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HULL-KERNEL TOPOLOGY ON A SPECIFIC CLASS OF TOPOLOGICAL ALGEBRAS

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Abstract. Let A be Fundamental Strongly Sequential algebra (FSS). It is known that carrier space Φ_A is compact and Hausdorff, In this paper we introduce a topology on Φ_A , called Hull-kernel topology, which coincides with the A-topology in certain circumstances.

1. Introduction

In 1979, T. Husain [7] introduced the concept of strongly sequential topological algebras. Also E.Ansari-Piri in [1] introduced fundamental topological algebras to generalized the famous Cohen factorization theorem in 1990. In [5], Bonsal and Duncan showed that if A is a complex commutative Banach algebra with or without unit , its carrier space Φ_A is locally compact Hausdorff space with respect to A-topology. They consider Hull-kernel topology on Φ_A which coincides with A-topology in a completely regular algebra. In this note, we consider fundamental strongly sequential algebras and show a result analogous to that of Bonsal and Duncan is true etc.

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2. PRELIMINARIES

We recall some basic definitions and known results.

In this article, an algebra A is called topological algebra if it is a complex algebra equipped with a Hausdorff topology such that:

(1) The map $(x, y) \longmapsto x + y$ from $A \times A$ to A is continuous.
(2) The map $(\alpha, x) \longmapsto \alpha x$ from $\mathbb{C} \times A$ to A is continuous.
(3) The map $(x.y) \longmapsto xy$ from $A \times A$ to A is continuous.

[7] Topological algebra A is said to be strongly sequential if there exists a neighborhood U of zero such that for all $x \in U$, $x^k \to 0$ as $k \to \infty$.

[1] A is said to be a fundamental topological algebra if there exists $b > 1$ such that for every sequence $(x_n)_n$ of A the convergence of $b^n(x_{n+1} - x_n)$ to zero in A implies that $(x_n)_n$ is a Cauchy sequence.

The topological algebra A is called FSS, if it is fundamental and strongly sequential.

In a unital algebra, the set of invertible elements of A is denoted by Inv(A). A unital topological algebra A is called Q-algebra if Inv(A) is an open set.

A multiplicative linear functional on A is a non-zero linear functional φ on A such that for all $x, y \in A$:

$$\varphi(xy) = \varphi(x)\varphi(y).$$

The set of all multiplicative linear functional on A with A-topology, is denoted by Φ_A, the carrier space of algebra A.

Theorem 1. *Suppose A is a complete metrisable FSS algebra. Then the carrier space of A is compact with A-topology.*

Proof. See [3, Proposition 3.5]. ∎

Theorem 2. *Suppose A is a complete metrisable Q-algebra. Then every multiplicative linear functional is continuous.*

Proof. See [6]. ∎

Theorem 3. *Every multiplicative linear functional is continuous on a complete metrisable FSS algebra.*

Proof. Every complete metrisable FSS algebra is a complete metrisable Q-algebra. See [2, Proposition 3.2]. ∎

3. Hull-kernel topology on complete metrisable FSS algebra

In this article, we suppose A is a unital commutative complete metrisable FSS algebra and Φ_A is its carrier space. By Theorem 1, Φ_A is a compact and Hausdorff space with respect to the A-topology. We now consider a second topology on Φ_A which in general differs from the A-topology but coincides with it for certain important algebras A.

Definition 4. *Given a subset E of Φ_A, the kernel of E, ker(E), is defined by*

$$ker(E) = \bigcap \{ker\varphi : \varphi \in E\}.$$

Given a subset J of A, the hull of J, is defined by

$$hul\,(J) = \{\varphi \in \Phi_A : ker\varphi \supset J\}\,.$$

We denote $hul\,(ker(E))$ by $\tilde{E}$.

Theorem 5. *In an algebra A we have*

1. *ker(E) is proper closed ideal of A.*
2. *hul(J) is a closed and therefore compact ideal of Φ_A with respect to A-topology, so $\tilde{E}$ is a closed ideal of Φ_A*
3. *If $\psi \in \Phi_A$, then $\psi \in \tilde{E}$ if and only if $ker\psi \supseteq \ker(E)$.*
4. *If $E_1 \subseteq E_2 \subseteq \Phi_A$, then $ker(E_1) \supseteq ker(E_2)$.*
5. *If $J_1 \subseteq J_2 \subseteq A$, then $hul\,(J_1) \supseteq hul\,(J_2)$.*
6. *If $E_1 \subseteq E_2 \subseteq \Phi_A$, then $\tilde{E_1} \subseteq \tilde{E_2}$.*

Proof. (1) By Theorem 3, since every linear multiplicative functional is continuous, its kernel is closed. Therefore, the $ker(E)$ is a closed ideal of A. Moreover, since every linear multiplicative functional is nonzero, the $ker(E)$ is a proper ideal of A.

