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ON THE INVARIANTS OF PARTIAL PARALLELIZABLE MANIFOLDS AND INTEGRABLE PARTIAL TRIVIAL STRUCTURES IN TANGENT BUNDLE

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Abstract. In this paper the concepts of "partial parallelizable manifold" and "partial trivial tangent bundle" are fundamental notions. The aim of this paper is to obtain some properties about partial parallelizable manifold (M, ρ_p) and their relations to integrable partial trivial structures in tangent bundle. In the study of the couple (M, ρ_p) an important operation on ρ_p is the construction of other geometric and topological objects: $G(M)-$invariants, homotopy invariants and their relations with Pontryagin classes of tangent bundle TM. We give a description of Pontryagin algebra of partial trivial tangent vector bundle of M: if $\mathcal{F}$ is a foliation of M we show that TM is null in dimension $d > 2\,(m-p)$, where $m = \dim M$ and $p = \dim \mathcal{F}$.

The case of integrable partial trivial structures is considered. If the subbundle $E\,(\rho_p)$ generated from ρ_p is integrable, then there is a $G\,(M)$-invariant $\left[\widetilde{M}\,(\rho_p)\right]$ associated with (M, ρ_p). Finally, we give some examples.

1. Introduction

This paper is a continuation of [1, 2, 3, 4, 5, 6]. Here, the manifolds are paracompact and $C^\infty-$differentiable. The morphisms are of constant rank and of $C^\infty-$class. Also, vector fields are of $C^\infty-$class.

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Definitions.

1) A differentiable manifold M is parallelizable if its tangent bundle TM admits a global fame $\rho_m = (X_1, X_2, ..., X_m)$, $m = \dim M$.

2) If TM admits some global vector fields $\rho_p = (X_1, ..., X_p)$, $1 \leq p < m$, in arbitrary $x \in M$, where $X_1(x)$, ,$X_p(x)$ are linearly independent, then we say that M is a partial parallelizable manifold.

If $p = m$ the notion of partial parallelizable manifold does not make sense.

We can obtain some properties of partial parallelizable manifolds using vector bundles and parallelizable Grassmann manifolds ([10, 12, 13, 14]). In order to study the geometry of integrable and non-integrable partial trivial structures, it is important to see certain properties of the partial parallelizable manifolds. A differentiable manifold M endowed with a global partial frame ρ_p of the tangent bundle TM is called a partial parallelizable manifold. To justify this terminology, we consider two arbitrary vector fields $v = \sum_{i=1}^{p} v^i X_i$, $w = \sum_{k=1}^{p} w^k X_k$, $\rho_p = (X_1, X_2, ..., X_p)$. Then v and w coincide if and only if $v^i = w^i$, $i = 1, 2, ..., p$. This fact proves that for $p = m$, the couple (M, ρ_m) is a parallelizable manifold. The tangent vector bundle TM that satisfies the preceding condition is called a partial trivial bundle.

The group consisting of global diffeomorphisms of M on M is denoted by $G(M)$.

We recall some results on $G(M)-$invariants and homotopy invariants that will be used later. In order to separate the geometric properties of the partial parallelizable manifold (M, ρ_p) from the topological properties, we use the isomorphism classes $\{E(\rho_k)\}$ and $\{TM/E(\rho_k)\}$ corresponding to global partial frames $\rho_1 \subset \rho_2 \subset .... \subset \rho_p$. There exist some relations between the partial parallelism of the manifold M and partial trivial structure of its tangent bundle TM.

In [5] we prove that the existence of a partial trivial structure in TM is reduced to the existence of a lift of Gauss map in a certain diagram. In this paper, we obtain some properties of a partial parallelizable manifold (M, ρ_p). We verify that the manifold M is partial parallelizable if and only if TM is partially trivial (Lemma 2.1 and Theorem 3.1).

The Pontryagin algebra of TM is considered, too (Section 3). For a partial parallelizable manifold (M, ρ_p) we have obtained $Pont^j(TM) =$

0 for $j > 2(m-p)$ (Theorem 3.1), where $Pont^j(TM)$ are the components of Pontryagin algebra of TM.

In Section 4 we investigate the particular case of integrable partial trivial structures. Consider the subbundle $E(\rho_p)$ spanned from $\rho_p = (X_1, ..., X_p)$. If $E(\rho_p)$ is integrable, then there exist some homotopy classes associated with TM and $E(\rho_p)$ (Theorem 4.4).

Finally, in Section 5, some explicit examples are given.

