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BOUNDEDNESS AND CONVERGENCE IN METRONLIKE STRUCTURES

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Abstract. Based on angle properties, Sonwane and Prasad [19] established a mathematical structure called metron as a generalization of metric space. Recently Sonwane, Prasad and Thakur [20] studied metronlike structures by weakening some properties of Metron. In the present paper we investigate, in the setting of metronlike structures, properties analogous to the boundedness of sets and counterparts to the convergence and Cauchy property of a sequence.

1. INTRODUCTION

Metric spaces [8] represent one of the basic structures in mathematics. During the past hundred and fifteen years many generalizations of metric spaces have been appeared in the literature [12, 13, 4, 10, 3, 15, 9, 21, 14, 7, 5, 1, 2, 16, 17, 18, 11, 6]. Based on angle properties, Sonwane and Prasad [19] introduced the concept of metron as a generalization of metric space. Recently, Sonwane, Prasad and Thakur [20] studied metronlike structures by weakening some properties of metron. In the present paper we investigate and study the concepts of boundedness of sets and convergence of sequences in metronlike structures.

Keywords and phrases: Metron, sur-bounded set, co-sur-bounded set, ide-bounded set, sur-convergent sequence, ide-convergent sequence, I –sur-convergent sequence.

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2. PRELIMINARIES

Definition 2.1. [19] Let X be a nonempty set and let $m : X \times X \times X \rightarrow \mathbb{R}$ (where $\mathbb{R}$ is the set of all real numbers) satisfying the following conditions:

- (M1) $m\langle x, y, z \rangle \geq 0$, for all $x, y, z \in X$ (property of non-negativity).
- (M2) $m\langle x, y, z \rangle = 0$, if $x = z$ (Vanishing property).
- (M3) if $x \neq z$, then there exists $y \in X$ such that $m\langle x, y, z \rangle > 0$ (Survival property).
- (M4) if $m\langle u, x, v \rangle = m\langle u, y, v \rangle$, for all $u, v \in X$ then $x = y$ (Identification property).
- (M5) $m\langle x, y, z \rangle = m\langle z, y, x \rangle$, for all $x, y, z \in X$ (Symmetric property).
- (M6) $m\langle x, y, z \rangle \leq m\langle x, y, u \rangle + m\langle u, y, z \rangle$, for all $x, y, z, u \in X$ (property of Triangle inequality).
- (M7) $m\langle x, y, z \rangle \leq m\langle x, u, z \rangle + m\langle u, z, x \rangle + m\langle z, x, u \rangle$, for all $x, y, z, u \in X$ (property of Triangular chain inequality).

Then the function $m : X \times X \times X \rightarrow \mathbb{R}$ is called a metron function on X and the couple (X, m) is called a metron with metron function m .

Remark 2.2. [19] Properties (M3) and (M4) in the definition of metron are equivalent to the following properties, respectively:

- (M3)(A) If $m\langle x, y, z \rangle = 0$, for all $y \in X \Rightarrow x = z$ (Implication of Equality property, IE-property for short).
- (M4)(A) Whenever $x \neq y$, then there exists $u, v \in X$, such that $m\langle u, x, v \rangle \neq m\langle u, y, v \rangle$ (Implication of Non equality property, IN-property for short).

Definition 2.3. [20] Let X be a nonempty set and $m : X \times X \times X \rightarrow \mathbb{R}$. We call (X, m) a metronlike structure if m has at least properties (M1) and (M2).

Example 2.4. [20] Let X be a nonempty set and $f : X \rightarrow \mathbb{R}$. Define $m : X \times X \times X \rightarrow \mathbb{R}$ by

$$m\langle x, y, z \rangle = ||f(x) - f(y)| - |f(y) - f(z)||, \text{ for all } x, y, z \in X.$$

Then (X, m) is a metronlike structure because m satisfies the properties M1 and M2.

Metronlike structures that are assumed to possess (at least) some properties of a metron are presented in the following Table I.

TABLE 1. Metronlike structures with their properties

Structure/ Properties	M1	M2	M3	M4	M5	M6	M7
Metron	✓	✓	✓	✓	✓	✓	✓
Metronlike	✓	✓					
Semi-metron	✓	✓			✓	✓	✓
Sur-semi-metron	✓	✓	✓		✓	✓	✓
Ide-semi-metron	✓	✓		✓	✓	✓	✓
Pre-metron	✓	✓	✓	✓	✓	✓	
Semi-pre-metron	✓	✓			✓	✓	
Sur-semi-pre-metron	✓	✓	✓		✓	✓	
Ide-semi-pre-metron	✓	✓		✓	✓	✓	

Definition 2.5. [20] *Let (X, m) be a metronlike structure. We say that this structure*

- (a) *satisfies the Existence of Base Supremum property (EBS-property) if $\sup\{m\langle x, u, y \rangle : u \in X\}$ exists in $\mathbb{R}$ for each choice of $x, y \in X$ or equivalently $\{m\langle x, u, y \rangle : u \in X\}$ is bounded for each choice of $x, y \in X$.*
- (b) *satisfies the Mark Binding property (MB-property) if $\sup\{m\langle u, x, v \rangle : u, v \in X\}$ exists in $\mathbb{R}$ for each choice of $x \in X$ or equivalently $\{m\langle u, x, v \rangle : u, v \in X\}$ is bounded for each choice of $x \in X$.*
- (c) *satisfies the Existence of Base Difference Supremum property (EBDS-property) if $\sup\{|m\langle x, u, y \rangle - m\langle x, v, y \rangle| : u, v \in X\}$ exists in $\mathbb{R}$ for each choice of $x, y \in X$ or equivalently $\{|m\langle x, u, y \rangle - m\langle x, v, y \rangle| : u, v \in X\}$ is bounded for each choice of $x, y \in X$.*
- (d) *satisfies the Existence of Mark Supremum property (EMS-property) if $\sup\{|m\langle u, x, v \rangle - m\langle u, y, v \rangle| : u, v \in X\}$ exists in $\mathbb{R}$ for each choice of $x, y \in X$ or equivalently, $\{|m\langle u, x, v \rangle - m\langle u, y, v \rangle| : u, v \in X\}$ is bounded for each choice of $x, y \in X$.*
- (e) *possesses the Existence of Nonsurvival property (EN-property) if for each choice of $x, y \in X$, there exists a point $u \in X$ (depending on x and y) such that $m\langle x, u, y \rangle = 0$.*

3. BOUNDEDNESS IN METRONLIKE STRUCTURES

Boundedness is a basic tool for the analytical study of metric spaces and their properties. The diameter of a set in a metric space is the supremum of distances between any two points within that set, and a

set whose diameter is finite is called a bounded set. In this section we define and study in the setting of metronlike structures concepts that correspond to the boundedness in metric spaces.