(2) Let ϕ be an element of Φ_A and let $(\phi_i)_i$ be a net of elements in hul(J) that converges to ϕ in the A-topology. It is clear that ϕ is an element of hul(J) . Therefore, hul(J) and consequently $\tilde{E} = hul\,(ker\,(E))$ are closed. By Theorem 1, since Φ_A is compact with the A-topology, it follows that hul(J) is also compact.

(3), (4), (5) are clear by Definition 4. ∎

Theorem 6. (1) $\ker(E) = ker\left(\tilde{E}\right)$.

(2) $E \subseteq \tilde{E} = \widetilde{\left(\tilde{E}\right)}$.

(3) $(\widetilde{E_1 \cup E_2}) = \tilde{E}_1 \cup \tilde{E}_2$.

Proof. See [5, page 115]. ∎

Definition 7. *Let X be a set with power set P(X), and c be a mapping from the P(X) to itself. Then c is a Kuratowski closure operator if and only if it satisfies the following Kuratowski closure axioms for all* $A, B \subseteq X$.

(1) $c(\emptyset) = \emptyset$.
(2) $A \subseteq c(A)$.
(3) $c(c(A)) \subseteq c(A)$.
(4) $c(A \cup B) = c(A) \cup c(B)$.

This operator naturally produces a topology on X with the following definition.
In this topology $F \subseteq X$ is closed, if and only if $c(F) = F$.

A set is open if its complement is closed. Note that (4) implies that c is monotone, therefore using (2) and (3) it follows that c is idempotent.

Definition 8. *If we consider* $X = \Phi_A$ *and* $c(E) = \tilde{E}$, *by Theorem 6, c will be a Kuratowski closure operator. The topology determined by c on* Φ_A *is called the hull-kernel topology.*

Theorem 9. (1) $\varphi_0 \in \Phi_A \backslash \tilde{E}$ *if and only if there exists* $a \in A$, *such that* $\hat{a}(\varphi) = 0\,(\varphi \in E)$ *and* $\hat{a}(\varphi_0) \neq 0$.

(2) *The A-topology contains the Hull-kernel topology.*

(3) *The hull of an ideal is compact in the hull-kernel topology.*

Proof. (1)

$$\tilde{E} = hul(ker(E)) = \{\varphi \in \Phi_A : ker\varphi \supseteq ker(E)\}$$

It means

$$\varphi_0 \in \tilde{E} \Leftrightarrow ker\varphi_0 \supseteq ker(E) = \bigcap \{ker\varphi : \varphi \in E\}.$$

So,

$$\varphi_0 \in \Phi_A \backslash \tilde{E} \Leftrightarrow ker\varphi_0 \not\supseteq \bigcap \{ker\varphi : \varphi \in E\} \Leftrightarrow$$

$$\exists a \in A : a \notin ker\varphi_0, a \in \bigcap \{ker\varphi : \varphi \in E\} \Leftrightarrow$$

$$\exists a \in A : \hat{a}(\varphi_0) \neq 0, \hat{a}(\varphi) = 0\,(\varphi \in E)$$

(2) Suppose τ_h and τ_A denote the Hull-kernel topology and A-topology respectively on Φ_A. We will show $\tau_h \subseteq \tau_A$. Let E is a τ_h-closed set of Φ_A, then we will show that it is τ_A-closed. Since E is τ_h-closed on Φ_A, by definition $E = \tilde{E}$. If $\varphi_0 \in \bar{E}$; we want to show $\varphi_0 \in E = \tilde{E}$.
Since $\varphi_0 \in \bar{E}$, so there exist a net $(\varphi_i)_i$ of E such that $\varphi_i \to \varphi_0$. If $\varphi_0 \in \Phi_A \backslash \tilde{E}$ by (1), there exists $a \in A$ such that $\hat{a}(\varphi) = 0\,(\varphi \in E)$ and $\hat{a}(\varphi_0) \neq 0$. Since $\hat{a}$ is a complex continuous function on Φ_A with A-topology, hence

$$\hat{a}(\varphi_i) \to \hat{a}(\varphi_0)$$

Left side of above equation is zero and the right side is non-zero which is a contradiction.

