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ON FILTERS AND POSITIVE IMPLICATIVE FILTERS IN PSEUDO-BI-ALGEBRAS

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Abstract The concept of (left distributive) pseudo-BI-algebra was introduced in 2023 by A. Radfar and A. Rezaei. In this paper, in addition to proving several, previously unregistered, properties of filters in (left distributive) pseudo-BI-algebras, we introduce and discuss the concept of positive implicative filters in these algebras.

1. Introduction

G. Georgescu and A. Iorgulescu ([5]) introduced the notion of a pseudo-BCK algebra as an extended notion of BCK-algebras. In [3], W. A. Dudek and Y. B. Jun introduced the notion of pseudo-BCI algebras as an extension of BCI-algebras, and investigated some properties. Y. B. Jun, H. S. Kim and J. Neggers in 2006 in [7] dealt with the concepts of pseudo-BCI ideals in pseudo BCI-algebras and, independently of them, in 2012, in [4], Dymek investigated the properties of ideals in that class of pseudo-algebras. Ideals in pseudo-BCI-algebras were also discussed K. J. Lee and C. H. Park ([8]). Also, pseudo-BCI-algerbas were the focus of the paper [2] written by I. Chajda and H. Länger.

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The notion of pseudo-BCH-algebras is introduced, and some of them properties are investigated in [12] by A. Walendziak. This author also studied ideals and congruences in this class of pseudo-algebras (see, [13, 14]).

BI-algebra was introduced in 2017 by A. Borumand Saeid, H. S. Kim and A. Rezaei ([1]) as a generalization of both a (dual) implication algebra and an implicative BCK-algebra. Right distributive BI-algebras were also the focus of this author in [11]. A generalization of this class of algebras, pseudo BI-algebras, is introduced and discussed in [9] by A. Radfar and A. Rezaei. The concept of filters in this class of logic algebras was introduced and discussed in [10] by A. Rezaei and D. A. Romano. In that paper, the connection between sub-algebras and filters was discussed as well as the connection between filters and left congruences in (left distributive) pseudo-BI-algebras.

This article is a continuation of articles [9, 10] in the literal sense. In this article, first we prove several new identifications of filters in (left distributive) pseudo-BI-algebras (Subsection 3.2), and then we introduce and analyze the concept of positive implicative filters in such algebras (Subsection 3.3). At the end of this paper, it was shown that every filter in a left distributive pseudo-BI-algebra is a positive implicative filter in that algebra.

2. Preliminaries

It should be emphasized here that the formulas in this text are written in a standard way, as is usual in mathematical logic, with the standard use of labels for logical functions. Thus, the labels $\wedge$, $\vee$, $\Longrightarrow$, and so on, are labels for the logical functions of conjunction, disjunction, implication, and so on. Brackets in formulas are used in the standard way, too. All formulas appearing in this paper are closed by some quantifier. If one of the formulas is open, then the variables that appear in it should be seen as free variables. In addition to the previous one, the sign =:, in the use of $A =: B$, should be understood in the sense that the mark A is the abbreviation for the formula B.

In this text, to mark recognizable formulas, we will use, as far as possible, their standard abbreviations that appear in a very well-known paper [6].

Definition 2.1. ([1], Definition 3.1) An algebra $\mathfrak{A} =: (A, \cdot, 0)$ of type $(2, 0)$ is called a BI-algebra if the following holds:

(Re) $(\forall x \in A)(x \cdot x = 0)$, (Im) $(\forall x, y \in A)(x \cdot (y \cdot x) = x)$.

A BI-algebra $\mathfrak{A}$ is said to be right distributive if the following

(RD) $(\forall x, y, z \in A)((x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z))$

is valid.

Left distributivity, according to [1], Proposition 3.9, in this class of logical algebras is possible only in the trivial case when $\mathfrak{A} = (\{0\}, \cdot, 0)$.

Let $\mathfrak{A} =: (A, \cdot, 0)$ be a BI-algebra. We introduce a relation $\preccurlyeq$ on the set A by

$$(\forall x, y \in A)(x \preccurlyeq y \iff x \cdot y = 0).$$

We note that $\preccurlyeq$ is not a partially order in the set A, since it is only reflexive. It is shown ([1], Proposition 3.14) that if $\mathfrak{A}$ is a right distributive BI-algebra, then the induced relation $\preccurlyeq$ is a transitive relation. So, if $\mathfrak{A}$ is a right distributive BI-algebra, then $\preccurlyeq$ is a quasi-order on A right compatible with the operation in $\mathfrak{A}$ ([1], Proposition 3.12(iv)). Some of the important properties of this class of logical algebras are given by [1], Proposition 3.7 and Proposition 3.12. Right distributive BI-algebras were also discussed in [11].

Definition 2.2. ([9], Definition 2.1) An algebra $\mathfrak{A} =: (A, \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called a pseudo-BI-algebra if it satisfies the following axioms:

(pRe) $(\forall x \in A)(x \to x = 1 = x \rightsquigarrow x)$.

