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APPLICATIONS OF ZORN'S LEMMA TO PRIMALS AND GRILLS

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Abstract. We investigate various connections between the notions of filter, ideal, grill and primal in topological spaces, then we characterize Hausdorff spaces, compact spaces and continuity of functions via limit points and cluster points of grills and primals. As a conclusion on our study on maximal primals, we provide a new proof of Tychonoff theorem.

1. Introduction and Preliminaries

The study of filters has two sides one is related to Zorn's Lemma [3] and other sides is convergent of Filters. However joint studies Zorn's Lemma as well as convergence (see [10, 11, 17]) in front of filter are also a remarkable part. The mathematical structures filter [8, 9, 10, 11, 12, 17], grill [12, 13, 14, 15, 16], ideal [5] and primal [1] are related to each other. Their related studied were established in [2, 7, 12, 14, 17].

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In this paper, we consider the further study of primals and grills in aspect of their convergence and as the application of Zorn's Lemma. Characterizations of Hausdorff space and Compact space through the convergence of primal, ultraprimal, grill and ultragrill are also an important phenomena. Throughout this paper, we denote 'iff' as if and only if.

Definition 1. *A collection $\mathbb{P}$ of subsets of a nonempty set Z is said to be a primal* [1] *on Z if it satisfies: (i) $Z \notin \mathbb{P}$, (ii) $I \in \mathbb{P}$ and $S \subseteq I$ implies $S \in \mathbb{P}$ and (iii) $I \cap S \in \mathbb{P}$ implies $I \in \mathbb{P}$ or $S \in \mathbb{P}$. Equivalently, a collection $\mathbb{P}$ of subsets of a nonempty set Z is said to be a primal on Z if it satisfies: (i) $Z \notin \mathbb{P}$, (ii) $S \notin \mathbb{P}$ and $S \subseteq I$ implies $I \notin \mathbb{P}$ and (iii) $I \notin \mathbb{P}$ and $S \notin \mathbb{P}$ implies $I \cap S \notin \mathbb{P}$.*

Definition 2. *A collection $\mathbb{I}$ of subsets of a nonempty set Z is called an ideal* [5] *on Z if $\mathbb{I}$ is closed under hereditary property and finite additivity property. If $Z \notin \mathbb{I}$, then $\mathbb{I}$ is called proper ideal. A proper ideal $\mathbb{I}$ is called an admissible ideal* [4] *if $\mathbb{I}$ contains every singleton.*

Definition 3. *A collection $\mathbb{G} \subseteq 2^Z$ on a set Z is said to be a grill on Z if (i) $\varnothing \notin \mathcal{G}$; (ii) $A \in \mathbb{G}$ and $A \subseteq B$ implies $B \in \mathbb{G}$; (iii) $A \cup B \in \mathbb{G}$ implies $A \in \mathbb{G}$ or $B \in \mathbb{G}$. In this manuscript, $\mathbb{G}$ denotes the grill.*

Definition 4. *A collection $\mathbb{F} \subseteq 2^Z$ on a set Z is said to be a filter* [6] *on Z if (i) $\varnothing \notin \mathcal{F}$; (ii) $A \in \mathbb{F}$ and $A \subseteq B$ implies $B \in \mathbb{F}$; (iii) $A \in \mathbb{F}$ and $B \in \mathbb{F}$ implies $A \cap B \in \mathbb{F}$. In this manuscript, $\mathbb{F}$ denotes the filter.*

Definition 5. *A collection $\mathbb{F}' \subseteq 2^Z$ on a set $Z \neq \varnothing$ is said to be a filter base (or base)* [6] *if the collection $\{F \in \mathcal{P}(Z) : F \text{contains a member of} \mathbb{F}'\}$ forms a filter on Z.*

Definition 6. *Let $\mathbb{F}$ be a filter on Z. Then the subfamily $\mathcal{S}$ of $\mathbb{F}$ is said to be a sub-base* [6] *for $\mathbb{F}$ if the family of all finite intersection of members of $\mathcal{S}$ is a base for $\mathbb{F}$.*

Definition 7. *Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two filters on Z with $\mathbb{F}_1 \subset \mathbb{F}_2$. Then $\mathbb{F}_2$ is called subfilter* [6] *if $\mathbb{F}_1$ converges to some point in Z implies $\mathbb{F}_2$ so.*

Definition 8. *Let Z be a set endowed with a topology l is called a topological space and it is written (Z, l) (or simply Z when there is no scope for misunderstanding). Further, if $\mathbb{P}$ is a primal on Z, then $(Z, l, \mathbb{P})$ is called a primal topological space (or simply **PTS**). Consequently $c(A)$ denotes the closure of the subset A of Z in the topological space (Z, l).*

Definition 9. *The collection $\mathbb{P}_Z$ of all primals on Z forms a partially ordered set with respect to $\subseteq$. Furthermore, every chain in $(\mathbb{P}_Z, \subseteq)$ has an upper bound. Thus, by Zorn's Lemma, $(\mathbb{P}_Z, \subseteq)$ has an maximal element. This maximal element is called ultraprimal.*

$\mathbb{F}$ stands for a filter; $\mathbb{U}$ stands for a ultraprimal; $\mathbb{F}_{\mathbb{U}}$ stands for a filter obtained from the primal $\mathbb{U}$; $\mathbb{P}_{\mathbb{F}}$ stands for a primal obtained from the filter $\mathbb{F}$. 2^Z denotes the set of all subsets of the set Z.

A collection $\circlearrowleft_Z$ of all grills on Z forms a partially ordered set with respect to $\subseteq$. Furthermore, every chain in $(\circlearrowleft_Z, \subseteq)$ has an upper bound. Thus, by Zorn's Lemma, $(\circlearrowleft_Z, \subseteq)$ has an maximal element. This maximal element is called ultragrill. In this manuscript $\mathbb{V}$ denotes the ultragrill.

Before entering the next section, we say that the σ-algebra of Borel sets [18] does not form a primal. Later we discuss why the σ-algebra does not form a primal.

2. Primals with Filters

In this section, we shall investigate more results of primals (resp. grills). Through this section, we also shall discuss about the convergence of primals (resp. grills) and its various properties. Continuity is also a part of this section.