Definition 3.1. Let (X, m) be a metronlike structure. A subset $A \subset X$ is said to be:

- (i) a *sur-bounded* subset of X if for each $a \in X$, there exists $M > 0$ such that $m\langle x, a, y \rangle \leq M$, for all $x, y \in A$.
- (ii) a *co-sur-bounded* subset of X if for each $a \in X$, there exists $N(a) > 0$ such that $m\langle a, x, y \rangle \leq N(a)$, for all $y \in A$ and $x \in X$.

Example 3.2. Let $X = \mathbb{R}$ and $\alpha : X \rightarrow \mathbb{R}$ defined by

$$\alpha(x) = \begin{cases} \frac{1}{x}, & \text{if } x \in \mathbb{Z} - \{0\} \\ x, & \text{if } x \notin \mathbb{Z} - \{0\} \end{cases}$$

where $\mathbb{Z}$ is the set of all integers. Let $m : X \times X \times X \rightarrow \mathbb{R}$ be defined by

$$m\langle x, y, z \rangle = ||\alpha(x) - y| - |y - \alpha(z)||, \text{ for all } x, y, z \in X.$$

Then (X, m) is a semi-pre-metron. Consider $\mathbb{Z} \subset X$. The set $\mathbb{Z}$ is a sur-bounded set in (X, m) as well as it is co-sur-bounded.

- (i) $\mathbb{Z}$ is sur-bounded.

Take $x \in \mathbb{Z}$. Then $-1 \leq \alpha(x) \leq 1$ and for each $a \in X$,

$$\begin{aligned} m\langle x, a, y \rangle &= ||\alpha(x) - a| - |a - \alpha(y)|| \\ &\leq |\alpha(x) - \alpha(y)| \\ &\leq |-1 - 1| = 2, \text{ for all } x, y \in \mathbb{Z} \end{aligned}$$

So, for each $a \in X$, $m\langle x, a, y \rangle \leq 2$ for all $x, y \in \mathbb{Z}$. This shows that $\mathbb{Z}$ is a sur-bounded set in (X, m) .

- (ii) $\mathbb{Z}$ is co-sur-bounded.

For, take an arbitrary point $a \in X$, then either $a \in \mathbb{Z}$ or $a \notin \mathbb{Z}$.

When $a \in \mathbb{Z}$, then we have

$$\begin{aligned} m\langle a, x, y \rangle &= ||\alpha(a) - x| - |x - \alpha(y)|| \\ &\leq |\alpha(a) - \alpha(y)| \\ &\leq |-1 - 1| = 2, \text{ for all } x \in X, y \in \mathbb{Z} \end{aligned}$$

When $a \notin \mathbb{Z}$ then $\alpha(a) = a$. Therefore,

$$\begin{aligned} m\langle a, x, y \rangle &= ||\alpha(a) - x| - |x - \alpha(y)|| \\ &= ||a - x| - |x - p||, \text{ for all } p \in [-1, 1] \\ &\leq |a + p| \\ &\leq ||a| + |p| \\ &\leq a + 1 \text{ for all } x \in X, y \in \mathbb{Z} \end{aligned}$$

Hence $\mathbb{Z}$ is a co-sur-bounded set.

Example 3.3. Let $X = \mathbb{R}$ and $\alpha : X \rightarrow \mathbb{R}$ defined by

$$\alpha(x) = \begin{cases} \frac{1}{x}, & \text{if } x \in \mathbb{Z} - \{0\} \\ x, & \text{if } x \notin \mathbb{Z} - \{0\} \end{cases}$$

where $\mathbb{Z}$ is the set of all integers. Let $m : X \times X \times X \rightarrow \mathbb{R}$ defined by

$$m\langle x, y, z \rangle = ||x - \alpha(y)| - |\alpha(y) - z|| \text{ for all } x, y, z \in X.$$

Then (X, m) is a semi-pre-metron. Consider $\mathbb{Z} \subset X$, then the set $\mathbb{Z}$ is neither a sur-bounded set nor a co-sur-bounded set in (X, m) .

Example 3.4. Let $X = \mathbb{R}$ and $\alpha : X \rightarrow \mathbb{R}$ defined by

$$\alpha(x) = \begin{cases} \frac{1}{x}, & \text{if } x \leq 0 \text{ and } x \neq 0 \\ 0, & \text{if } x = 0 \\ x, & \text{if } x > 0 \end{cases}$$

where $\mathbb{Z}$ is the set of all integers. Let $m : X \times X \times X \rightarrow \mathbb{R}$ defined by

$$m\langle x, y, z \rangle = ||\alpha(x) - y| - |y - \alpha(z)|| \text{ for all } x, y, z \in X.$$

Then (X, m) is a semi-pre-metron. Let $A = \{\frac{1}{n} : n \in \mathbb{Z} - \{0\}\}$, then the set A is neither a sur-bounded set, nor a co-sur-bounded set in (X, m) .

Theorem 3.5. Let (X, m) be a metronlike structure with the property of triangle inequality. Then:

- (i) Finite union of sur-bounded sets in X is a sur-bounded set, if (X, m) possesses the EBS-property.
- (ii) Arbitrary intersection of sur-bounded sets in X is a sur-bounded set.
- (iii) Finite union of co-sur-bounded sets in X is a co-sur-bounded set.

- (iv) *Arbitrary intersection of co-sur-bounded sets in X is a co-sur-bounded set.*

Proof. (i) Suppose A and B are two sur-bounded sets in X . Then there exists $M_A, M_B > 0$ such that

$$m\langle x, a, y \rangle \leq M_A, \text{ for all } x, y \in A, a \in X,$$

and

$$m\langle u, a, v \rangle \leq M_B, \text{ for all } u, v \in B, a \in X.$$

Now

$$\begin{aligned} m\langle u, a, y \rangle &\leq m\langle u, a, v_0 \rangle + m\langle v_0, a, x_0 \rangle + m\langle x_0, a, y \rangle \\ &\leq M_A + m\langle v_0, a, x_0 \rangle + M_B \\ &= M_{A \cdot B}, \text{ for all } u \in B, y \in A, \text{ and } a \in X. \end{aligned}$$

When $x_0 \in A$ and $v_0 \in B$, then $\{m\langle x_0, a, v_0 \rangle\}$ is bounded by EBS-property of X . Thus

$$m\langle u, a, y \rangle \leq M_{A \cdot B}, \text{ for all } u, y \in A \cup B \text{ and } a \in X.$$

This shows that $A \cup B$ is a sur-bounded set in X . Induction over the number of sets shows that every finite union of sur-bounded sets is also sur-bounded.