(pI$_4$) $(\forall x, y \in A)(x \to y = 1 \iff x \rightsquigarrow y = 1)$.

(pBI) $(\forall x, y \in A)((x \to y) \rightsquigarrow x = x = (x \rightsquigarrow y) \to x)$.

A pseudo-BI-algebra $\mathfrak{A} =: (A, \to, \rightsquigarrow, 1)$ is called left distributive ([9], Definition 2.6) if it satisfies the following axioms:

(pLD$_1$) $(\forall x, y, z \in A)(z \to (x \rightsquigarrow y) = (z \to x) \rightsquigarrow (z \to y))$,

(pLD$_2$) $(\forall x, y, z \in A)(z \rightsquigarrow (x \to y) = (z \rightsquigarrow x) \to (z \rightsquigarrow y))$.

Some of the important properties of this class of logical algebras are expressed by statements (P$_1$)-(P$_{11}$) and (d$_1$)-(d$_{18}$) in [9], Theorem 2.5 and Theorem 2.8. For our purposes in this article, we will repeat some of them:

Proposition 2.1 ([9], Theorem 2.5)**.** *Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 1)$ be a pseudo-BI-algebra. Then the following hold:*

(P$_1$) $(\forall x \in A)(1 \to x = x = 1 \rightsquigarrow x)$.

(P$_2$) $(\forall x \in A)(x \to 1 = 1 = x \rightsquigarrow 1)$.

(P$_3$) $(\forall x, y \in A)(x \to y = x \to (x \to y) \wedge x \rightsquigarrow y = x \rightsquigarrow (x \rightsquigarrow y))$, Throughout this paper, we define a binary relation $\preccurlyeq$ on a (left distributive) pseudo-BI-algebra by

$$(\forall x, y \in A)(x \preccurlyeq y \iff x \to y = 1 = x \rightsquigarrow y).$$

We note that by (pRe), the relation $\preccurlyeq$ is reflexive.

Proposition 2.2 ([9], Theorem 2.8)**.** *Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 1)$ be a left distributive pseudo-BI-algebra. Then the following hold:*

(d_8) $(\forall x, y, z \in A)(x \rightsquigarrow y \preccurlyeq (z \to x) \rightsquigarrow (z \to y))$.

(d_9) $(\forall x, y, z \in A)(x \to y \preccurlyeq (z \rightsquigarrow x) \to (z \rightsquigarrow y))$.

(d_{10}) *The induced relation $\preccurlyeq$ is a quasi-order on the set A.*

(d_{12}) $(\forall x, y, z \in A)((x \rightsquigarrow y) \to z \preccurlyeq x \rightsquigarrow (y \to z))$.

(d_{13}) $(\forall x, y, z \in A)((x \to y) \rightsquigarrow z \preccurlyeq x \to (y \rightsquigarrow z))$.

(d_{18}) $(\forall x, y \in A)(x \to y \preccurlyeq x \rightsquigarrow y \wedge x \rightsquigarrow y \preccurlyeq x \to y)$.

3. The main results: Some new results on filters

This section is the central part of this article. In the first subsection, in order to comfortably follow the exposition in this report, we remind a reader on the concept of filter in pseudo-BI-algebras introduced by A. Rezaei and D. A. Romano in [10]. In Subsection 3.2 more new results about filters in this type of logic algebras are proved. The last subsection is devoted both to the concept of positive implicative filters in (left distributive) pseudo-BI-algebras and to the criteria for recognizing this class of filters.

3.1. **About filters.** In this subsection we repeat some information about filters in (left distributive) pseudo-BI-algebras, taken from [10]. The concept of filters in pseudo-BI-algebras is introduced by the following definition:

Definition 3.1 ([10], Definition 3.2)**.** Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 1)$ be a (left distributive) pseudo-BI-algebra. A subset F of A is called a filter of $\mathfrak{A}$ if it satisfies the following conditions:

(F0) $1 \in F$.

(F1) $(\forall x, y \in A)((x \to y \in F \wedge x \in F) \implies y \in F)$.

(F2) $(\forall x, y \in A)((x \rightsquigarrow y \in F \wedge x \in F) \implies y \in F)$.

The family $\mathfrak{F}(A)$ of all filters in the pseudo-bI-algebra $\mathfrak{A}$ is not empty because $\{1\} \in \mathfrak{F}(A)$ and $A \in \mathfrak{F}(A)$.

Further on, we have:

Proposition 3.1 ([10], Proposition 3.3). *Let F be a filter of a (left distributive) pseudo-BI-algebra $\mathfrak{A} = (A, \rightarrow, \rightsquigarrow, 1)$. Then, holds*

(F3) $(\forall x, y \in A)((x \preccurlyeq y \wedge x \in F) \Longrightarrow y \in F)$.