Lemma 10. *Suppose Z is a non-empty set and $\mathcal{S} \subseteq 2^Z$ has the finite intersection property. Then for $\mathbb{F} := \{A \supseteq S_1 \cap S_2 \cap \cdots \cap S_l |\ S_1, S_2, \cdots, S_l \in \mathcal{S},\ l \text{ is arbitrary}\}$,*

(1) $\mathbb{P} := \{A \in 2^Z |\ A \notin \mathbb{F}\}$ *is a primal on Z.*
(2) $\mathbb{G} := \{A \in 2^Z |\ Z \setminus A \notin \mathbb{F}\}$ *is a grill on Z.*

Proof. 1. As $Z \in \mathbb{F}$, then $Z \notin \mathbb{P}$. Let $A \subseteq B \in \mathbb{P}$. Then $B \notin \mathbb{F}$. If possible suppose $S_1, S_2, \cdots, S_l \in \mathcal{S}$ such that $S_1 \cap S_2 \cap \cdots \cap S_l \subset A$ implies $S_1 \cap S_2 \cap \cdots \cap S_l \subset B$ and hence $B \in \mathbb{F}$, a contradiction. Thus $A \in \mathbb{P}$.
Let $A, B \notin \mathbb{P}$. Then, $A, B \in \mathbb{F}$. Thus, there exists $S_1, S_2, \cdots, S_l \in \mathcal{S}$ and $S_1', S_2', \cdots, S_l' \in \mathcal{S}$ such that $A \supseteq S_1 \cap S_2 \cap \cdots \cap S_l$ and $B \supseteq S_1' \cap S_2' \cap \cdots \cap S_l'$. This implies, $A \cap B \supseteq S_1 \cap S_2 \cap \cdots \cap S_l \cap S_1' \cap S_2' \cap \cdots \cap S_l'$ and hence $A \cap B \in \mathbb{F}$. Thus, $A \cap B \notin \mathbb{P}$. Contrapositively, $A \cap B \in \mathbb{P}$ implies either $A \in \mathbb{P}$ or $B \in \mathbb{P}$. ∎

Lemma 11. [7] *Let $\mathbb{P}$ be a primal and $\mathbb{G}$ be a grill on a set Z. Then*

(1) $\mathbb{F} := \{A \subseteq Z |\ A \notin \mathbb{P}\}$ *is a filter on Z.*
(2) $\mathbb{F}' := \{A \subseteq Z |\ Z \setminus A \notin \mathbb{G}\}$ *is a filter on Z.*

One may denote these by $\mathbb{F}_{\mathbb{P}}$ and $\mathbb{F}_{\mathbb{G}}$ called these by filter associated with the primal $\mathbb{P}$ and filter associated with the primal $\mathbb{P}$, respectively.

Lemma 12. [7] *Let $\mathbb{F}$ be filter on a set Z. Then*

(1) $\mathbb{P}_{\mathbb{F}} = \{A \subseteq Z |\ A \notin \mathbb{F}\}$ *is a primal on Z.*
(2) $\mathbb{G}_F = \{A \subseteq Z |\ Z \setminus A \notin \mathbb{F}\}$ *is a grill on Z.*

Note that, the duality between ideals and filters, as well as the duality between primal and grills could be made. Thus, the complements of the sets forming one family belonging to a one of the category form a family belonging to the dual category. A filter can be derived by an ideal on a set, analogous to Lemma 12 and conversely and the related research has been elaborately studied by Matejdes in [7]. However, in [7] the author used this converse but he did not mention that every proper ideal gives a filter.

Lemma 13. *Let $\mathbb{P}$ be a primal on a set Z. Then $D := \{t \times P |\ P \notin \mathbb{P}\ and\, t \in P\}$. For $t \times H,\ s \times L \in D$, we define*
$t \times H \geq s \times L$ iff $H \subseteq L$.
Then $(D, \geq)$ is a directed set.

Proof. Let $t \times H,\ s \times L \in D$ and $t \times H \geq s \times L \in D$. Then $H \subseteq L$, $H \cap L \notin \mathbb{P}$. Thus $H \cap L \neq \varnothing$. Pick $z \in H \cap L$. Then $z \times (H \cap L) \in D$ and $z \times (H \cap L) \geq t \times H$ and $z \times (H \cap L) \geq s \times L$. ∎

In this case, we always get a net $S : D \longrightarrow Z$ in Z by the rule $S(t \times H) = t$.

Corollary 14. *Let $\mathbb{G}$ be a grill on a set Z. Then $D := \{t \times G |\ Z \setminus G \notin \mathbb{G}\ and\, t \in (Z \setminus G)\}$. For $t \times H,\ s \times L \in D$, we define*
$t \times H \geq s \times L$ iff $H \subseteq L$.
Then $(D, \geq)$ is a directed set.

It is noteworthy that a net can be derived from an ideal on a set, and the related research has been explored by the authors Modak et al. in [17].

Theorem 15. *Let $S : D \longrightarrow Z$ be a net, and for every $m \in D$, let $B_m := \{S(n) |\ n \in D\ and\, n \geq m\}$. Then*

(1) $\mathbb{I}^S := \{A |\ A \subseteq Z \setminus B_m\ \ for\ \ \ some\, m \in D\}$ *is a proper ideal on Z* [17].
(2) $\mathbb{P}^S := \{A |\ Z \setminus A \notin \mathbb{I}^S\}$ *is a primal on Z.*
(3) $\mathbb{G}^S := \{A |\ A \notin \mathbb{I}^S\}$ *is a grill on Z.*

Example 16. *Let (Z, l) be a topological space and $z \in Z$. Then*

(1) $\mathbb{P}_{\mathcal{N}_z} := \{A \subseteq Z |\ A \notin \mathcal{N}_z\}$ *is a primal on* Z.
(2) $\mathbb{G}_{\mathcal{N}_z} := \{A \subseteq Z |\ Z \setminus A \notin \mathcal{N}_z\}$ *is a grill on* Z.
where $\mathcal{N}_z$ *denotes the collection of all neighbourhoods of* z.

Definition 17. *If* $\mathbb{P}_1$ *(resp.* $\mathbb{G}_1$*) and* $\mathbb{P}_2$ *(resp.* $\mathbb{G}_2$*) are primals (resp. grills) on a set* Z *such that* $\mathbb{P}_1 \subseteq \mathbb{P}_2$ *(resp.* $\mathbb{G}_1 \subseteq \mathbb{G}_2$*) , then we call* $\mathbb{P}_2$ *(resp.* $\mathbb{G}_2$ *) a refinement of* $\mathbb{P}_1$ *(resp.* $\mathbb{G}_1$ *). We say that* $\mathbb{P}_2$ *(resp.* $\mathbb{G}_2$ *) is finer than* $\mathbb{P}_1$ *(resp.* $\mathbb{G}_1$*).*

Definition 18. *Let* (Z, l) *be a topological space. A primal* $\mathbb{P}$ *(resp.* $\mathbb{G}$*) on* Z *converges to a point* $z \in Z$ *if* $\mathbb{P}_{\mathcal{N}_z}$ *(resp.* $\mathbb{G}_{\mathcal{N}_z}$*) is a refinement of* $\mathbb{P}$ *(resp.* $\mathbb{G}$*). We say that* z *is a limit point of* $\mathbb{P}$ *(resp.* $\mathbb{G}$*).*

In this connection, $A_{\mathbb{P}}$ (resp. $A_{\mathbb{G}}$) denotes the set of all limits of $\mathbb{P}$ (resp. $\mathbb{G}$) on the topological space (Z, l).