(3) Let J be an ideal of A. By Theorem 5(2), $hul\,(J)$ is compact with A-topology. Suppose $(U_i)_{i \in I}$ is an open cover for $hul\,(J)$ with Hull-kernel topology. By (2), since U_i is also τ_A-open, $hul\,(J)$ is also compact with A-topology. Then, there is finite subcover of $(U_i)_{i \in I}$ for $hul\,(J)$. It means $hul\,(J)$ is compact with Hull-kernel topology too.

∎

In the following, we recall that the algebra A is called semi-simple when

$$\bigcap \{ker\varphi : \varphi \in \Phi_A\} = \{0\}$$

Theorem 10. *Let E be a non-empty subset of Φ_A that is closed with hull-kernel topology. Let $B = A/ker(E)$, let π denote the canonical mapping of A onto B, and π^* denote the dual mapping of Φ_B into Φ_A given by*

$$(\pi^*\psi)(a) = \psi(\pi a) \quad (\psi \in \Phi_B, a \in A)$$

Then π^* is a homeomorphism of Φ_B with B-topology onto E with the A-topology.

Proof. By Theorem 5(1) It is easy to see that ker(E) is a proper closed ideal of A. Also, B is a unital commutative complete metrisable FSS.[8] Let $\psi \in \Phi_B$. Since π is a linear multiplicative functional of A onto B therefore $\pi\psi$ is a complex linear multiplicative functional of A. ψ is non-zero so, there exists π a of B such that $\psi(\pi a) \neq 0$. Therefore $\pi\psi$ is non-zero and then it is in Φ_A.

Hence, $\pi^* : \Phi_B \to \Phi_A$ given by $\pi^*(\psi) = \pi\psi$ is well defined. For $\psi \in \Phi_B$ and $a \in ker(E)$, πa is the zero of B, then we have

$$(\pi^*\ \psi)(a) = \psi(\pi a) = 0,$$

therefore $a \in ker(\pi^*\psi)$ and so $ker(\pi^*\ \psi) \supseteq ker(E)$.
since $\pi^*\ \psi \in \Phi_A$, then $\pi^*\ \psi \in hul(\ker(E)) = \tilde{E}$. Moreover E is closed in the Hull-kernel topology So, $\tilde{E} = E$ and therefore $\pi^*\psi \in E$.
Now let $\varphi \in E$, by definition it is clear that $ker(E) \subseteq ker\varphi$. For $b \in B = A/ker(E)$, there exists $a \in A$, such that $\pi a = b$. Define a complex mapping ψ of B, given by $\psi(b) = \varphi(a)$. Let $b_1 = a_1 + ker(E) = b_2 = a_2 + ker(E)$. Then $a_1 - a_2 \in ker(E)$ and according to definition $\varphi(a_1 - a_2) = 0$ and so $\varphi(a_1) = \varphi(a_2)$. It means ψ is well defined. It is clear that ψ is linear. Also, it is multiplicative because

$$\psi((\pi a_1)(\pi a_2)) = \psi(\pi(a_1 a_2)) = \varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2) = \psi(\pi a_1)\psi(\pi a_2)$$

So $\psi \in \Phi_B$ and we have

$$\pi^*(\psi) = \pi\psi = \varphi$$

Therefore π^* is a mapping of Φ_B onto E.
It is clear π^* is linear. Let $\pi^*(\psi) = \pi\psi = 0$, So for all $\pi a \in B$ we have $\pi(\psi(a)) = (\pi\psi)a = 0$. It means ψ is zero mapping and therefore $ker(\pi^*) = 0$, hence π^* is injective and so bijective function.
Since Φ_B is compact in the B-topology, to show that π^* is a homeomorphism it is enough to show π^* is continuous. By definition of A-topology, the below sets create a local base for all $\varphi \in \Phi_A$

$$\{V(\varphi; x_1, \ldots, x_n; \varepsilon) : n\epsilon N\ \ ,\ x_1, \ldots, x_n \epsilon A\ ,\ \varepsilon > 0\}$$

Which

$$V(\varphi; x_1, \ldots, x_n; \varepsilon) = \{\psi \in \Phi_A :\ |\psi(x_i) - \varphi(x_i)| < \varepsilon \quad i : 1, 2, \ldots, n\}.$$

Now suppose ψ_0 is an arbitrary element of Φ_B and $V = V(\pi^*(\psi_0); a_1, \ldots, a_n; \varepsilon)$ is a neighborhood of $\pi^*(\psi_0)$. We will find a neighborhood U of ψ_0 such that for all $\psi \in U, \pi^*(\psi) \in V$.