Also, we have:

Proposition 3.2 ([10], Proposition 3.4). *Let F be a filter of a left distributive pseudo-BI-algebra $\mathfrak{A} = (A, \rightarrow, \rightsquigarrow, 1)$. Then, holds*

(F4) $(\forall x, y \in A)(x \in F \Longrightarrow y \rightarrow x \in F)$.

(F5) $(\forall x, y \in A)(x \in F \Longrightarrow y \rightsquigarrow x \in F)$.

An important property of filters in pseudo-BI-algebras is given in the following proposition:

Proposition 3.3 ([10], Proposition 3.7). *For any filter F in a left distributive pseudo-BI-algebra $\mathfrak{A} =: (A, \rightarrow, \rightsquigarrow, 1)$ the following holds*

(F) $(\forall x, y \in A)(x \rightarrow y \in F \Longleftrightarrow x \rightsquigarrow y \in F)$.

Example 3.1. Let $A = \{1, a, b, c\}$ with the operations

$\rightarrow$	1	a	b	c
1	1	a	b	c
a	1	1	c	c
b	1	a	1	c
c	1	b	b	1

$\rightsquigarrow$	1	a	b	c
1	1	a	b	c
a	1	1	c	b
b	1	c	1	c
c	1	a	b	1

.

Then $\mathfrak{A} =: (A, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BI-algebra ([9], Example 2.3(i)). Subsets $F_0 = \{1\}, F_1 = \{1, a\}$, $F_2 = \{1, b\}$, $F_3 = \{1, c\}$, $F_4 = \{1, a, b\}$ are filters in $\mathfrak{A}$. Subset $F_5 = \{1, a, c\}$ is not a filter in $\mathfrak{A}$ because, for example, we have $a \in F_5$ and $a \rightarrow b = c \in F_5$ but $b \notin F_5$. The subset $F_6 = \{1, b, c\}$ also is not a filter in $\mathfrak{A}$ because, for example, we have $c \rightarrow a = b \in F_6$ but $a \notin F_6$.

Example 3.2. Let $A = \{1, a, b, c, d\}$ with the operations.

$\rightarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	1
b	1	d	1	1	d
c	1	a	1	1	a
d	1	1	c	c	1

$\rightsquigarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	c	1
b	1	a	1	1	a
c	1	d	1	1	d
d	1	1	b	b	1

.

Then $\mathfrak{A} =: (A, \rightarrow, \rightsquigarrow, 1)$ is a left distributive pseudo-BI-algebra ([9], Example 2.7(i)). Subsets F_0, $F_1 = \{1, a\}$, $F_2 = \{1, b\}$, $F_3 = \{1, c\}$,

$F_4 = \{1, d\}$, $F_7 = \{1, a, d\}$ and $F_8 = \{1, b, c\}$ are filters in $\mathfrak{A}$. Subsets $F_5 = \{1, a, b\}$, $F_6 = \{1, a, c\}$, $F_9 = \{1, b, d\}$ and $F_{10} = \{1, c, d\}$ are not filters in $\mathfrak{A}$.

Notions and notations used in this paper and not previously defined here, were taken from [9, 10, 11].

3.2. **Something else about filters.** The following theorem gives criteria for recognizing filters in pseudo-BI-algebras.

Theorem 3.1. *Let $A =: (A, \to, \rightsquigarrow, 1)$ be a pseudo-BI-algebra. A subset F of A is a filter in $\mathfrak{A}$ if and only if it holds*

(F6) $(\forall x, y, z \in A)((x \in F \wedge z \in F \wedge (x \to y) \rightsquigarrow z = 1) \implies y \in F)$.

(F7) $(\forall x, y, z \in A)((x \in F \wedge z \in F \wedge (x \rightsquigarrow y) \to z = 1) \implies y \in F)$.

Proof. Let F be a filter in $\mathfrak{A}$ and let $x, y, z \in A$ be such that $(x \to y) \rightsquigarrow z = 1$ and $y, z \in F$. Then $x \to y \preccurlyeq z \in F$. Then $x \to y \in F$ by (F3). Hence, $y \in F$ by (F1). Validity of the formula (F7) can be demonstrated analogously to the previous proof.

Conversely, let (F6) and (F7) be valid and let $x, y \in A$ be such that $x \in F$ and $x \to y \in F$. Then $(x \to y) \rightsquigarrow (x \to y) = 1$ by (pRe). Thus, $y \in F$ according (F6). So, (F1) holds. The validity of (F2) can be proved similarly to the previous demonstration. ∎

If we transform formulas (F1) and (F2) using (pBI) and putting $x = z$, we get

(IF1) $(\forall x, y, z \in A)(((z \to ((y \rightsquigarrow x) \to y) \in F \wedge z \in F) \implies y \in F)$.