Definition 19. *Let* (Z, l) *be a topological space. A point* $z \in Z$ *is a cluster point of a primal* $\mathbb{P}$ *(resp. grill* $\mathbb{P}$*) if* $z \in \bigcap\{c(P) |\ P \notin \mathbb{P}\}$ *(resp.* $z \in \bigcap\{c(G) |\ Z \setminus G \notin \mathbb{G}\}$*).*

Evidently, every limit point of a primal (resp. grill) is a cluster point of that primal (resp. grill).

Lemma 20. *Let* (Z, l) *be a topological space. Then the primal* $\mathbb{P}$ *on* Z *converges to* $p \in Z$ *iff* $\mathbb{F}_{\mathbb{P}}$ *converges to* p.

Proof. Given that $\mathcal{N}_p \nsubseteq \mathbb{P}$. This implies that $\mathcal{N}_p \subseteq \mathbb{F}_{\mathbb{P}}$. ∎

Lemma 21. *Let* (Z, l) *be a topological space. Then the filter* $\mathbb{F}$ *on* Z *converges to* $p \in Z$ *iff* $\mathbb{P}_{\mathbb{F}}$ *converges to* p.

Lemma 22. *Let* (Z, l) *be a topological space. Then* $p \in Z$ *is a cluster point of a primal* $\mathbb{P}$ *iff* p *is a cluster point of the negation filter* $\mathbb{F}_{\mathbb{P}}$.

Lemma 23. *Let* (Z, l) *be a topological space. Then* $p \in Z$ *is a cluster point of a filter* $\mathbb{F}$ *iff* p *is a cluster point of the negation primal* $\mathbb{P}_{\mathbb{F}}$.

Lemma 24. *Let* (Z, l) *be a topological space and* $A \subseteq Z$. *Then* $z \in c(A)$ *iff one of the condition holds*

(1) *there exists a primal* $\mathbb{P}$ *on* Z *such that* $A \notin \mathbb{P}$ *and* $z \in A_{\mathbb{P}}$.
(2) *there exists a grill* $\mathbb{G}$ *on* Z *such that* $Z \setminus A \notin \mathbb{G}$ *and* $z \in A_{\mathbb{G}}$.
(3) $z \in c(A)$ *iff there exists a primal* $\mathbb{P}$ *on* Z *such that* $A \notin \mathbb{P}$ *and* $\mathbb{P} \xrightarrow{Z} p$.

Proof. 1. Given that $p \in c(A)$. Define $\mathbb{P} := \{B \subseteq Z |\ B \cap N = \varnothing,\ N \in \mathcal{N}_p\}$. Since $A \cap N \neq \varnothing$, $A \notin \mathbb{P}$. It is obvious that $\mathbb{P}$ is a primal on Z and $\mathcal{N}_x \nsubseteq \mathbb{P}$. Thus $\mathbb{P} \xrightarrow{Z} p$.

Conversely suppose a primal $\mathbb{P}$ on Z such that $A \notin \mathbb{P}$ and $\mathbb{P} \xrightarrow{\overline{Z}} p$. Then for $N \in \mathcal{N}_x$, $A \cap N \neq \varnothing$, otherwise $A \cap N \in \mathbb{P}$ implies $N \in \mathbb{P}$, a contradiction. ∎

Proposition 25 ([6]). *Let $\mathcal{S}$ be a family of subsets of a set Z. Then there exists a filter on Z having $\mathcal{S}$ as a sub-base iff $\mathcal{S}$ has the finite intersection property.*

Theorem 26. *A topological space (Z, l) is Hausdorff iff no primal on Z has more than one limit.*

Proof. Suppose that (Z, l) is Hausdorff. Let $\mathbb{P}$ be a primal that converges more than one limit say p and q where $p \neq q$. Since the primal $\mathbb{P}$ converges to p and q, then by Lemma 20, the filter $\mathbb{F}_{\mathbb{P}}$ converges to p and q. Then, $N_p, N_q \in \mathbb{F}_{\mathbb{P}}$ for each $N_p \in \mathcal{N}_p$ and $N_q \in \mathcal{N}_q$. This implies, $N_p \cap N_q \in \mathbb{F}_{\mathbb{P}}$ for each $N_p \in \mathcal{N}_p$ and $N_q \in \mathcal{N}_q$ and hence $N_p \cap N_q \neq \varnothing$ for each $N_p \in \mathcal{N}_p$ and $N_q \in \mathcal{N}_q$ which contradicts the fact that (Z, l) is Hausdorff. Thus, we must have $p = q$.
Conversely, assume that no primal on Z has more than one limit. If (Z, l) is not Hausdorff, then there exist $p, q \in Z$ with $p \neq q$ such that $N_p \cap N_q \neq \varnothing$ for each $N_p \in \mathcal{N}_p$ and $N_q \in \mathcal{N}_q$. From this, it follows that the family $\mathcal{N}_p \cup \mathcal{N}_q$ has the finite intersection property. So by Proposition 25, there exists a filter $\mathbb{F}$ on Z containing $\mathcal{N}_p \cup \mathcal{N}_q$. Evidently, the filter $\mathbb{F}$ converges both to p and q. Hence, by Lemma 21, the primal $\mathbb{P}_{\mathbb{F}}$ converges both to p and q which contradicts the given condition. So (Z, l) is Hausdorff. ∎

Corollary 27. *A topological space (Z, l) is Hausdorff iff no grill on Z has more than one limit.*

Theorem 28. *A topological space (Z, l) is Hausdorff iff a primal $\mathbb{P}$ on Z converges to z implies z is the only cluster point of $\mathbb{P}$.*