- (ii) Let $\{A_\alpha : \alpha \in \Lambda\}$ be an arbitrary family of sur-bounded sets in X . Then $\bigcap_{\alpha \in \Lambda} A_\alpha \subset A_\alpha$, for all $\alpha \in \Lambda$. So $\bigcap_{\alpha \in \Lambda} A_\alpha$ is a sur-bounded set.
- (iii) Suppose A and B be two co-sur-bounded sets in X . Then for each $a \in X$ there exists $N_{A(a)} > 0$ corresponding to A and there exists $N_{B(a)}$ corresponding to B such that

$$m\langle a, x, y \rangle \leq N_{A(a)}, \text{ for all } y \in A, \text{ and } x \in X,$$

and

$$m\langle a, x, v \rangle \leq N_{B(a)}, \text{ for all } v \in B, \text{ and } x \in X.$$

Now choose $z \in A \cup B$, then $z \in A$ or $z \in B$. So clearly,

$$m\langle a, x, y \rangle \leq \max\{N_{A(a)}, N_{B(a)}\}, \text{ for all } z \in A \cup B, \text{ and } x \in X.$$

This shows that $A \cup B$ is a co-sur-bounded set. Induction over the number of sets shows that every finite union of co-sur-bounded sets is also co-sur-bounded.

- (iv) Follows on the similar lines as in (ii).

■

Definition 3.6. Let (X, m) be a metronlike structure and $A \subset X$. Then A is said to be:

- (i) *ide-bounded subset of X if for each $a \in X$, there exists $M > 0$ such that*

$$|m\langle u, x, v \rangle - m\langle u, y, v \rangle| \leq M, \text{ for all } x, y \in A, \text{ and } u, v \in X.$$

- (ii) *co-ide-bounded subset of X if for each $a \in X$, there exists $M(a) > 0$ such that*

$$|m\langle u, a, v \rangle - m\langle u, x, v \rangle| \leq M(a), \text{ for all } x \in A, \text{ and } u, v \in X.$$

Definition 3.7. Let (X, m) be a metronlike structure and $A \subset X$. Then A is said to be:

- (i) *I -sur-bounded subset of X if for each $a \in X$, there exists $M > 0$ such that*

$$|m\langle x, u, y \rangle - m\langle x, v, y \rangle| \leq M, \text{ for all } x, y \in A \text{ and } u, v \in X.$$

- (ii) *I -cosurebounded subset of X if for each $a \in X$, there exists $M(a) > 0$ such that*

$$|m\langle a, u, x \rangle - m\langle a, v, x \rangle| \leq M(a), \text{ for all } x \in A, \text{ and } u, v \in X.$$

Lemma 3.8. Let $\mathbb{R}$ be the set of all the real numbers and $|x|$ denotes the absolute value of x , then

$$||u - x_n| - |x_n - v|| - ||u - x_0| - |x_0 - v|| \leq 2|x_n - x_0|,$$

for all $x_n, x_0, u, v \in \mathbb{R}$.

Proof. By the usual triangle inequality in $\mathbb{R}$, we know that

$$||u - x_n| - |x_n - v|| \leq |u - v|, \text{ for all } x_n, u, v \in \mathbb{R}.$$

Let $x_n, u, v \in \mathbb{R}$ and assume that $u < v$. Then either $x_n \in]u, v[$ or $x_n \notin]u, v[$.

Case I: Let $x_n \in]u, v[$. We know that $\frac{1}{2}(u + v)$ is the middle point of the open interval $]u, v[$. Assume, without the loss of generality that x_n is nearer to v , then the distance between v and x_n will be given by $(v - x_n)$ and distance between u and x_n is given by $(x_n - u)$. Since x_n is nearer to v ,

$$(v - x_n) \leq \frac{1}{2}(u + v) - u \leq (x_n - u).$$

Hence

$$\begin{aligned}
 ||u - x_n| - |x_n - v|| &= (x_n - u - (v - x_n)) \\
 &= (2x_n - u - v) \\
 &= 2(x_n - \frac{1}{2}(u + v)) \\
 &= 2|\frac{1}{2}(u + v) - x_n| \\
 &= |(u + v) - 2x_n|.
 \end{aligned}$$

Case II: Let $x_n \notin]u, v[$. Then

$$\begin{aligned}
 |\frac{1}{2}(u + v) - x_n| &\geq \frac{1}{2}(v - u) \\
 &\geq |\frac{1}{2}(v - u)|.
 \end{aligned}$$

But

$$\begin{aligned}
 ||u - x_n| - |x_n - v|| &\leq |u - v| \\
 &\leq 2|\frac{1}{2}(u - v)| \\
 &\leq 2|\frac{1}{2}(u + v) - x_n| = |(u + v) - 2x_n|.
 \end{aligned}$$

Thus in either of the cases, we get

$$||u - x_n| - |x_n - v|| \leq |(u + v) - 2x_n|$$

Now apply above inequality for different positions of x_n, x_0 with respect to $]u, v[$: either (i) $x_n, x_0 \in]u, v[$, or (ii) $x_n \in]u, v[$ but $x_0 \notin]u, v[$, or (iii) $x_n, x_0 \notin]u, v[$.

(i) If $x_n, x_0 \in]u, v[$, then

$$\begin{aligned}
 &||u - x_n| - |x_n - v|| - ||u - x_0| - |x_0 - v|| \\
 &\leq |(u + v) - 2x_n| - |(u + v) - 2x_0| \\
 &\leq |2x_n - 2x_0| = 2|x_n - x_0|.
 \end{aligned}$$

(ii) If $x_n \in]u, v[$ but $x_0 \notin]u, v[$, then

$$||u - x_n| - |x_n - v|| = |(u + v) - 2x_n|$$

and

$$||u - x_0| - |x_0 - v|| \leq |(u + v) - 2x_0|.$$

Thus

$$\begin{aligned} & |||u - x_n| - |x_n - v|| - ||u - x_0| - |x_0 - v|| \\ &= |||u - x_0| - |x_0 - v|| - ||u - x_n| - |x_n - v|| \\ &\leq |(u + v) - 2x_0| - |(u + v) - 2x_n| = 2|x_n - x_0|. \end{aligned}$$

(iii) If $x_n, x_0 \notin]u, v[$, then

$$||u - x_n| - |x_n - v|| = |u - v|$$

and

$$||u - x_0| - |x_0 - v|| = |u - v|.$$

Thus

$$\begin{aligned} & |||u - x_n| - |x_n - v|| - ||u - x_0| - |x_0 - v|| \\ &\leq ||u - v| - |u - v|| = 0 \leq 2|x_n - x_0|. \end{aligned}$$

Hence, in all the situations, we obtain

$$|||u - x_n| - |x_n - v|| - ||u - x_0| - |x_0 - v|| \leq 2|x_n - x_0|,$$

for all $x_n, x_0, u, v \in \mathbb{R}$. ■

Example 3.9. Let $\mathbb{R}$ be the set of all real numbers and $m : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$m\langle x, y, z \rangle = ||x - y| - |y - z||.$$

Then $(\mathbb{R}, m)$ is a metron. Then the set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is an ide-bounded set as well and it is co-ide-bounded also.