2. Invariants of partial parallelizable manifolds

2.1. **Preliminaries.** In this section our purpose is to describe relations between some algebraic invariants and homotopy invariants of a partial parallelizable manifold. Let M be a differentiable paracompact manifold of dimension m, TM its tangent bundle and p is a fixed natural number, $1 \leq p < m$. Let $\rho_p = (X_1, X_2, ..., X_p)$ a global partial frame of TM, i.e. the vectors $(X_1(x), ..., X_p(x))$ are linearly independent for all $x \in M$. Also we say that a global partial frame ρ_p is a partial parallelism on manifold M. The frame ρ_p enables us to define some $G(M)-$invariants and homotopy invariants of (M, ρ_p). An element $\varphi \in G(M)$ induces a bijection φ' on the vector field X defined by:

$$\varphi'(X)(\varphi(x)) = (\varphi(x); \varphi'_x(X(x))),$$

where φ'_x denotes the derivative of φ at $x \in M$. Then, the action of φ on ρ_p is defined by relation $(\varphi, \varphi')(\rho_p) = (\varphi(x); \varphi'_x(X_1(x)), ..., \varphi'_x(X_p(x)))$, $x \in M$. Now, we consider the subbundle spanned from $\rho_p : \xi(\rho_p) = (E(\rho_p), \Pi^p, M)$ with total space $E(\rho_p) = \cup_{x\in M} E_x(\rho_p)$, where $E_x(\rho_p)$ is the vector subspace of T_xM spanned from $\rho_p(x) = (X_1(x), ..., X_p(x))$, $x \in M$. Because M is paracompact manifold, the following sequence $O \to E(\rho_p) \to TM \to TM/E(\rho_p) \to O$ is exact. Now, for fixed p, let $\Theta^p = (M \times \mathcal{R}^p \to M)$ be the product bundle over M and fiber $\mathcal{R}^p$. We observe that it is a global frame of $E(\rho_p)$. Hence $E(\rho_p)$ and Θ^p are isomorphic, $E(\rho_p) \simeq \Theta^p$. Finally, preceding considerations give the following isomorphism of vector bundles over M: $TM \simeq (M \times \mathcal{R}^p) \oplus TM/E(\rho_p)$. Conversely, let $TM \simeq (M \times \mathcal{R}^p) \oplus TM/E$ be the structure of TM, where E isomorphic to Θ^p. Then E admits a global partial frame σ_p, where σ_p defines a partial parallelism on M. Then it follows:

Lemma 1. *Let M be a $C^\infty-$differentiable and paracompact manifold of dimension m, TM its tangent bundle and p an integer, $1 \leq p < m$. Then M is a partial parallelizable manifold (M, ρ_p) if and only if*

TM is a partial trivial bundle. The structure of TM is given by the isomorphism $TM \simeq (M \times \mathbb{R}^p) \oplus TM/E(\rho_p)$, $E(\rho_p) \simeq M \times \mathbb{R}^p$.

Remark. The vector bundle $TM/E(\rho_p)$ describes the deviation of the tangent bundle TM from trivialization. Hence, $TM/E(\rho_p)$, $1 \leq p < m$, gives an obstruction of (M, ρ_p) from a parallelizable manifold.

In this paper, we identify a partial parallelism on M with a global partial frame of TM. In the next section, we emphasize this idea.

2.2. **Homotopy invariants of partial parallelizable manifolds.** Let G_k be the Grassmann variety consisting of the set of $k-$subspaces of $\mathbb{R}^t$, $t = 1, 2, ...$ Denote by $Vect_k(M)$ the set of isomorphism classes of vector bundles over M and of fiber $\mathbb{R}^k$, $k = 1, 2, ..., p$. Let $[M, G_k]$ be the set of homotopy classes of maps $f : M \to G_k$.

Lemma 2.2 ([[11], p. 33], homotopy classification) There exists a bijection $f_k : Vect_k(M) \to [M, G_k]$.

For fixed f_k, there exists a unique homotopy class $c_k(M) \in [M, G_k]$ defined by $c_k(M) = f_k(\{M \times \mathbb{R}^k\})$, where $\{M \times \mathbb{R}^k\}$ denotes the isomorphism class of vector bundles isomorphic to $M \times \mathbb{R}^k$.