Proof. Let (Z, l) be Hausdorff and the primal $\mathbb{P}$ on Z converges to z. Then, z is a limit point of $\mathbb{P}$ and hence z is a cluster point of $\mathbb{P}$. Assume that z_1 is another cluster point of $\mathbb{P}$. Then, $z, z_1 \in \bigcap\{c(P) |\ P \notin \mathbb{P}\}$. This implies, $z, z_1 \in c(P)$ for all $P \notin \mathbb{P}$ and hence for each $N_z \in \mathcal{N}_z$ and $N_{z_1} \in \mathcal{N}_{z_1}$ such that $N_z \cap P \neq \varnothing$ and $N_{z_1} \cap P \neq \varnothing$ for all $P \notin \mathbb{P}$. Thus, $(N_z \cap N_{z_1}) \cap P = (N_z \cap P) \cap (N_{z_1} \cap P) \neq \varnothing$ for each $N_z \in \mathcal{N}_z$ and $N_{z_1} \in \mathcal{N}_{z_1}$. This implies, $N_z \cap Nz_1 \neq \varnothing$ for each $N_z \in \mathcal{N}_z$ and $N_{z_1} \in \mathcal{N}_{z_1}$ which contradicts the fact that (Z, l) is Hausdorff. Hence, z is the only cluster point of $\mathbb{P}$.
Conversely, assume that a primal $\mathbb{P}$ on Z converges to z implies z is

the only cluster point of $\mathbb{P}$. We have to show that (Z, l) is Hausdorff. If possible let (Z, l) is not Hausdorff. Then, by Theorem 26, there exists a primal on Z has more than one limit. Also, since every limit points are cluster point, then there exists a primal on Z has more than one cluster point which contradicts the given condition. So (Z, l) is Hausdorff. ∎

Theorem 29. *A topological space (Z, l) is Hausdorff iff a grill $\mathbb{G}$ on Z converges to z implies z is the only cluster point of $\mathbb{G}$.*

Theorem 30 ([6])**.** *For a topological space (Z, l), the following statements are equivalent:*
1. *Z is compact;*
2. *Every filter on Z has a cluster point in Z;*
3. *Every filter on Z has a convergent subfilter.*

Theorem 31. *For a topological space (Z, l), the following statements are equivalent:*
1. *Z is compact;*
2. *Every primal (resp. grill) on Z has a cluster point in Z;*
3. *Every primal (resp. grill) on Z has a convergent refinement primal (resp. grill).*

Proof. 1. $\Rightarrow$ 2.: Assume that Z is compact. We have to show that every primal on Z has a cluster point in Z. If possible, let there exists a primal $\mathbb{P}$ on Z which has no cluster point. Then by Lemma 22, the negation filter $\mathbb{F}_{\mathbb{P}}$ has no cluster point which contradicts the Theorem 30. Hence our assumption is wrong. Thus Every primal on Z has a cluster point in Z.
2. $\Rightarrow$ 3.: Assume that every primal on Z has a cluster point in Z. We have to show that every primal on Z has a convergent refinement primal. If possible let there exists a primal $\mathbb{P}$ on Z which has no convergent refinement primal. This implies that the negation filter $\mathbb{F}_{\mathbb{P}}$ has no convergent refinement negation filter. Thus, there exists a filter on Z which has no convergent subfilter which contradicts the Theorem 30. Hence our assumption is wrong. Thus, Every primal on Z has a convergent refinement primal.
3. $\Rightarrow$ 1.: Assume that every primal on Z has a convergent refinement primal. We have to show that Z is compact. If possible, let Z is not compact. Then by Theorem Theorem 30, there exists filter $\mathbb{F}$ on Z which has no convergent subfilter. Thus, the primal $\mathbb{P}_{\mathbb{F}}$ has no convergent refinement primal which contradicts our given condition.

Thus, Z is compact.
This completes the proof. ∎

Theorem 32. *Let $\hbar : Z \to Z'$ be a bijective mapping. Then,*

1. *For a primal $\mathbb{P}$ on Z, then $\hbar(\mathbb{P}) := \{\hbar(P)| \ P \in \mathbb{P}\}$ is a primal on Z'* [1].
2. *For a grill $\mathbb{G}$ on Z, $\hbar(\mathbb{G}) := \{\hbar(G)| \ G \in \mathbb{G}\}$ is also a grill on Z'.*

Proof. (i) $\varnothing = \hbar(\varnothing) \notin \hbar(G)$ as $\varnothing \notin G$ and $\varnothing \subset Z$.
(ii) Let $P, Q \subset Z'$ with $P \in \hbar(G)$ and $P \subset Q$. Then, there exists $A \subset Z$ with $A \in G$ such that $\hbar(A) = P$. This implies $\hbar(A) \subset Q$ and hence $\{\hbar(x)| \ x \in A\} \subset Q$. Let $S = \{x \in X| \ \hbar(x) \in Q\}$. This implies $A \subset S$. Then, $\hbar(S) = \{\hbar(s)|s \in S\} = Q$ and $A \subset S$. This implies, $S \in G$ since G is a grill and hence $\hbar(S) \in \hbar(G)$. Thus, $Q \in \hbar(G)$.
(iii) Let $P, Q \notin \hbar(G)$. Then, there does not exist $A, B \subset Z$ with $A, B \in G$ such that $\hbar(A) = P$ and $\hbar(B) = Q$. Thus, $A, B \notin G$ implies $A \cup B \notin G$. So, $\hbar(A \cup B) \notin \hbar(G)$ and hence $\hbar(A) \cup \hbar(B) \notin \hbar(G)$ implies $P \cup Q \notin \hbar(G)$. Contrapositively, $P \cup Q \in \hbar(G)$ implies either $P \in \hbar(G)$ or $Q \in \hbar(G)$.
Hence $\hbar(\mathbb{G}) := \{\hbar(G)| \ G \in \mathbb{G}\}$ is a grill on Z'. This completes the proof. ∎

Theorem 33. *Let (Z_1, l_1), (Z_2, l_2) be two topological spaces, $z \in Z_1$, and $\hbar : Z_1 \to Z_2$ be a function. Then, following statements are hold.*

1. *$\hbar$ is continuous at z iff whenever a primal $\mathbb{P}$ converges to z, the image primal $\hbar(\mathbb{P})$ converges to $\hbar(z)$.*
2. *$\hbar$ is continuous at z iff whenever a grill $\mathbb{G}$ converges to z, the image grill $\hbar(\mathbb{G})$ converges to $\hbar(z)$.*

Proof. Suppose $\hbar$ is continuous at z and $\mathbb{P}$ converges to z. Let N be any neighbourhood of $\hbar(z)$ in Z_2. By continuity of $\mathbb{P}$, $\hbar^{-1}(N)$ is a neighbourhood of z in Z_2. By the condition, $\hbar^{-1}(N) \notin \mathbb{P}$. This implies that $N \notin \hbar(\mathbb{P})$. So, $\hbar(\mathbb{P})$ converges to $\hbar(z)$.