(i) A is ide-bounded: by Lemma 3.8 we have

$$|||u - x| - |x - v|| - ||u - y| - |y - v|| \leq 2|x - y|, \text{ for all } x, y, u, v \in \mathbb{R}.$$

So,

$$|m\langle u, x, v \rangle - m\langle u, y, v \rangle| \leq 2|x - y|, \text{ for all } x, y, u, v \in \mathbb{R}.$$

Since $|x - y| \leq 1$, for all $x, y \in A$, we obtain that

$$|m\langle u, x, v \rangle - m\langle u, y, v \rangle| \leq 2, \text{ for all } x, y \in A, \text{ and for all } u, v \in \mathbb{R}.$$

This shows that A is an ide-bounded set in $(\mathbb{R}, m)$.

(ii) A is co-ide-bounded: By Lemma 3.8 we have

$$\begin{aligned} |m\langle u, a, v \rangle - m\langle u, x, v \rangle| &= |||u - a| - |a - v|| - ||u - x| - |x - v|| \\ &\leq 2|a - x|. \end{aligned}$$

and $|x - x_0| \leq 1$ for all $x, x_0 \in A$. Thus we obtain that

$$\begin{aligned} |m\langle u, a, v \rangle - m\langle u, x, v \rangle| &\leq |m\langle u, a, v \rangle - m\langle u, x_0, v \rangle| \\ &\quad + |m\langle u, x_0, v \rangle - m\langle u, x, v \rangle| \\ &\leq 2|a - x_0| + 2|x - x_0|, \text{ for all } x, x_0 \in A, \text{ and } a \in X. \end{aligned}$$

Hence, in particular, for $x_0 = 1$,

$$\begin{aligned} |m\langle u, a, v \rangle - m\langle u, x, v \rangle| &\leq 2|a - 1| + 2|x - 1| \\ &\leq 2|a - 1| + 2.1 = 2(|a - 1| + 1). \end{aligned}$$

Put $M(a) = 2(|a - 1| + 1)$. Then we get for each $a \in \mathbb{R}$, there exists $M(a) > 0$ such that

$$|m\langle u, a, v \rangle - m\langle u, x, v \rangle| \leq M(a), \text{ for all } x \in A, \text{ and } u, v \in X.$$

This shows that A is an co-ide-bounded set in $(\mathbb{R}, m)$.

Example 3.10. The set of all natural numbers $\mathbb{N}$ in the usual metron $(\mathbb{R}, m)$ is neither ide-bounded nor co-ide-bounded.

Example 3.11. The set $B = \{-1 + \frac{1}{n} : n \in \mathbb{N}\} \cup \{1 - \frac{1}{n} : n \in \mathbb{N}\}$ is an I -ide-bounded set, as well as I -co-ide-bounded set in the usual metron $(\mathbb{R}, m)$.

(i) B is an I -ide-bounded set. We know from a well-known inequality that

$$||x - u| - |u - y|| = ||x - v| - |v - y|| \leq |x - y|, \text{ for all } x, y, u, v \in \mathbb{R}.$$

So,

$$|m\langle x, u, y \rangle - m\langle x, v, y \rangle| \leq |x - y|, \text{ for all } x, y, u, v \in \mathbb{R}.$$

Further, $|x - y| \leq |-1 - (+1)| = 2$, gives that

$$|m\langle x, u, y \rangle - m\langle x, v, y \rangle| \leq 2, \text{ for all } x, y \in B \text{ and for all } u, v \in X.$$

This shows that B is an I -sur-bounded set.

(ii) B is an I -co-ide-bounded set: clearly, we have

$$|m\langle a, u, x \rangle - m\langle a, v, x \rangle| \leq |a - x|,$$

so

$$|m\langle a, u, x \rangle - m\langle a, v, x \rangle| \leq |a - x_0| + |x_0 - x|,$$

for all $x_0, x \in B$ and for all $u, v \in \mathbb{R}$. In particular put $x_0 = 0$, then

$$\begin{aligned} |m\langle a, u, x \rangle - m\langle a, v, x \rangle| &\leq |a - 0| + |0 - x| \\ &\leq |a| + 1, \text{ for all } x \in B. \end{aligned}$$

Put $M(a) = |a| + 1$. Then we get that, for each $a \in B$, there exists $M(a) > 0$ such that

$$|m\langle a, u, x \rangle - m\langle a, v, x \rangle| \leq M(a),$$

for all $x \in B$ and $u, v \in \mathbb{R}$. This shows that B is an I -co-sur-bounded set in $(\mathbb{R}, m)$

Example 3.12. Let U be an unbounded set in usual metron $(\mathbb{R}, m)$. Then U is neither sur-bounded, ide-bounded and I -sur-bounded, nor co-sur-bounded, co-ide-bounded and I -co-sur-bounded.

Theorem 3.13. Let (X, m) be a metronlike structure with the property of triangle inequality. Then:

- (i) Finite union of ide-bounded sets in X is an ide-bounded set, if (X, m) possesses the EMS-property.
- (ii) Arbitrary intersection of ide-bounded sets in X is an ide-bounded set.
- (iii) Finite union of co-ide-bounded sets in X is a co-ide-bounded set.
- (iv) Arbitrary intersection of co-ide-bounded sets in X is a co-ide-bounded set.

Proof. (1)

- (i) Suppose that A and B are two ide-bounded sets in X . Then for the set A , there exists an $M_A > 0$ and for the set B , there exists an $M_B > 0$ such that

$$|m\langle u, x, v \rangle - m\langle u, y, v \rangle| \leq M_A, \text{ for all } x, y \in A \text{ and } u, v \in X,$$

and

$$|m\langle u, s, v \rangle - m\langle u, t, v \rangle| \leq M_B, \text{ for all } s, t \in B \text{ and } u, v \in X.$$

Now

$$\begin{aligned} |m\langle u, x, v \rangle - m\langle u, s, v \rangle| &\leq |m\langle u, x, v \rangle - m\langle u, x_0, v \rangle| \\ &\quad + |m\langle u, x_0, v \rangle - m\langle u, s_0, v \rangle| \\ &\quad + |m\langle u, s_0, v \rangle - m\langle u, s, v \rangle| \\ &\leq M_A + |m\langle u, x_0, v \rangle - m\langle u, s_0, v \rangle| + M_B, \\ &= M_{A \cdot B}, \text{ for all } x \in A, s \in B \text{ and } u, v \in X. \end{aligned}$$

When $x_0 \in A, s_0 \in B$, the set $\sup_{u,v \in X} \{|m\langle u, x_0, v \rangle - m\langle u, s_0, v \rangle|\}$ is bounded by EMS-property. Thus,

$$|m\langle u, x_0, v \rangle - m\langle u, s_0, v \rangle| \leq M_{A \cdot B},$$

for all $x, s \in A \cup B$ and $u, v \in X$. This shows that $A \cup B$ is an ide-bounded set. Induction over the number of sets shows that every finite union of ide-bounded sets is also ide-bounded .

