corollary 1.3 leads to

$$f(G_\alpha)=1+\frac{7p_\alpha^2+6p_\alpha+2}{2p_\alpha^3+p_\alpha^2}\,,$$

which tends to 1 when α tends to ∞. Thus Conjecture 3.1 holds under the assumption that there are infinitely many Sophie Germain primes.

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