Conversely suppose that the condition holds. If possible suppose that $\hbar$ is not continuous at z. Then there exists a neighbourhood N of $\hbar(z)$ such that $\hbar^{-1}(N)$ is not a neighbourhood of z in Z_1. This means that every neighbourhood of z in Z_1 intersects the complement $Z_1 \setminus \hbar^{-1}(N)$ (if $M \cap (Z_1 \setminus \hbar^{-1}(N)) = \varnothing$, then $M \subseteq \hbar^{-1}(N)$ a contradiction).

Now $\mathcal{S} := \mathcal{N}_z \cup \{X \setminus \hbar^{-1}(N)\}$ has the finite intersection property. Then by Lemma 10 $\mathbb{P} := \{A \in 2^Z| \ A \not\supseteq S_1 \cap S_2 \cap \cdots \cap S_l, \ S_1, S_2, \cdots, S_l \in \mathcal{S}, \ l \text{ is arbitrary}\}$ is a primal on Z_1. By the construction of the primal $\mathbb{P}$, $\mathcal{N}_z \nsubseteq \mathbb{P}$, thus $\mathbb{P}$ converges to z. But $\hbar(\mathbb{P})$ does not converge to $\hbar(z)$. Indeed since $Z_1 \setminus \hbar^{-1}(N) \notin \hbar(\mathbb{P})$ implies $\hbar(Z_1 \setminus \hbar^{-1}(N)) \notin \hbar(\mathbb{P})$. But $Z_2 \setminus N$ contains $\hbar(Z_1 \setminus \hbar^{-1}(N))$ and hence $Z_2 \setminus N \notin \hbar(\mathbb{P})$. Thus $N \in \hbar(\mathbb{P})$, contradiction as $\mathbb{P}$ converges to z. ∎

Theorem 34. *Let Z be the topological product of an indexed family of topological spaces $\{Z_i |\ i \in I\}$. Let $\mathbb{P}$ be a primal on Z and $z \in Z$. Then $\mathbb{P}$ converges to z in Z iff for each $i \in I$, the primal $\pi_i(\mathbb{P})$ converges to $\pi_i(z)$ in Z_i.*

Proof. The necessity of the condition follows from Theorem 33.

Sufficiency: Let $N \in \mathcal{N}_z$. Then N contains a basic open set V containing z.

Let $V = \Pi_i(V_i)$

where each V_i is an open set in Z_i and $V_i = Z_i$ for all $i \in I$ except for $i = i_1, i_2, \cdots, i_n$ (say). Given that $\pi_{i_k}(\mathbb{P})$ converges to $\pi_{i_k}(z)$ for all $k = 1, 2, \cdots, n$. So $V_{i_k} \notin \pi_{i_k}(\mathbb{P})$ and hence there exists $P_k \notin \mathbb{P}$ such that $V_{i_k} \supseteq \pi_{i_k}(P_k)$ implies $\pi_{i_k}^{-1}(V_{i_k}) \supseteq P_k$ for $k = 1, 2, \cdots, n$. So $N \supseteq V = \bigcap\limits_{k=1}^{n} \pi_{i_k}^{-1}(V_{i_k}) \supseteq \bigcap\limits_{k=1}^{n} P_k$, and $\bigcap\limits_{k=1}^{n} P_k \notin \mathbb{P}$ implies $N \notin \mathbb{P}$. Thus $\mathbb{P}$ converges to z. ∎

Corollary 35. *Let Z be the topological product of an indexed family of topological spaces $Z_i |\ i \in I\}$. Let $\mathbb{G}$ be a grill on Z and $z \in Z$. Then $\mathbb{G}$ converges to z in Z iff for each $i \in I$, the grill $\pi_i(\mathbb{G})$ converges to $\pi_i(z)$ in Z_i.*

3. Zorn's Lemma followed by primals and grills

In this section, we shall discuss about Zorn's Lemma and its application to primals and grills.

Lemma 36. *Let Z be a nonempty set. Then the collection $\mathbb{P}_Z$ (resp. $\circlearrowleft_Z$) of all primals (resp. grills) on Z forms a partially ordered set with respect to $\subseteq$ (set inclusion).*

Proof. The proof is straightforward and hence omitted. ∎

Theorem 37. *Considering partially ordered set $(\mathbb{P}_Z, \subseteq)$ (resp. $(\circlearrowleft_Z, \subseteq)$) of the Lemma 36. and let $\{\mathbb{P}_i |\ i \in I\}$ (resp. $\{\mathbb{G}_i |\ i \in I\}$) be a chain in $(\mathbb{P}_Z, \subseteq)$ (resp. $(\circlearrowleft_Z, \subseteq)$). Then $\mathbb{P} = \bigcup\limits_{i \in I} \mathbb{P}_i$ (resp. $\mathbb{G} = \bigcup\limits_{i \in I} \mathbb{G}_i$) is an upper bound of the chain $\{\mathbb{P}_i |\ i \in I\}$ (resp. $\{\mathbb{G}_i |\ i \in I\}$).*

Proof. Since $Z \notin \mathbb{P}_i$ for all $i \in I$, it follows that $Z \notin \mathbb{P}$. Let $S \in \mathbb{P}$ and $T \subseteq S$. Then there exists $k \in I$ such that $S \in \mathbb{P}_k$, and hence

$T \in \mathbb{P}_k$. This implies $T \in \mathbb{P}$. Now let $S, T \in 2^Z$ such that $S \notin \mathbb{P}$ and $T \notin \mathbb{P}$. Then $S, T \notin \bigcup_{i \in I} \mathbb{P}_i$. Thus $S, T \notin \mathbb{P}_i$ for all $i \in I$ implies $S \cap T \notin \mathbb{P}_i$ for all $i \in I$, and hence $S \cap T \notin \mathbb{P}$. Contrapositively, $S \cap T \in \mathbb{P}$ implies either $S \in \mathbb{P}$ or $T \in \mathbb{P}$. Hence, $\mathbb{P}$ is a primal on Z and as a result $\mathbb{P} \in \mathbb{P}_Z$. Thus, by construction $\mathbb{P}$ is an upper bound of the chain $\{\mathbb{P}_i | \, i \in I\}$. ∎

In view of the Zorn's Lemma, we conclude that $(\mathbb{P}_Z, \subseteq)$ (resp. $(\circlearrowleft_Z, \subseteq)$) has a maximal element, and we call it maximal primal or ultraprimal (respectively, maximal grill or ultragrill).