(ii) Let $\{A_\alpha : \alpha \in \Lambda\}$ be an arbitrary family of ide-bounded sets in X . Then $\bigcap_{\alpha \in \Lambda} A_\alpha \subset A_\alpha$ for all $\alpha \in \Lambda$. So $\bigcap_{\alpha \in \Lambda} A_\alpha$ is an ide-bounded set.

(iii) Suppose A and B are two co-ide-bounded sets in X , so for each $a \in X$, there exists an $M_{A(a)} > 0$ corresponding to A and $M_{B(a)} > 0$ corresponding to B such that

$$|m\langle u, a, v \rangle| - |m\langle u, x, v \rangle| \leq M_{A(a)}, \text{ for all } x \in A \text{ and } u, v \in X,$$

and

$$|m\langle u, a, v \rangle| - |m\langle u, s, v \rangle| \leq M_{B(a)}, \text{ for all } s \in B \text{ and } u, v \in X.$$

Now choose $t \in A \cup B$, then either $t \in A$ or $t \in B$. So,

$$|m\langle u, a, v \rangle| - |m\langle u, t, v \rangle| \leq \max\{M_{A(a)}, M_{B(a)}\},$$

for all $t \in A \cup B$ and $u, v \in X$.

Put $M(a) = \max\{M_{A(a)}, M_{B(a)}\}$. Thus for each $a \in X$, there exists $M(a) > 0$ such that

$$|m\langle u, a, v \rangle| - |m\langle u, t, v \rangle| \leq M(a),$$

for all $t \in A \cup B$ and $u, v \in X$. This shows that $A \cup B$ is a co-ide-bounded set. Induction over the number of sets shows that every finite union of co-ide-bounded sets in X is also co-ide-bounded.

(iv) Obvious.

■

Theorem 3.14. *Let (X, m) be a metronlike structure. Then :*

- (i) *Finite union of I -sur-bounded sets is I -sur-bounded if (X, m) possesses the triangle inequality of difference and EBDS-property.*
- (ii) *Arbitrary intersection of I -sur-bounded sets is I -isur-bounded.*
- (iii) *Finite union of I -co-sur-bounded sets is I -co-sur-bounded.*

- (iv) *Arbitrary intersection I -co-sur-bounded sets is I -co-sur-bounded.*

Proof. (i) Suppose A and B are two I -sur-bounded sets in X . Then for A , there exists $M_A \geq 0$ and for B , there exists $M_B > 0$ such that $|m\langle x, u, y \rangle - m\langle x, v, y \rangle| \leq M_A$, for all $x, y \in A$, and for all $u, v \in X$, and

$|m\langle s, u, t \rangle - m\langle s, v, t \rangle| \leq M_B$, for all $s, t \in B$, and for all $u, v \in X$. To the I -sur-boundedness of $A \cup B$, when we choose two points from $A \cup B$, then either both the points belong to A or to B or one belongs to A and the other belongs to B . So, by the consideration of the third situation, we have that

$$\begin{aligned} |m\langle x, u, t \rangle - m\langle x, v, t \rangle| &\leq |m\langle x, u, x_0 \rangle - m\langle x, v, x_0 \rangle| \\ &\quad + |m\langle x_0, u, t_0 \rangle - m\langle x_0, v, t_0 \rangle| \\ &\quad + |m\langle t_0, u, t \rangle - m\langle t_0, v, t \rangle| \end{aligned}$$

for all $x_0, x \in A$, for all $t_0, t \in B$ and for all $u, v \in X$ (by the triangle inequality of difference property). But, as (X, m) possesses the EBDS-property,

$$M(x_0, t_0) = \sup_{u, v \in X} \{|m\langle x_0, u, t_0 \rangle - m\langle x_0, v, t_0 \rangle|\}$$

exists and bounded. Thus

$$|m\langle x, u, t \rangle - m\langle x, v, t \rangle| \leq M_A + M(x_0, t_0) + M_B,$$

for all $s, t \in A \cup B$ and for all $u, v \in X$. This shows that $A \cup B$ is I -sur-bounded. Induction over the number of sets shows that every finite union of I -sur-bounded sets is also I -sur-bounded

Proofs of (ii), (iii) and (iv) are easy and left to the reader. ■

The following Theorems can be easily verified.

Theorem 3.15. *Let (X, m) be a metronlike structure then :*

- (i) *(X, m) possesses the EMS-property if and only if sets having two points are sur-bounded in (X, m) .*
- (ii) *(X, m) possesses the EMS-property if and only if sets having two points are ide-bounded in (X, m) .*
- (iii) *(X, m) possesses the EBDS-property if and only if sets having two points are I -sur-bounded in (X, m) .*
- (iv) *Every sur-bounded set in (X, m) is I -sur-bounded.*

- (v) Every I –sur-bounded set in (X, m) is sur-bounded if (X, m) possesses the existence of nonsurvival property.

Theorem 3.16. Let (X, m) be a semi–pre-metron with EBS-property and (X, d_s) be the semimetric space, where

$$d_s(x, y) = \sup_{a \in X} \{m\langle x, a, y \rangle\}.$$

Then a subset A of X is bounded in (X, d_s) if A is sur-bounded (co-sur-bounded) in (X, m) .

Theorem 3.17. Let (X, m) be a metronlike structure which possesses M_4 and EMS-property and (X, d_i) be the metric space, where

$$d_i(x, y) = \sup_{u, v \in X} \{|m\langle u, x, v \rangle - m\langle u, y, v \rangle|\}.$$

Then a subset A of X is bounded in (X, d_i) if A is ide-bounded (co-ide-bounded) in (X, m) .

Theorem 3.18. Let (X, m) be a metronlike structure which possesses the I –survival property, triangle inequality of difference property and EBDS-property, and (X, d_{is}) be the metric space, where

$$d_{is}(x, y) = \sup_{u, v \in X} \{|m\langle x, u, y \rangle - m\langle x, v, y \rangle|\}.$$

Then a subset A of X is bounded in (X, d_{is}) if A is I –sur-bounded (I –co-sur-bounded) in (X, m) .