Analogously, we consider the isomorphisms classes of vector bundles over M, $\{TM/E(\rho_k)\} \in Vect_{m-k}(M)$, $\{TM\} \in Vect_m(M)$ and the bijections $f_{m-k} : Vect_{m-k}(M) \to [M, G_{m-k}]$, $f_m : Vect_m(M) \to [M, G_m]$. The partial parallelizable manifold (M, ρ_k) has associated the homotopy invariants $c_{m-k}(M) = f_{m-k}(\{TM/E(\rho_k)\})$, $c_m(M) = f_m(\{TM\})$ [6]. So we conclude that the preceding relations compute these homotopy classes in terms of isomorphisms classes of vector bundles over M. The homotopy classes $c_k(M) = f_k(\{M \times \mathbb{R}^k\})$ are uniquely determined by bijections f_k and the condition to contain a map $h_k : M \to G_k$, $h_k =$ constant [6]. We can verify that $c_m(M) = (c_k(M)), c_{m-k}(M)$ [6]. As a consequence, these homotopy classes permit distinguishing between the tangent bundle of a parallelizable manifold and the tangent bundle of a partial parallelizable manifold.

Remark. A change bijection $f_s \to \widetilde{f}_s$, $s = p, m-p, m$ modifies the classes $c_s(M)$ by relations of the form $c_p(M) = f_p \widetilde{f}_p^{-1}(\widetilde{c}_p(M))$ etc. These classes are invariants of transformations $\rho_s \to \widetilde{\rho}_s$.

3. Pontryagin algebra and homotopy classes of partial parallelizable manifolds

3.1. **Vanishing theorem for Pontryagin algebra of tangent bundle.** Our next goal of this section is to establish some relations between the $G(M)$-invariants, the homotopy invariants and the Pontryagin algebra of (M,ρ_p). We use the geometric definition of the Pontryagin classes as real cohomology classes represented by differential forms. The isomorphism classes $\{TM\}$, $\{TM/E(\rho_k)\}$, $\{E(\rho_k)\}$ constitute a link between the homotopy classes $c_m(M)$, $c_{m-k}(M)$, $c_k(M)$ and the total Pontryagin classes $p(TM)$, $p(TM/E(\rho_k))$, $p(E(\rho_k))$, $k=1,2,...,p$, respectively. These classes are topological invariants of M and depend essentially only on ρ_p and on characteristic homomorphism [9]. In other words, two different homotopy classes give rise to the same total Pontryagin class.

Theorem 2. *Let (M,ρ_p) be a paracompact, $C^\infty-$ differentiable and partial parallelizable manifold, $1\le p<m$. Let $E(\rho_p)$ be the tangent subbundle spanned by $\rho_p=(X_1,...,X_p)$. Then $Pont^j(TM)=0$ for $j>2(m-p)$.*

Proof. We use the isomorphism theorem $TM\simeq E(\rho_p)\oplus TM/E(\rho_p)$, $E(\rho_p)\simeq M\times\mathbb{R}^p$, and cup-multiplication between Pontryagin classes. Remark that if two vector bundles are isomorphic, then their Pontryagin algebras are isomorphic. The total Pontryagin class of TM is defined by $p(TM)=\left[\det\left(I_m+\frac{\sqrt{-1}}{2\pi}\Omega\right)\right]\in H^*(M;\mathbb{R})$ where Ω is the curvature form of a connection in $TM\simeq(M\times R^p)\oplus TM/E(\rho_p)$. Therefore

$$p(TM)=\left[\det\left(I_m+\frac{\sqrt{-1}}{2\pi}\begin{pmatrix}\Omega_1 & 0\\ 0 & \Omega_2\end{pmatrix}\right)\right]=p(E(\rho_p))p(TM/E(\rho_p)),$$

where Ω_2 is the curvature form of a connection of $TM/E(\rho_p)$. The symbol $[;]$ denotes the cohomology classes. Since $E(\rho_p)$ is a trivial vector bundle there exists a flat connection of $TM/E(\rho_p)$, i.e. its curvature is null, $\Omega_1=0$. Then, precedent relation gives:

(1)
$$p(E(\rho_p))=1 \text{ and } p(TM)=p(E(\rho_p)\oplus TM/E(\rho_p))=p(TM/E(\rho_p)).$$

Now, we use the components of total Pontryagin classes:

$$(2)\quad p(TM)=1+p_1(TM)+...+p_{m-p}(TM)+...+p_m(TM).$$

$$(3)\quad p(TM/E(\rho_p))=1+p_1(TM/E(\rho_p))+...+p_{m-p}(TM/E(\rho_p)).$$

Comparing the relations (2) and (3), then relation (1) gives:

$$p_{m-p+1}(TM)=...=p_m(TM)=0.$$

Consequence: Pontryagin algebra vanishes in dimension greater than $2(m-p)$. ☐

Remark. If $p = m$ then $TM \simeq M \times \mathbb{R}^m$ and hence $p(TM) = 1$.