(IF2) $(\forall x, y, z \in A)(((z \rightsquigarrow ((y \to x) \rightsquigarrow y) \in F \wedge z \in F) \implies y \in F)$.

Therefore, we have the conclusion:

Theorem 3.2. *Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 1)$ be a pseudo-BI-algebra and F be a subset of A such that $1 \in F$. Then F is a filter in $\mathfrak{A}$ if and only if it satisfies the conditions (IF1) and (IF2).*

In addition to the previous one, we have another way of recognizing filters in pseudo-BI-algebras.

Theorem 3.3. *Let F be a subset of a pseudo-BI-algebra $\mathfrak{A} =: (A, \to, \rightsquigarrow, 1)$ satisfying (F0) and*

(Fc) $(\forall x, y, z \in A)((z \rightsquigarrow (x \to (x \to y)) \in F \wedge z \in F) \implies$
$x \to y \in F)$.

(Fd) $(\forall x, y, z \in A)((z \to (x \rightsquigarrow (x \rightsquigarrow y)) \in F \wedge z \in F) \implies$
$x \rightsquigarrow y \in F)$.

Then F is a filter in $\mathfrak{A}$ and vice versa.

Proof. Let $u, v \in A$ be such that $u \in F$ and $u \to v \in F$. Then $u \in F$ and $u \to (1 \rightsquigarrow (1 \rightsquigarrow v)) \in F$ by (P_1). This $1 \to v \in F$ according (Fc). Hence $v \in F$ by (P_1). Implication (F2) can be proved analogously to the previous demonstration. So, F is a filter in $\mathfrak{A}$.

Assume that F is a filter in $\mathfrak{A}$. Let us prove that F satisfies conditions (Fc) and (Fd). Let $x, y, z \in A$ be such that $z \rightsquigarrow (x \to (x \to y)) \in F$ and $z \in F$. Then $x \to (x \to y) \in F$ by (F2). From here, according to (P_3), we get $x \to y \in F$, which means that (Fc) is valid. The validity of (Fd) can be proved similarly. ∎

The following proposition gives a necessary and sufficient condition for recognizing filters in left distributive pseudo-BI-algebras.

Theorem 3.4. *Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 1)$ be a left distributive pseudo-BI-algebra and F be a subset in A containing 1. For F to be a filter in $\mathfrak{A}$, it is necessary and sufficient that the following holds*

(Fa) $(\forall x, y, z \in A)(((x \rightsquigarrow y) \to z \in F \wedge y \in F) \Longrightarrow x \rightsquigarrow z \in F)$.

(Fb) $(\forall x, y, z \in A)(((x \to y) \rightsquigarrow z \in F \wedge y \in F) \Longrightarrow x \to z \in F)$.

Proof. (i) Taking $x = 1$, $y = x$ and $z = y$ in (Fa), we have

$$((1 \rightsquigarrow x) \to y \in F \wedge x \in F) \Longrightarrow 1 \rightsquigarrow y \in F.$$

From here, taking into account (P_1), we get (F1). Using a procedure similar to the one above, it is possible to demonstrate obtaining (F2) from (Fb).

(ii) Assume that F is a filter in $\mathfrak{A}$ and let $x, y, z \in A$ be such that $(x \rightsquigarrow y) \to z \in F$ and $y \in F$. According to (d_{12}), we have $(x \rightsquigarrow y) \to z \preccurlyeq x \rightsquigarrow (y \to z)$. From $(x \rightsquigarrow y) \to z \in F$ we get $x \rightsquigarrow (y \to z) \in F$ in accordance with (F3). This last fact is, according to (pLD_2), equivalent to $(x \rightsquigarrow y) \to (x \rightsquigarrow z) \in F$. On the other hand, from $y \in F$ it follows $x \rightsquigarrow y \in F$ according to (F5). Finally, from the previous facts, we get $x \rightsquigarrow z \in F$ by (F1). Obtaining (Fb) can be demonstrated by an analogous way. ∎

Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 1)$ be a pseudo-BI-algebra, K be a non-empty subset of A and $a \in A$ be an arbitrary element. Let us define

$$K_a =: \{y \in A : a \to y \in K \wedge a \rightsquigarrow y \in K\}.$$

Theorem 3.5. *Let F be a filter in a left distributive pseudo-BI-algebra $\mathfrak{A}$. Then for any $a \in A$, the set F_a is the smallest filter in $\mathfrak{A}$ containing F and a.*

Proof. It is clear that $1 \in F_a$ holds because $a \to 1 = 1 = a \rightsquigarrow 1 \in F$ by (P2) and (F0).

Let $x, y \in A$ be such that $x \to y \in F_a$ and $x \in F_a$. Then $a \rightsquigarrow (x \to y) \in F$ and $a \rightsquigarrow x \in F$. Hence $(a \rightsquigarrow x) \to (a \rightsquigarrow y) \in F$ according to (pLD$_2$). Thus, $a \rightsquigarrow y \in F$ by (F2). So, $y \in F_a$. Implication (F2) for F_a can be proved analogously.