Theorem 3.19. Let (X, m) be a metronlike structure. Then:

- (a) The sur-boundedness and co-sur-boundedness in X are equivalent if (X, m) possesses the EBS-property and the property of triangle inequality.
- (b) The ide-boundedness and co-ide-boundedness in X are equivalent if (X, m) possesses the EMS-property.
- (c) The I –sur-boundedness and I –co-sur-boundedness in X are equivalent if (X, m) possesses the EBDS-property and the triangular inequality of difference property.

Proof. (a) Let (X, m) be a metronlike structure which possesses the EBS-property and the property of triangle inequality. Suppose $A \subset X$ is sur-bounded in X , then for A , there exists an $M > 0$ such that,

$$m\langle x, u, y \rangle \leq M, \text{ for all } x, y \in A, \text{ for all } u \in X.$$

Now, take $a \in X$, then

$$\begin{aligned} m\langle a, u, y \rangle &\leq m\langle a, u, x_0 \rangle + m\langle x_0, u, y \rangle \\ &\leq m\langle a, u, x_0 \rangle + M, \text{ for all } x_0, y \in A; \text{ for all } u \in X. \\ &\leq M(a), \text{ for all } y \in A, u \in X, \end{aligned}$$

because x_0 is a fixed point of A and so $m\langle a, u, x_0 \rangle$ is bounded for all $u \in X$, by EBS-property. Equivalently, for each $a \in X$, there exists $M(a) > 0$ such that

$$m\langle a, u, y \rangle \leq M(a), \text{ for all } y \in A, u \in X.$$

Hence, A is a co-sur-bounded set in X .

Conversely, assume that A is co-sur-bounded in X . Then for each $a \in X$, there exists $M(a) > 0$ such that

$$m\langle a, u, x \rangle \leq M(a), \text{ for all } x \in A, u \in X.$$

Then

$$\begin{aligned} m\langle x, u, y \rangle &\leq m\langle x, u, a_0 \rangle + m\langle a_0, u, y \rangle \\ &\leq M(a_0) + M(a_0), \text{ for all } x, y \in A, u \in X \\ &= M, \text{ for all } x, y \in A, u \in X. \end{aligned}$$

Thus, A is sur-bounded in X .

Similarly, the results (b) and (c) can be proved as we have done for the result (a)

■

4. CONVERGENCE OF SEQUENCES IN METRONLIKE STRUCTURES

In this section we define and study some types of convergence for sequences in metronlike structures.

Definition 4.1. Let (X, m) be an ide-semi-pre-metron and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is said to be an ide-Cauchy sequence in (X, m) if for each $\epsilon > 0$ and each pair $u, v \in X$, there exists a positive integer $N(\epsilon, u, v)$ such that

$$|m\langle u, x_n, v \rangle - m\langle u, x_m, v \rangle| \leq \epsilon, \text{ for all } n, m \geq N(\epsilon, u, v).$$

Definition 4.2. Let (X, m) be an sur-semi-pre-metron and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is said to be an sur-Cauchy sequence in (X, m) if for each $\epsilon > 0$ and $a \in X$, there exists a positive integer $N(\epsilon, a)$ such that

$$m\langle x_n, a, x_m \rangle \leq \epsilon, \text{ for all } n, m \geq N(\epsilon, a).$$

Definition 4.3. Let (X, m) be a sur-semi-pre-metron with I -survival property and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is said to be an I -sur-Cauchy sequence in (X, m) if for each $\epsilon > 0$ and each pair $u, v \in X$, there exists a positive integer $N(\epsilon, u, v)$ such that

$$|m\langle x_n, u, x_m \rangle - m\langle x_n, v, x_m \rangle| \leq \epsilon, \text{ for all } n, m \geq N(\epsilon, u, v).$$

Definition 4.4. Let (X, m) be an ide-semi-pre-metron. A sequence $\{x_n\}$ in X is said to be an ide-convergent sequence in (X, m) , that ide-converges to $x_0 \in X$, if for each $\epsilon > 0$ and each pair $u, v \in X$, there exists an integer $N(\epsilon, u, v)$ such that

$$|m\langle u, x_n, v \rangle - m\langle u, x_0, v \rangle| \leq \epsilon, \text{ for all } n \geq N(\epsilon, u, v).$$

If $\{x_n\}$ ide-converges to x_0 , we write $I - \lim_{n \rightarrow \infty} x_n = x_0$, and the point x_0 is called an ide-limit of $\{x_n\}$.

Definition 4.5. Let (X, m) be a sur-semi-pre-metron. A sequence $\{x_n\}$ in X is said to be a sur-convergent sequence in (X, m) , that sur-converges to $x_0 \in X$, if for each $\epsilon > 0$, and $a \in X$, there exists an integer $N(\epsilon, a)$ such that,

$$m\langle x_n, a, x_0 \rangle \leq \epsilon, \text{ for all } n \geq N(\epsilon, a).$$

If $\{x_n\}$ sur-converges to x_0 , we write $s - \lim_{n \rightarrow \infty} x_n = x_0$, and the point x_0 is called a sur-limit of $\{x_n\}$.

Definition 4.6. Let (X, m) be an sur-semi-pre-metron with I -survival property. A sequence $\{x_n\}$ in X is said to be I -sur-convergent to $x_0 \in X$ in (X, m) if for each $\epsilon > 0$ and for each pair $u, v \in X$, there exists an integer $N(\epsilon, u, v)$ such that,

$$|m\langle x_0, u, x_n \rangle - m\langle x_0, v, x_n \rangle| \leq \epsilon, \text{ for all } n \geq N(\epsilon, u, v).$$

If $\{x_n\}$ I -sur-converges to x_0 , we write this as $I - \lim_{n \rightarrow \infty} x_n = x_0$, and the point x_0 is called an I -sur-limit of $\{x_n\}$.