$$\begin{gathered} V(\pi^*(\psi_0); \pi a_1, \ldots, \pi a_n; \varepsilon) \\ = \{\pi^*\ \psi \in E :\ |\pi^*\ \psi(a_i) - \pi^*(\psi_0)(a_i)| < \varepsilon \quad i : 1, 2, \ldots, n\} \\ = \{\pi^*\ \psi \in E :\ |\psi(\pi a_i) - \psi_0(\pi a_i)| < \varepsilon \quad i : 1, 2, \ldots, n\} \end{gathered}$$

According to this

$$U = \{\psi \in \Phi_B :\ |\psi(\pi a_i) - \psi_0(\pi a_i)| < \varepsilon \quad i : 1, 2, \ldots, n\}$$

is an open set in Φ_B that for all $\psi \in U$ we have

$$\pi^*\ \psi \in V = \pi^* U.$$

By definition, to show B is semi-simple it is enough to show that

$$\bigcap \{ker\varphi : \ \varphi \in \Phi_B\} = \{0\}$$

Suppose $b = \pi a \in \bigcap \{ker\varphi : \ \varphi \in \Phi_B\} \subseteq B$. It means for all $\psi \in \Phi_B$ we have

$$\psi(b) = \psi(\pi a) = 0.$$

And so, for all $\psi \in \Phi_B$, $(\pi^* \ \psi)(a) = \psi(\pi a) = 0$. Let $\varphi \in E$. Since π^* is a surjective mapping of Φ_B onto E, there exists $\psi \in \Phi_B$, such that $\varphi = \pi^* \psi$. Hence $\varphi(a) = \pi^* \psi(a) = 0$. So

$$a \in ker(E) = \bigcap \{ker\varphi : \ \varphi \in E\}$$

Therefore $b = \pi a = 0_B$. ∎

Remark 11. *Given E is a compact set in the A-topology. We know that E is compact in Hull-kernel topology, but we cannot conclude that E is closed in the Hull-kernel topology, since that topology need not be Hausdorff.*

Definition 12. *A unital topological algebra A is said to be completely regular if Φ_A is Hausdorff space in the Hull-kernel topology.*[5]

Theorem 13. *Let A be a unital commutative complete metrisable FSS algebra. Then A is completely regular if and only if the A-topology coincides with the Hull- kernel topology in Φ_A.*

Proof. In this proof τ_A and τ_h will denote the A-topology and the Hull-kernel topology on Φ_A respectively.

Suppose A is completely regular topological algebra. We will show $\tau_h = \tau_A$. By Theorem 9(2), $\tau_h \subseteq \tau_A$, so it is enough to show $\tau_A \subseteq \tau_h$. Let F be a τ_A-closed subset of Φ_A. We will show F is τ_h-closed, it means $F = \tilde{F}$. According to Theorem 6(2) $F \subseteq \tilde{F}$, so it is enough to show $\tilde{F} \subseteq F$.

Since F is τ_A-closed and Φ_A is τ_A-compact, so F is τ_A-compact set and therefore τ_h-compact.

Suppose $f \in F$ and $\varphi_0 \in \Phi_A \backslash F$. It is obvious that $f \neq \varphi_0$. By definition of completely regular algebra, since Φ_A is τ_h- Hausdorff, so f and φ_0 have τ_h- neighborhood W_f and W_{φ_0} such that $W_f \cap W_{\varphi_0} = \emptyset$. It is clear that complement of W_{φ_0} is a τ_h-closed set and so $W^c_{\varphi_0} = \tilde{W^c_{\varphi_0}}$. We have

$$W_f \subseteq W^c_{\varphi_0} \quad \Rightarrow \quad \tilde{W_f} \subseteq \tilde{W^c_{\varphi_0}} = W^c_{\varphi_0}$$

Therefore $\varphi_0 \notin \tilde{W_f} = hul\,(ker\,(W_f))$.
Based on $\{W_f : f \in F\}$ is a τ_h-open cover for F, and F is a τ_h-compact, So there exists τ_h-open sets $U_1, \dots, U_n$ such that $F \subseteq \bigcup_{i=1}^{n} U_i$ and $\varphi_0 \notin hul\,(ker\,(U_i))$. Therefore, for each $i = 1, 2, ..., n$, there exists $v_i \in ker\,(U_i)$ which $\varphi_0(v_i) \neq 0$. Let $v = v_1 \dots v_n$, then $\varphi_0(v) \neq 0$ and so, we will have $\varphi_0 \notin \tilde{F} = hul\,(ker\,(F))$.
Conversely, suppose that $\tau_h = \tau_A$, since τ_A is Hausdorff topology then obviously, τ_h is Hausdorff. So, algebra A is completely regular. ∎

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