Therefore, F_a is a filter in $\mathfrak{A}$.

Since $\mathfrak{A}$ is a left distributive pseudo-BI-algebra, for all $x \in F$ we have $F \ni x = 1 \to x = (a \rightsquigarrow a) \to x \preccurlyeq a \rightsquigarrow (a \to x)$ by (P$_1$), (pRe) and (d$_{12}$). Then $a \rightsquigarrow (a \to x) \in F$ by (F3). Hence $F \ni a \rightsquigarrow (a \to x) \preccurlyeq a \to (a \to x)$ according with (d$_{18}$). Therefore, $a \to (a \to x) \in F$ by (F3). Finally, we have $a \to x \in F$ with respect to (P$_3$). This means $x \in F_a$. So, $J \subseteq F_a$. Also, it is clear that $a \in F_a$ because $a \to a = 1 \in F$.

Suppose that K is a filter in $\mathfrak{A}$ containing F and a. If $x \in F_a$, then $a \to x \in F \subseteq K$. Thus $a \to x \in K$. Hence $x \in K$ by (F1) since $a \in J \subseteq K$. So, $F_a \subseteq K$. This means that F_a is the least filter containing F and a. ∎

Definition 3.2. Let $\mathfrak{A} =: (A, \to_A, \rightsquigarrow_A, 1_A)$ and $\mathfrak{B} =: (B, \to_B, \rightsquigarrow_B, 1_B)$ be pseudo-BI algebras. A mapping $f : A \longrightarrow B$ is called a homomorphism if the following holds:

(f1) $(\forall x, y \in A)(f(x \to_A y) = f(x) \to_B f(y))$ and
(f2) $(\forall x, y \in A)(f(x \rightsquigarrow_A y) = f(x) \rightsquigarrow_B f(y))$.

We write this homomorphism as $f : \mathfrak{A} \longrightarrow \mathfrak{B}$.

Note that if $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a homomorphism of BI-algebras, then holds

(f0) $f(1_A) = 1_B$.

Indeed, for arbitrary $x \in A$, we have $f(1_A) = f(x \to_A x) = f(x) \to_B f(x) = 1_B$ according to (pRe) and (f1).

Proposition 3.4. *Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism of pseudo-BI-algebras. Then:*

(0) $(\forall x, y \in A)(x \preccurlyeq_A y \Longrightarrow f(x) \preccurlyeq_B f(y))$.

(i) *If S is a sub-algebra in $\mathfrak{A}$, then*

$$f(S) =: \{u \in B : (\exists x \in S)(f(x) = u)\}$$

is a sub-algebra in $\mathfrak{B}$.

(ii) *If K is a filter in $\mathfrak{B}$, then $f^{-1}(K) =: \{x \in A : f(x) \in K\}$ is a filter in $\mathfrak{A}$.*

(iii) *The subset* $Ker(f) =: f^{-1}(\{1_B\})$ *is a filter in* $\mathfrak{A}$.

Proof. (0) Let $x, y \in A$ be such that $x \preccurlyeq_A y$. Then $x \to_A y = 1_A$. Thus $f(x) \to_B f(y) = f(x \to_A y) = f(1_A) = 1_B$. Hence, $f(x) \preccurlyeq_B f(y)$.

(i) Let $u, v \in f(S)$ ve arbitrary elements. Then there exist elements $x, y \in S$ such that $f(x) = u$ and $f(y) = v$. Thus $u \to_B v = f(x) \to_B f(y) = f(x \to_A) \in f(S)$ since $x \to_A \in S$ because S is a sub-algebra in $\mathfrak{A}$. The statement $u \rightsquigarrow_B v \in f(S)$ can be proved analogously. So, $f(S)$ is a sub-algebra in $\mathfrak{B}$.

(ii) Let $x \in A$ be such that $x \in f^{-1}(K)$ and $x \to_A y \in f^{-1}(K)$. This means $f(x) \in K$ and $f(x \to_A y) \in K$. Then $f(x) \in K$ and $f(x) \to_B f(y) \in K$. Thus $f(y) \in K$ by (F1). Hence, $y \in f^{-1}(K)$. The validity of the condition (F2) can be proved analogously. So, $f^{-1}(K)$ is a filter in $\mathfrak{A}$.

(iii) Let $x, y \in A$ be such that $x \in f^{-1}(\{1_B\})$ and $x \to y \in f^{-1}(\{1_B\})$. This means $f(x) = 1_B$ and $f(x \to_A y) = 1_B$. Then $F \ni 1_B = f(x) \to_B f(y) = 1_B \to_B f(y) = f(y)$. Hence, $y \in f^{-1}(\{1_B\})$. The validity of the condition (F2) can be proved analogously. So, $f^{-1}(\{1_B\})$ is a filter in $\mathfrak{A}$. ∎

In what follows, we need the following lemma:

Lemma 3.1. *Let* $\mathfrak{A} =: (A, \to, \rightsquigarrow, 1)$ *be a left distributive pseudo BI-algebra. Then:*

(1) $(\forall x \in A)(x \preccurlyeq 1)$.