Lemma 4.7. Let $\mathbb{R}$ be the set of real numbers. Then

$$||x - u| - |u - y|| - ||x - v| - |v - y|| \leq |x - y|, \text{ for all } x, y, u, v \in \mathbb{R}.$$

Proof. We know that $||x - u| - |u - y|| \leq |x - y|$ and the absolute value of difference of two positive real numbers does not exceed the maximum value of these real numbers. Hence

$$||x - u| - |u - y|| - ||x - v| - |v - y|| \leq |x - y|, \text{ for all } x, y, u, v \in \mathbb{R}.$$

■

Example 4.8. Let $(\mathbb{R}, m)$ be the distance metron on $\mathbb{R}$. Consider the sequence $\{\frac{1}{n}\}$ in $\mathbb{R}$. This sequence: (i) is an ide-Cauchy sequence in $\mathbb{R}$, (ii) is a sur-Cauchy sequence in $\mathbb{R}$, (iii) is an I -sur-Cauchy sequence in $\mathbb{R}$. By Lemma 3.8,

$$\begin{aligned}
 |m\langle u, \frac{1}{n}, v \rangle - m\langle u, \frac{1}{m}, v \rangle| &= |||u - \frac{1}{n}| - |\frac{1}{n} - v|| - ||u - \frac{1}{m}| - |\frac{1}{m} - v|| \\
 &\leq 2|\frac{1}{n} - \frac{1}{m}| \\
 &= 2|\frac{m-n}{mn}| \\
 &< \frac{m}{mn} = \frac{2}{n}, \text{ for all } m \geq n \\
 &< \epsilon, \text{ for all } m \geq n > \frac{2}{\epsilon}
 \end{aligned}$$

Hence, for each $\epsilon > 0$, and for each pair $u, v \in \mathbb{R}$, there exists an integer $N(\epsilon) \geq \frac{2}{\epsilon}$ such that

$$|m\langle u, \frac{1}{n}, v \rangle - m\langle u, \frac{1}{m}, v \rangle| < \epsilon, \text{ for all } m \geq n > \frac{2}{\epsilon}$$

This shows that $\{\frac{1}{n}\}$ is an ide-Cauchy sequence in $(\mathbb{R}, m)$. Similarly, by the use of Lemma 4.7 and the inequality

$$||x - u| - |u - y|| \leq |x - y|, \text{ for all } x, y, u \in \mathbb{R},$$

we get that the sequence $\{\frac{1}{n}\}$ is sur-Cauchy as well as I -sur-Cauchy sequence in $(\mathbb{R}, m)$.

Example 4.9. Let $(\mathbb{R}, m)$ be the distance metron on $\mathbb{R}$, and $\{\frac{1}{n}\} \subset \mathbb{R}$ be a sequence in $\mathbb{R}$, then it is (i) sur-converges to 0 (ii) ide-converges to 0 and (iii) I -sur-converges to 0. For (iii)

$$\begin{aligned}
 |m\langle \frac{1}{n}, u, 0 \rangle - m\langle \frac{1}{n}, v, 0 \rangle| &= |||\frac{1}{n} - u| - |u - 0|| - ||\frac{1}{n} - v| - |v - 0|| \\
 &\leq |\frac{1}{n} - 0| = \frac{1}{n} < \epsilon, \text{ for all } n > \frac{1}{\epsilon}.
 \end{aligned}$$

Thus for each $\epsilon > 0$ and each pair $u, v \in \mathbb{R}$, there exists an integer $N(\epsilon, u, v) > \frac{1}{\epsilon}$ such that

$$|m\langle \frac{1}{n}, u, 0 \rangle - m\langle \frac{1}{n}, v, 0 \rangle||v - 0|| < \epsilon, \text{ for all } n \geq N(\epsilon, u, v).$$

Hence the sequence $\{\frac{1}{n}\}$ I -sur-converges to 0. Similarly, by the use of Lemma 3.8 and the inequality,

$$||x - u| - |u - y|| \leq |x - y|, \text{ for all } x, y, u \in \mathbb{R},$$

we get that the sequence $\{\frac{1}{n}\}$ sur-converges to 0 and also $\{\frac{1}{n}\}$ ide-converges to 0.

Theorem 4.10. *Every sur-convergent sequence in a sur-semi-pre-metron (X, m) is a sur-Cauchy sequence.*

Proof. Let $\{x_n\}$ be a sur-convergent sequence which sur-converges to x_0 in a sur-semi-pre-metron (X, m) . Then for each $\epsilon > 0$, and $a \in X$, there exists an integer $N(\epsilon, a)$ such that,

$$m\langle x_n, a, x_0 \rangle < \frac{\epsilon}{2}, \text{ for all } n \geq N(\epsilon, a).$$

Now,

$$\begin{aligned} |m\langle x_n, a, x_m \rangle| &\leq m\langle x_n, a, x_0 \rangle + m\langle x_0, a, x_m \rangle \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ for all } n, m \geq N(\epsilon, a). \end{aligned}$$

Hence $\{x_n\}$ is a sur-Cauchy sequence. ■

Theorem 4.11. *Every ide-convergent sequence in a ide-semi-pre-metron (X, m) is a ide-Cauchy sequence.*

Theorem 4.12. *Let (X, m) be a sur-semi-pre-metron with I -survival property and the property of triangle inequality of difference. Then each I -sur-convergent sequence is an I -sur-Cauchy sequence.*

The examples given below show that an ide-Cauchy (sur-Cauchy) sequence is not necessarily an ide-convergent (resp. sur-convergent) sequence.

Example 4.13. Let $X =]0, \infty[\cup \{-1\}$ and $f : X \rightarrow \mathbb{R}$ given by $f(x) = |x|$, for all $x \in X$. Define $m : X \times X \times X \rightarrow \mathbb{R}$ by

$$m\langle x, y, z \rangle = ||x - f(y)| - |f(y) - z||, \text{ for all } x, y, z \in X.$$

Then (X, m) is a sur-semi-pre-metron. Consider a sequence $\{\frac{1}{n}\}$ in X , Then $\{\frac{1}{n}\}$ is a sur-Cauchy sequence in X but it is not sur-convergent in X because $0 \notin X$.

Example 4.14. Let $X =]0, \infty[\cup \{-1\}$ and $f : X \rightarrow \mathbb{R}$ given by $f(x) = |x|$, for all $x \in X$. Define $m : X \times X \times X \rightarrow \mathbb{R}$ by

$$m\langle x, y, z \rangle = ||f(x) - y| - |y - f(z)||, \text{ for all } x, y, z \in X.$$

Then (X, m) is a ide-semi-pre-metron. Consider a sequence $\{\frac{1}{n}\}$ in X . Then $\{\frac{1}{n}\}$ is a ide-Cauchy sequence in X but it is not ide-convergent in X because $0 \notin X$.

Definition 4.15. Let (X, m) be a sur-semi-pre-metron. Then $A \subset X$ is said to be sur-complete if each sur-Cauchy sequence in A is sur-converges in A .

Definition 4.16. Let (X, m) be a ide-semi-pre-metron. Then $A \subset X$ is said to be ide-complete if each ide-Cauchy sequence in A is ide-converges in A .

Definition 4.17. Let (X, m) be a sur-semi-metron, with I -survival property and the triangle inequality of difference property. Then $A \subset X$ is said to be I -suromplete if each I -sur-Cauchy sequence in A is I -sur-converges in A .