Theorem 38. *Let Z be a nonempty set. Then, every primal on Z is contained in an ultraprimal on Z.*

Proof. Let P be a primal on a set Z. Let $\mathbb{P}_Z$ be the collection of all primals on Z containing P. Then, $P \in \mathbb{P}_Z$ and hence $\mathbb{P}_Z$ is nonempty. Also, the collection $\mathbb{P}_Z$ with respect to the set inclusion ($\subseteq$) forms a partially ordered set. Let $\{P_i | i \in I\}$ be a non- empty chain in $\mathbb{P}_Z$ and let $\mathbb{P} = \bigcup_{i \in I} P_i$. Then, $\mathbb{P}$ is a primal by the Theorem 37. Obviously, $P \in \mathbb{P}$ as $P \in P_i$ for each $i \in I$. So, $\mathbb{P} \in \mathbb{P}_Z$ and by its construction, it is an upper bound for the chain $\{P_i | i \in I\}$. So by Zorn's Lemma $\mathbb{P}_Z$ contains a maximal element (i.e., ultraprimal) say $\mathbb{P}_1$. Hence, $\mathbb{P}_1$ is an ultraprimal containing P. ∎

Theorem 39. *Let Z be a nonempty set. Then, every grill on Z is contained in a ultragrill on Z.*

Lemma 40. *Let Z be a nonempty set. Then, for any primal $\mathbb{P} \in \mathbb{P}_Z$, the following arguments are equivalent:*

(1) *$\mathbb{P}$ is a maximal primal;*
(2) *for any $A \subseteq Z$, either $A \notin \mathbb{P}$ or $Z \setminus A \notin \mathbb{P}$;*
(3) *for any $A, B \subseteq Z$, $A \cup B \notin \mathbb{P}$ iff either $A \notin \mathbb{P}$ or $B \notin \mathbb{P}$.*

Proof. $1 \Longrightarrow 2$: Let $A \notin \mathbb{P}$. Then by the above lemma, $\mathbb{U} := \{M \in 2^Z | \; M \notin \mathbb{P}\}$ is a ultrafilter, and hence $A \in \mathbb{U}$. Thus $Z \setminus A \notin \mathbb{U}$, and hence $Z \setminus A \in \mathbb{P}$.

$2 \Longrightarrow 1$: If possible suppose that $\mathbb{P}$ is not a ultraprimal in $(\mathbb{P}_Z, \subseteq)$. Then $\mathbb{F}_{\mathbb{P}}$ is not a ultrafilter. Then there exists a filter $\mathbb{F}$ on Z such that $\mathbb{F}_{\mathbb{P}}$ properly contained in $\mathbb{F}$. Then there exists $A \in \mathbb{F} \setminus \mathbb{F}_{\mathbb{P}}$ implis $A \notin \mathbb{F}_{\mathbb{P}}$ implies $A \in \mathbb{P}$. Then by the given condition $Z \setminus A \notin \mathbb{P}$ implies $Z \setminus A \in \mathbb{F}_{\mathbb{P}}$. As $\mathbb{F}_{\mathbb{P}} \subseteq \mathbb{F}$, then $Z \setminus A \in \mathbb{F}$. As A and $Z \setminus A$ both are the members of $\mathbb{F}$, then $A \cap (Z \setminus A) \in \mathbb{F}$, a contradiction. Therefore $\mathbb{P}$ is a ultraprimal in $(\mathbb{P}_Z, \subseteq)$.

$2 \implies 3$: Firstly, suppose $A, B \in 2^Z$ and $A \cup B \notin \mathbb{P}$ but both $A \in \mathbb{P}$ and $B \in \mathbb{P}$. Then by assumption, $Z \setminus A \notin \mathbb{P}$ and $Z \setminus B \notin \mathbb{P}$ implies $(Z \setminus A) \cap (Z \setminus B) \notin \mathbb{P}$ and hence $Z \setminus (A \cup B) \notin \mathbb{P}$. This implies that $Z \setminus (A \cup B) \in \mathbb{F}_{\mathbb{P}}$. Again $A \cup B \notin \mathbb{P}$ implies $A \cup B \in \mathbb{F}_{\mathbb{P}}$. Thus,$(A \cup B)$ and $Z \setminus (A \cup B)$ both are the members of $\mathbb{F}_{\mathbb{P}}$ and hence $(A \cup B) \cap (Z \setminus (A \cup B)) \in \mathbb{F}_{\mathbb{P}}$, a contradiction. Hence either $A \notin \mathbb{P}$ or $B \notin \mathbb{P}$.
For converse, $A \subset A \cup B$ as well as $B \subset A \cup B$ and either $A \notin \mathbb{P}$ or $B \notin \mathbb{P}$. Then from of definition of primal, we have $A \cup B \notin \mathbb{P}$.
$3 \implies 2$: Since for any $A \subset Z$, $A \cup (Z \setminus A) = Z \notin \mathbb{P}$, by assumption we have, either $A \notin \mathbb{P}$ or $Z \setminus A \notin \mathbb{P}$. ∎

Condition number (2.) of the Lemma 40 tells us that Algebra does not form a primal and hence does not form a grill as well as a filter also.

Corollary 41. *Let Z be a nonempty set. Then for $\mathbb{G} \in \circlearrowleft_Z$, the following arguments are equivalent:*

(1) *$\mathbb{G}$ is a maximal grill;*
(2) *for any $A \subseteq Z$, either $A \notin \mathbb{G}$ or $Z \setminus A \notin \mathbb{G}$*
(3) *for any $A, B \subseteq Z$, $A \cap B \notin \mathbb{G}$ iff either $A \notin \mathbb{G}$ or $B \notin \mathbb{G}$.*