(2) $(\forall x \in A)(1 \preccurlyeq x \implies x = 1)$.

(3) $(\forall x, y \in A)(x \preccurlyeq y \implies z \to x \preccurlyeq z \to y)$.

(4) $(\forall x \in A)(x \preccurlyeq y \implies z \rightsquigarrow x \preccurlyeq z \rightsquigarrow y)$.

(5) $(\forall x, y, z \in A)(z \to (x \to y) \preccurlyeq z \rightsquigarrow (x \rightsquigarrow y))$.

(6) $(\forall x, y, z \in A)(z \rightsquigarrow (x \rightsquigarrow y) \preccurlyeq z \to (x \to y))$.

Proof. (1) is equivalent to (pRe).

(2) Let $x \in A$ be such that $1 \preccurlyeq x$. This means $1 \to x = 1$. From here, due to (P_1), it follows that $x = 1$.

(3) Let $x, y \in A$ be such that $x \preccurlyeq y$. This means $x \to y = 1 = x \rightsquigarrow y$. Since, according to ($d_8$), we have $1 = x \rightsquigarrow y \preccurlyeq (z \to x) \rightsquigarrow (z \to y)$, from here we have $1 = (z \to x) \rightsquigarrow (z \to y)$ in accordance with (2). Thus $z \to x \preccurlyeq z \to y$.

(4) The proof for (4) can be demonstrated analogously to the proof for (3).

(5) For arbitrary elements $x, y, z \in A$, we have $x \to y \preccurlyeq x \rightsquigarrow y$ by (d_{18}). Then,

$$z \to (x \to y) \preccurlyeq z \to (x \rightsquigarrow y) \preccurlyeq z \rightsquigarrow (x \rightsquigarrow y)$$

with respect to (3) and (d_{18}).

(6) Claim (6) can be proved similar to the proof of (5). ∎

It is common knowledge that if the relation $\preccurlyeq$ is a quasi-order on a set A, then the relation $\equiv_{\preccurlyeq} = \preccurlyeq \cap \preccurlyeq^{-1}$ is an equivalence on A. Thus, since $\preccurlyeq$ is a quasi-order by (d_{10}), left compatible, by (3) and (4), with the operations on a left distributive pseudo-BI-algebra $\mathfrak{A} =: (A, \to, \rightsquigarrow, 1)$, we conclude that $\equiv_{\preccurlyeq}$ is a left congruence on the algebra $\mathfrak{A}$.

As a consequence of the previous lemma, we have:

Proposition 3.5. *Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 1)$ be a left distributive pseudo BI-algebra. Then:*

(pLD_3) $(\forall x, y, z \in A)(z \to (x \to y) \equiv_{\preccurlyeq} (z \to x) \to (z \to y))$.

(pLD_4) $(\forall x, y, z \in A)(z \rightsquigarrow (x \rightsquigarrow y) \equiv_{\preccurlyeq} (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y))$.

Proof. We will prove (pLD_4). The formula (pLD_3) can be proved analogously. Let $x, y, z \in A$ be arbitrary elements. Then

$$\begin{aligned} z \rightsquigarrow (x \rightsquigarrow y) \preccurlyeq z \to (x \to y) \preccurlyeq z \rightsquigarrow (x \to y) \quad & \text{by (6) and } (d_{18}) \\ = (z \rightsquigarrow x) \to (z \rightsquigarrow y) \quad & \text{by } (pLD_2) \\ \preccurlyeq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y) \quad & \text{by } (d_{18}) \\ \preccurlyeq (z \rightsquigarrow x) \to (z \rightsquigarrow y) \quad & \text{by } (d_{18}) \\ = z \rightsquigarrow (x \to y) \quad & \text{by } (pLD_2) \\ \preccurlyeq z \to (x \to y) \quad & \text{by } (d_{18}) \\ \preccurlyeq z \rightsquigarrow (x \rightsquigarrow y) \quad & \text{by (5)} \end{aligned}$$ ∎

Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 1)$ be a pseudo-BI-algebra and let $a, b \in A$ be arbitrary elements. Let us define

$$A(a, b) =: \{x \in A : a \to (b \rightsquigarrow x) = 1 \ \wedge \ a \rightsquigarrow (b \to x) = 1\}.$$

By direct checking, without major difficulties, it can be concluded that the following is valid:

(7) $1 \in A(a, b)$

because $a \to (b \rightsquigarrow 1) = a \to 1 = 1$ and $a \rightsquigarrow (b \to 1) = a \rightsquigarrow 1 = 1$ due to validity of (P_2).