The sur-semi-pre-metron (X, m) defined in Example 4.13 is sur-complete and the ide-semi-pre-metron (X, m) defined in Example 4.14 is ide-complete.

The following theorems can be easily verified.

Theorem 4.18. Let (X, m) be a sur-semi-pr-metron and let $\{x_n\}$ be a sequence in X which sur-converges in X , then $\{x_n\}$ sur-converges to an unique point in X .

Theorem 4.19. Let (X, m) be a ide-semi-pre-metron and let $\{x_n\}$ be a sequence in X which ide-converges in X , then $\{x_n\}$ ide-converges to an unique point in X .

Theorem 4.20. Let (X, m) be a sur-semi-pre-metron with I -survival property and the triangle inequality of difference property, and let $\{x_n\}$ be a sequence in X which I -sur-converges in X , then $\{x_n\}$ I -sur-converges to an unique point in X .

Theorem 4.21. Let (X, m) be a semi-pre-metron with EBS-property and (X, d_s) be the metric space, established by the metric

$$d_s(x, y) = \sup_{a \in X} \{m\langle x, a, y \rangle\}.$$

If $\{x_n\}$ is a Cauchy sequence (resp. a convergent sequence) that converges to x_0 in (X, d_s) , then it is a sur-Cauchy sequence (resp. a sur-convergent sequence) that sur-converges to x_0 .

Theorem 4.22. Let (X, m) be a an ide-semi-pre-metron with EMS-property, and (X, d_i) be a metric space, where

$$d_i(x, y) = \sup_{u, v \in X} \{|m\langle u, x, v \rangle - m\langle u, y, v \rangle|\}.$$

If $\{x_n\}$ is a Cauchy sequence (resp. a convergent sequence) that converges to x_0 in (X, d_i) , then it is an ide-Cauchy sequence (resp. an ide-convergent sequence) that ide-converges to x_0 .

Theorem 4.23. Let (X, m) be a sur-semi-pre-metron with I -survival property, the triangle inequality of difference property and the EBDs-property. Let (X, d_{is}) be a metric space, where

$$d_{is}(x, y) = \sup_{u, v \in X} \{|m\langle x, u, y \rangle - m\langle x, v, y \rangle|\}.$$

If $\{x_n\}$ is a Cauchy sequence (resp. a convergent sequence) that converges to x_0 in (X, d_{is}) , then it is a I -sur-Cauchy sequence (resp. an I -sur-convergent sequence) that I -sur-converges to x_0 in (X, m) .

Definition 4.24. Let (X, m) be a sur-semi-pre-metron and $A \subset X$. A point $p \in X$ is called a sur-cluster point of A if there exists a sequence $\{x_n\} \subset A$ of distinct points such that $\{x_n\}$ sur-converges to p .

Definition 4.25. Let (X, m) be a sur-semi-pre-metron and $F \subset X$, then F is said to be a sur-closed set in X if F contains all its sur-cluster points in X .

Theorem 4.26. Let (X, m) be a sur-complete sur-semi-pre-metron and $A \subset X$. Then A is a sur-closed set in X if and only if A is sur-complete in X .

Proof. Let (X, m) be a sur-complete sur-semi-pre-metron and $A \subset X$. Suppose that A is a sur-closed set in X . Then A contains all its sur-cluster points in X . If $\{x_n\}$ is a sur-Cauchy sequence in A , then $\{x_n\}$ sur-converges to x_0 in X , as X is sur-complete. Further, x_0 is a sur-cluster point of $\{x_n\}$, so $x_0 \in A$, as A is sur-closed. Hence each sur-Cauchy sequence in A , sur-converges in A . This shows that A is sur-complete in X .

Conversely, let A is sur-complete in X , so each sur-Cauchy sequence in A sur-converges in A . Suppose that p is a sur-cluster point of A , so there is a sequence $\{x_n\}$ in A which sur-converges to p . But each sur-convergent sequence is a sur-Cauchy sequence in A with $s - \lim_{n \rightarrow \infty} \{x_n\} = p$. Then A is sur-complete $p \in A$, hence A contains all its sur-cluster points. Thus, A is sur-closed in X . ■

Definition 4.27. Let (X, m) be a ide-semi-pre-metron and $A \subset X$, then a point $p \in X$ is called a ide-cluster point of A if there exists a

sequence $\{x_n\} \subset A$ of distinct points such that $\{x_n\}$ ide-converges to p .

Definition 4.28. Let (X, m) be a ide-semi-pre-metron and $F \subset X$, then F is said to be a ide-closed set in X if F contains all its ide-cluster points in X .

Theorem 4.29. Let (X, m) be a ide-complete ide-semi-pre-metron and $A \subset X$. Then A is a ide-closed set in X if and only if A is ide-complete in X .

Definition 4.30. Let (X, m) be a sur-semi-pre-metron with I -survival property and the triangle inequality of difference property, and $A \subset X$, then a point $p \in X$ is called a I -sur-cluster point of A if there exists a sequence $\{x_n\} \subset A$ of distinct points such that $\{x_n\}$ I -sur-converges to p .

Definition 4.31. Let (X, m) be a sur-semi-pre-metron with I -survival property and the triangle inequality of difference property. Then $F \subset X$ is said to be a I -sur-closed set in X if F contains all its I -sur-cluster points in X .

Theorem 4.32. Let (X, m) be a I -sur-complete sur-semi-pre-metron with I -survival property and the triangle inequality of difference property and $A \subset X$. Then A is a I -sur-closed set in X if and only if A is I -sur-complete in X .

Proof. Let (X, m) be a I -sur-complete sur-semi-pre-metron and $A \subset X$. Suppose that A is a I -sur-closed set in X , then A contains all its I -sur-cluster points in X . If $\{x_n\}$ is a I -sur-Cauchy sequence in A , then $\{x_n\}$ I -sur-converges to x_0 in X , as X is I -sur-complete. Further, x_0 is a I -sur-cluster point of $\{x_n\}$, so $x_0 \in A$, as A is I -sur-closed. Hence, each I -sur-Cauchy sequence in A , I -sur-converges in A . This shows that A is I -sur-complete in X .

Conversely, let A be I -sur-complete in X . Since A is I -sur-complete in X , each I -sur-Cauchy sequence in A , I -sur-converges in A . Suppose p is a I -sur-cluster point of A , so there is a sequence $\{x_n\}$ in A which I -sur-converges to p . But each I -sur-convergent sequence is a I -sur-Cauchy sequence in A with $\lim_{n \rightarrow \infty} \{x_n\} = p$. As A is I -sur-complete, $p \in A$. Hence A contains all its I -sur-cluster points. Thus, A is I -sur-closed in X . ■

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