Proof. $1 \implies 2$: Let $\mathbb{U}_{\mathbb{G}} = \{A \in 2^Z | \ Z \setminus A \notin G\}$ be the ultrafilter associated with the ultragrill $\mathbb{G}$. Then for $A \in 2^Z$, either $A \in \mathbb{U}_{\mathbb{G}}$ or $Z \setminus A \in \mathbb{U}_{\mathbb{G}}$. That is either $Z \setminus A \notin \mathbb{G}$ or $Z \setminus (Z \setminus A) \notin \mathbb{G}$ implies either $A \notin \mathbb{G}$ or $Z \setminus A \notin \mathbb{G}$.
$2 \implies 1$: If possible suppose that $\mathbb{G}$ is not a ultragrill in $(\circlearrowleft_Z, \subseteq)$. Then $\mathbb{F}_{\mathbb{G}}$ is not a ultrafilter. Then there exists a filter $\mathbb{F}$ on Z such that $\mathbb{F}_{\mathbb{G}}$ properly contained in $\mathbb{F}$. Then there exists $A \in \mathbb{F} \setminus \mathbb{F}_{\mathbb{G}}$. This implies, $A \notin \mathbb{F}_{\mathbb{G}}$ and hence $Z \setminus A \in \mathbb{G}$. Then by the given condition $A \notin \mathbb{G}$ implies $Z \setminus A \in \mathbb{F}_{\mathbb{G}}$. As $\mathbb{F}_{\mathbb{G}} \subseteq \mathbb{F}$, then $Z \setminus A \in \mathbb{F}$. As A and $Z \setminus A$ both are the members of $\mathbb{F}$, then $A \cap (Z \setminus A) \in \mathbb{F}$ implies $\varnothing \in \mathbb{F}$, a contradiction. Therefore $\mathbb{G}$ is a ultra grill in $(\circlearrowleft_Z, \subseteq)$.
$2 \implies 3$: Firstly, suppose $A, B \in 2^Z$ and $A \cap B \notin \mathbb{G}$ but both $A \in \mathbb{P}$ and $B \in \mathbb{P}$. Then by assumption, $Z \setminus A \notin \mathbb{G}$ and $Z \setminus B \notin \mathbb{G}$ implies $A \in \mathbb{F}_{\mathbb{G}}$ and $A \in \mathbb{F}_{\mathbb{G}}$. This implies that $Z \setminus (A \cap B) \in \mathbb{F}_{\mathbb{G}}$. Again $A \cap B \notin \mathbb{G}$ implies $Z \setminus (A \cap B) \in \mathbb{F}_{\mathbb{G}}$. Thus,$(A \cap B)$ and $Z \setminus (A \cap B)$ both are the members of $\mathbb{F}_{\mathbb{G}}$ and hence $(A \cap B) \cap (Z \setminus (A \cap B)) \in \mathbb{F}_{\mathbb{G}}$ implies $\varnothing \in \mathbb{F}_{\mathbb{G}}$, a contradiction. Hence either $A \notin \mathbb{G}$ or $B \notin \mathbb{G}$.
For converse, $A \cap B \subset A$ as well as $A \cap B \subset B$ and either $A \notin \mathbb{G}$ or $B \notin \mathbb{G}$. Then from of definition of grill, we have $A \cap B \notin \mathbb{G}$.

$3 \Longrightarrow 2$: Since for any $A \subset Z$, $A \cap (Z \setminus A) = \varnothing \notin \mathbb{G}$, by assumption we have, either $A \notin \mathbb{G}$ or $Z \setminus A \notin \mathbb{G}$. ∎

Theorem 42. *Let $\mathbb{U}$ be a primal on a nonempty set Z. Then $\mathbb{U}$ is an ultraprimal iff $A \cap F \neq \varnothing$, for all $F \notin \mathbb{U}$, implies $A \in \mathbb{U}$.*

Corollary 43. *Let Z be a nonempty set. Then*

(1) $\Theta_Z := \{\mathbb{F}_{\mathbb{P}} | \ \mathbb{P} \in \mathbb{P}_Z\}$ *is the collection of all filters on Z*
(2) $(\Theta_Z, \subseteq)$ *is a partially ordered set.*
(3) *the set of maximal filter (or ultrafilter) of $(\Theta_Z, \subseteq)$ coincides with the maximal primal (or ultraprimal) of $(\mathbb{P}_Z, \subseteq)$*

Proof. Let $F \notin \mathbb{U}$ such that $A \cap F \neq \varnothing$. As $F \subseteq A \cup F$, then $A \cup F \notin \mathbb{U}$. Now by Theorem 40 (3), $A \in \mathbb{U}$.

By the given condition it is not possible that $Z \setminus A \in \mathbb{U}$. If $Z \setminus A \in \mathbb{U}$ and from $A \in \mathbb{U}$, $(Z \setminus A \in \mathbb{U}) \cap A = \varnothing \in \mathbb{U}$ this implies (from the Definition of primal) either $A \in \mathbb{U}$ or $Z \setminus A \in \mathbb{U}$, a contradiction. Hence $Z \setminus A \notin \mathbb{U}$ and $\mathbb{U}$ is an ultraprimal. ∎

Corollary 44. *Let Z be a nonempty set. Then*

(1) $\Theta_Z := \{\mathbb{F}_{\mathbb{G}} | \ \mathbb{G} \in \circlearrowright_Z\}$ *is the collection of all filter on Z*
(2) $(\Theta_Z, \subseteq)$ *is a partially ordered set.*
(3) *the set of maximal filter (or ultrafilter) of $(\circlearrowright_Z, \subseteq)$ coincides with the maximal grill (or ultragrill) of $(\circlearrowright_Z, \subseteq)$*

Theorem 45. *Let (Z, l) be a topological space. An ultraprimal $\mathbb{U}$ on Z converges to a point $z \in Z$ iff z is a cluster point of $\mathbb{U}$.*

Proof. Since every limit point is a cluster point of a primal, then direct implication is true for any primal.
For the converse, suppose that (Z, l) is a topological space and $z \in Z$ is a cluster point of an ultraprimal $\mathbb{U}$ on Z. If $\mathbb{U}$ does not converges to z, then the negation filter $\mathbb{F}_{\mathbb{U}}$ does not converges to z, then there a neighbourhood N of Z such that $N \notin \mathbb{F}_{\mathbb{U}}$. This implies $Z \setminus N \in \mathbb{F}_{\mathbb{U}}$. Since z is cluster point of $\mathbb{U}$, then z is cluster point of $\mathbb{F}_{\mathbb{U}}$, thus every neighbourhhod of z intersects every member of $\mathbb{F}_{\mathbb{U}}$ whereas $N \cap (Z \setminus N) = \varnothing$, a contradiction. Hence $\mathbb{U}$ converges to z. ∎

Theorem 46. *Let (Z, l) be a topological space. An ultragrill $\mathbb{V}$ on Z converges to a point $z \in Z$ iff z is a cluster point of $\mathbb{V}$.*

Theorem 47. *Suppose that (Z, l) is a topological space and $\mathbb{U}$ an ultraprimal on Z. Then each of the following holds:*

(1) *If C is a closed subset of Z and $C \notin \mathbb{U}$, then $A_{\mathbb{U}} \subseteq C$.*

(2) *If* $z \in A_{\mathbb{U}}$, *then* $c(\{z\}) \subseteq A_{\mathbb{U}}$.
(3) $A_{\mathbb{U}}$ *is closed.*

Proof. 1. Let $z \in A_{\mathbb{U}}$, then $N_z \cap C \neq \varnothing$ for every $N_z \in \mathcal{N}_z$. Then $z \in c(C) = C$.