(8) $b \in A(a, b)$

because $a \to (b \rightsquigarrow b) = a \to 1 = 1$ and $a \rightsquigarrow (b \to b) = a \rightsquigarrow 1 = 1$ due to validity of (pRe) and (P_2).

$A(a,1) = \{a \in A : a \to x = 1 \wedge a \rightsquigarrow x = 1\} = A(a,a)$,

$A(1,b) = \{x \in A : b \to x = 1 \wedge b \rightsquigarrow x = 1\} = A(b,b)$

$A(1,1) = \{x \in A : 1 \to (1 \rightsquigarrow x) = 1 \wedge 1 \rightsquigarrow (1 \to x) = 1\} = \{1\}$.

In connection with the previous one, it can be proved:

Theorem 3.6. *Let F be a subset of a pseudo-BI-algebra $\mathfrak{A} =: (A, \to, \rightsquigarrow, 1)$. Then F is a filter in $\mathfrak{A}$ if and only if the following holds:*

(9) $(\forall u, v \in F)(A(u,v) \subseteq F)$.

Proof. Let F be a filter in $\mathfrak{A}$. Then for any $a, b \in F$ and $x \in A(a,b)$ we have $a \to (b \rightsquigarrow x) = 1 \in F$ and $a \rightsquigarrow (b \to x) = 1 \in F$. Thus $b \rightsquigarrow x \in F$ and $b \to x \in F$ by (F1) and (F2) because $a \in F$. Hence, $x \in F$ by (F2) and (F1) respectively, since $b \in F$. So, $A(a,b) \subseteq F$.

Conversely, suppose that formula (9) is valid. Let us prove, in that case, that F is a filter in $\mathfrak{A}$. First, by (7), we have $1 \in F$. Let us take $u, v \in A$ such that $u \in F$, $u \to v \in F$ and $u \rightsquigarrow v \in F$. On the other hand, since $(u \to v) \to (u \rightsquigarrow v) = 1$ and $(u \rightsquigarrow v) \rightsquigarrow (u \to v) = 1$, according to ($d_{18}$), we conclude that $v \in A(u \to v, u) \subseteq F$ and $v \in A(u \rightsquigarrow v, u) \subseteq F$ holds. Therefore, F is a filter in $\mathfrak{A}$. ∎

Further on, we have:

Theorem 3.7. *Let $\mathfrak{A} = (A, \to, \rightsquigarrow, 1)$ be a left distributive pseudo-BI-algebra. Then for any elements $a, b \in A$, the subset $A(a,b)$ is a filter in $\mathfrak{A}$.*

Proof. Let $u, v \in A$ be such that $u \in A(a,b)$ and $u \rightsquigarrow v \in A(a,b)$. This means $a \to (b \rightsquigarrow u) = 1$, $a \rightsquigarrow (b \to u) = 1$ and $a \to (b \rightsquigarrow (u \to v)) = 1$. Since $\mathfrak{A}$ is a left distributive, we have

$1 = a \rightsquigarrow (b \to (u \rightsquigarrow v)) = a \to ((b \rightsquigarrow u) \to (b \rightsquigarrow v))$ by (pLD_1)

$\equiv_{\preccurlyeq} (a \to ((b \rightsquigarrow u) \to (a \to (b \rightsquigarrow u))$ by (pLD_3)

$= 1 \to (a \to (b \rightsquigarrow u))$ in accordance with the hypothesis

$= a \to (b \rightsquigarrow u)$ according to (P_1).

Since $1 \equiv_{\preccurlyeq} a \rightsquigarrow (b \to v)$ can be demonstrated similarly to the previous one, we have $v \in A(a,b)$. Therefore, $A(a,b)$ is a filter in $\mathfrak{A}$. ∎

3.3. **Positive implicative filters.** In this subsection, we introduce the concept of positive implicative filter in pseudo-BI-algebras and show some of its basic properties. While in the general case of pseudo-BI-algebras, this concept differs from the concept of filter and them, in the case of left distributive pseudo-BI-algebras, the concept of positive implicative filter coincides with the concept of filter.

Definition 3.3. A subset F of a pseudo-BI algebra $\mathfrak{A} =: (A, \rightarrow, \rightsquigarrow, 1)$ is called a positive implicative filter in $\mathfrak{A}$ if it satisfies (F0) and the following conditions:

(PIF1) $(\forall x, y, z \in A)(z \rightarrow (x \rightsquigarrow y) \in F \wedge z \rightarrow y \in F) \Longrightarrow z \rightarrow x \in F)$.

(PIF2) $(\forall x, y, z \in A)(z \rightsquigarrow (x \rightarrow y) \in F \wedge z \rightsquigarrow y \in F) \Longrightarrow z \rightsquigarrow x \in F)$.

Proposition 3.6. *Any positive implicative filter in a pseudo-BI-algebra $\mathfrak{A} = (A, \rightarrow, \rightsquigarrow, 1)$ is a filter in $\mathfrak{A}$.*

Proof. If in (PIF1) and (PIF2) we choose $z = 1$, with respect to (P_1), we get (F2) and (F1) respectively. ∎

The following example shows that the converse of the Proposition 3.3 need not be valid.

Example 3.3. Let $A = \{1, a, b, c\}$ as in Example 3.1. Then $\mathfrak{A} =: (A, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BI-algebra. The filter $F_0 = \{1\}$ is not a positive implicative filter in $\mathfrak{A}$ because, for example, we have $a \rightarrow (b \rightsquigarrow 1) = a \rightarrow 1 = 1 \in F_0$ and $a \rightarrow 1 = 1 \in F_0$ but $a \rightarrow b = c \notin F_0$. The filter $F_3 = \{1, c\}$ also is not a positive implicative filter in $\mathfrak{A}$ because, for example, we have $b \rightarrow (a \rightsquigarrow c) = b \rightarrow b = 1 \in F_3$ and $b \rightarrow c = c \in F_3$ but $b \rightarrow a = a \notin F_3$.

Theorem 3.8. *Let $\mathfrak{A} = (A, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BI-algebra. Then a subset F of A is a positive implicative filter in $\mathfrak{A}$ if and only if the following holds:*

(PIF3) $(\forall x, y, z \in A)(z \rightarrow (x \rightsquigarrow y) \in F \wedge z \rightarrow y \in F) \Longrightarrow z \rightarrow (z \rightarrow x) \in F)$.

(PIF4) $(\forall x, y, z \in A)(z \rightsquigarrow (x \rightarrow y) \in F \wedge z \rightsquigarrow y \in F) \Longrightarrow z \rightsquigarrow (z \rightsquigarrow x) \in F)$.

Proof. The validity of the formulas (PIF3) and (PIF4) follows directly from the validity of the formulas (PIF1) and (PIF2), respectively, with respect to (P_3), and vice versa. ∎

Theorem 3.9. *Let F be a filter in a BI-algebra $\mathfrak{A} =: (A, \rightarrow, \rightsquigarrow, 1)$. If for all $a \in A$ the subset F_a is a filter in $\mathfrak{A}$, then F is a positive implicative filter in $\mathfrak{A}$.*

Proof. Let $x, y, z \in A$ be such that $z \rightarrow (x \rightsquigarrow y) \in F$ and $z \rightarrow z \in F$. Then $x \rightsquigarrow y \in F_z$ and $y \in F_z$. Since F_z is a filter in $\mathfrak{A}$, then $x \in F_z$ by (F2). So, $z \rightarrow x \in F$. Therefore, F is a positive implicative filter in $\mathfrak{A}$. ∎

Example 3.4. Let $A = \{1, a, b, c\}$ as in Example 3.1. Then $\mathfrak{A} =: (A, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BI-algebra. Applying the criterion in Theorem 3.7, it can be shown that the filter $F_1 = \{1, a\}$ is a positive implicative filter in $\mathfrak{A}$, because, for the filter $K =: \{1, a\}$ we have $K_1 = F$, $K_a = \{1, a\}$, $K_b = \{1, b\}$ and $K_c = \{1, c\}$.

However, if $\mathfrak{A}$ is a left distributive pseudo-bi-algebra, then we have:

Theorem 3.10. *In left distributive pseudo-BI-algebras, any filter is a positive implicative filter.*

Proof. Let F be a positive implicative filter in a left distributive pseudo-bi-algebra $\mathfrak{A} = (A, \rightarrow, \rightsquigarrow, 1)$ and let $x, y, z \in A$ be such that $z \rightarrow (x \rightsquigarrow y) \in F$ and $z \rightarrow y \in F$. Then $(z \rightarrow x) \rightsquigarrow (z \rightarrow y) \in F$ by (pLD$_1$). From here and from $z \rightarrow y \in F$, we get $z \rightarrow x \in F$ according to (F2). The validity of the formula (PIF2) can be proved analogously. ∎

4. Conclusion

Although the concept of BI-algebras was introduced recently (more precisely, in 2017), a significant number of researchers took part in looking at the properties of this newly introduced class of logical algebras. The algebraic structure of pseudo-BI-algebras was determined in 2023 by A. Radfar and A. Rezaei ([9]). The concept of filters in this class of algebras was introduced by A. Rezaei and D. A. Romano ([10]). Here, in this paper, as a continuation of the previously mentioned two papers, the phenomenon of filters in (left distributive) pseudo-BI-algebras is considered much more. Since each filter is a positive implicative filter in left distributive pseudo-BI-algebras (Theorem 3.10), the distinguishing specificities between the concept of filters and the concept of positive implicative filters can only be registered in pseudo-BI-algebras that are not left distributive. As a continuation of these researches, it could be, among other things, a consideration of some other special types of filters in this class of logic algebras.

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