2. Let $z' \in c(\{z\})$. Then $z \in N_{z'}$ for each $N_{z'} \in \mathcal{N}_{z'}$. As $z \in A_{\mathbb{U}}$, $N_{z'} \notin \mathbb{U}$ for each $N_{z'} \in \mathcal{N}_{z'}$ implies $z' \in A_{\mathbb{U}}$.

3. Let $z \in c(A_{\mathbb{U}})$. Then $A_{\mathbb{U}} \cap N_z \neq \varnothing$, for each $N_z \in \mathcal{N}_z$. It follows that $c(U) \cap N_z \neq \varnothing$, for any $U \notin \mathbb{U}$ and for all N_z. Observe that $c(U) \notin \mathbb{U}$, for each $U \notin \mathbb{U}$. Since $\mathbb{U}$ is an ultraprimal, each N_z does not belong to $\mathbb{U}$. This belongs $A_{\mathbb{U}}$. ∎

Corollary 48. *Suppose that* (Z, l) *is a topological space and* $\mathbb{V}$ *be an ultragrill on* Z. *Then each of the following holds:*

(1) *If* C *is a closed subset of* Z *and* $C \notin \mathbb{V}$, *then* $A_{\mathbb{V}} \subseteq C$.
(2) *If* $z \in A_{\mathbb{V}}$, *then* $c(\{z\}) \subseteq A_{\mathbb{V}}$.
(3) $A_{\mathbb{V}}$ *is closed.*

Theorem 49. *A topological space* (Z, l) *is compact iff every ultraprimal (resp. ultragrill) on* Z *is convergent.*

We give an alternative proof of the Tychonoff theorem.

Theorem 50. *Let* $\{Z_i \mid i \in I\}$ *be a collection of nonempty topological spaces and let* Z *be its topological product. Then* Z *is compact iff each* Z_i *is so for* $i \in I$.

Proof. Sufficiency: Let $\mathbb{P}$ be an ultraprimal on Z. For each $i \in I$, let $\mathbb{P}_i = \pi_i(P)$. Then $\mathbb{P}_i$ is a primal on Z_i by Theorem 32,

Claim: $\pi_i(P)$ is an ultraprimal:

Let $A \subseteq Z_i$. Put $B = \pi_i^{-1}(A)$. Note that $Z \setminus B = \pi_i^{-1}(Z_i \setminus A)$. Since $\mathbb{P}$ is an ultraprimal, either $A \notin \mathbb{P}$ or $Z \setminus A \notin \mathbb{P}$. In first case, $A = \pi_i(B) \notin \pi_i(P)$ while in the other case we get similarly that $Z \setminus A \notin \mathbb{P}_i$. Hence $\mathbb{P}_i$ is an ultraprimal on Z_i. By compactness of Z_i, $\mathbb{P}_i$ converges to z_i (say). By Theorem 34, $\mathbb{P}$ converges to z where $z \in Z$ is defined by $z(i) = z_i$ for $i \in I$. Thus every ultraprimal on Z is convergent and so Z is compact (by Theorem 49). ∎

Convergence of the ultragrill may also be a tool for proving of the Tychonoff Theorem.

Theorem 51. *Assume* (Z, l) *be a topological space, then for an ultraprimal (resp. ultragrill)* $\mathbb{U}$ *(resp.* $\mathbb{V}$*) on* Z. *Then followings hold:*

(1) (Z, l) *is Hausdorff iff* $A_{\mathbb{U}}$ *(resp.* $A_{\mathbb{V}}$ *) has at most one point.*

(2) (Z, l) *is compact iff* $A_{\mathbb{U}}$ *(resp.* $A_{\mathbb{V}}$*) has at least one element.*

Corollary 52. *A topological space* (Z, l) *is compact Hausdorff iff* $A_{\mathbb{U}}$ *(resp.* $A_{\mathbb{V}}$*) has exactly one element.*

Proposition 53. *Let* $\hbar : Z \longrightarrow Z'$ *be a bijective function, and* $\mathbb{U}$ *(resp.* $A_{\mathbb{V}}$*) an ultraprimal (resp. ultragrill) on* Z*. Then image of* $\mathbb{U}$ *(resp.* $A_{\mathbb{V}}$*) under* f *is an ultraprimal (resp. ultragrill) on* $\mathbb{Z}'$*.*

Proof. Let $\mathbb{U}$ be an ultraprimal on Z. Then $\mathbb{F}_{\mathbb{U}}$ is an ultrafilter on Z.

If $\mathbb{F}$ is a filter on Z' such that $\hbar(\mathbb{F}_{\mathbb{U}}) \subseteq \mathbb{F}$, but $\hbar(\mathbb{F}_{\mathbb{U}}) \neq \mathbb{F}$, then there is $A \in \mathbb{F}$ such that $A \notin \hbar(\mathbb{F}_{\mathbb{U}})$. Therefore $\hbar^{-1}(A) \notin \mathbb{F}_{\mathbb{U}}$. By the characterization of ultrafilter [6], $Z \setminus \hbar^{-1}(A) \in \mathbb{F}_{\mathbb{U}}$. From $\hbar(\mathbb{F}_{\mathbb{U}}) \subseteq \mathbb{F}$, $\hbar^{-1}(\hbar(\mathbb{F}_{\mathbb{U}})) \subseteq \hbar^{-1}(\mathbb{F}_{\mathbb{U}})$ implies $Z \setminus \hbar^{-1}(A) \in \hbar^{-1}(\mathbb{F})$. Therefore $\hbar(Z \setminus \hbar^{-1}(A)) \in \hbar(\hbar^{-1}(\mathbb{F})) \subseteq \mathbb{F}$ implies $\hbar(Z \setminus \hbar^{-1}(A)) \in \mathbb{F}$. Again $\hbar(Z \setminus \hbar^{-1}(A)) \subseteq Z' \setminus A$, so $Z' \setminus A$, so $Z' \setminus A \in \mathbb{F}$ it is not possible, otherwise $\varnothing \in \mathbb{F}$. Thus $\hbar(\mathbb{F}_{\mathbb{U}})$ is an ultrafilter and hence $\hbar(\mathbb{U})$ is an ultraprimal. ∎

Conclusion

The article highlights the significance of Zorn's Lemma in relation to grills, primals and ideals. The Tychonoff product theorem can be demonstrated using Zorn's Lemma through the concepts of grills, primals, and ideals. The exploration of Zorn's Lemma is relevant for the research on the axiom of choice.

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