3.2. **Relations between Pontryagin classes and homotopy classes of partial parallelizable manifolds.** In this subsection we associate to a homotopy class and a bijection, a certain Pontryagin class of (M, ρ_p).

Theorem 3. *Let (M, ρ_p) be a paracompact, $C^\infty-$differentiable and partial parallelizable manifold, TM its tangent bundle. Then:*

1) There exists a $G(M)-$invariant $A_3 : [M, G_p \times G_{m-p}] \to Pont(TM)$, $A_3(c_m(M)) = (1, h_2 f_{m-p}^{-1}(c_{m-p}(M)))$;

2) The total Pontryagin class of $TM - TM/E(\rho_k)$ is given by the relation:
$p(TM - TM/E(\rho_k)) = 1 + p_1(TM - TM/E(\rho_k)) + ... + p_{m/2}(TM - TM/E(\rho_k))$, *$m =$even number, $k = 1, 2, ..., p$.*

Proof. 1) Using the isomorphisms $TM \simeq E(\rho_p) \oplus TM/E(\rho_p)$, $E(\rho_p) \simeq M \times R^p$ and the definition of the fiber product [[9], p. 15], we identify $\{TM\} = (\{E(\rho_p)\}, \{TM(E(\rho_p))\})$.

Let h_1, h_2, h_3 be the characteristic homomorphisms of the bundles $E(\rho_p)$, $TM/E(\rho_p)$ and TM, respectively [8].

Consider the following maps:
$A_1 : [M, G_p] \to Pont(E(\rho_p))$, $A_1(c_p(M)) = h_1 f_p^{-1}(c_p(M)) = 1$;
$A_2 : [M, G_{m-p}] \to Pont(TM/E(\rho_p))$, $A_2(c_{m-p}(M)) = p_2(\{TM/E(\rho_p)\}) = h_2 f_{m-p}^{-1}(c_{m-p}(M)) = h_2(TM/E(\rho_p))$;
$A_3 : [M, G_m] \to Pont(TM)$, such that
$A_3(c_m(M)) = h_3 f_m^{-1}(c_m(M)) = p_3(TM) = (A_1 \times A_2)(c_p(M), c_{m-p}(M)) = (A_1(c_p(M)), A_2(c_{m-p}(M))) = (1,\ h_2 f_{m-p}^{-1}(c_{m-p}(M)))$.
This result proves that the expression of the total Pontryagin class of TM does not depend on the class $\{E(\rho_p)\}$.

2) In the ring $H^*(M; \mathbb{R}) = H^\circ(M; \mathbb{R}) \oplus ... \oplus H^m(M; \mathbb{R})$ the total Pontryagin class of the virtual bundle $TM - TM/E(\rho_k)$ is invertible [7] and hence $p(TM - TM/E(\rho_k)) = p(TM)p(TM/E(\rho_k))$. Then Pontryagin classes $p_s(TM - TM/E(\rho_k))$, $s = 1, 2, ..., m/2$, are determined by $p_s(TM - TM/E(\rho_k)) \in H^{2s}(M; \mathbb{R})$, $p(TM)/p(TM/E(\rho_k)) = 1 + p_1(TM - TM/E(\rho_k)) + ... + p_{m/2}(TM - TM/E(\rho_k))$, $k = 1, 2, ..., p$, where m is an even number. □

Remark. There exists a result similar to 1) for $\rho_k \subset \rho_p$ where k is a fixed number, $1 \leq k < p$.

4. Integrable partial trivial structures

4.1. **Foliation defined by a integrable partial trivial structure.** By the lemma 1, the partial parallelizability of a manifold M is equivalent to the partial triviality of the tangent bundle TM. Using this fact, we present the notion of an integrable partial trivial structure and some properties of the associated invariants.

Definition. An integrable partial trivial structure in the tangent bundle is defined by a subbundle of TM that is trivial andintegrable.

In this case, the integral manifold of subbundles that define the integral partial trivial structure are parallelizable submanifolds of M. The invariants of (M, ρ_p) determine certain invariants of the integral manifolds.

Denote by $\widetilde{M}(\rho_p)$ the foliation defined by integrable tangent subbundle $E(\rho_p)$ for a fixed natural number p, $1 \leq p < m$. Suppose that the leaves of $\widetilde{M}(\rho_p)$ are embedded submanifolds of M (topology of leaves of $\widetilde{M}(\rho_p)$ is induced by topology of M, i.e. $\widetilde{M}(\rho_p)$ is an embedded foliation). We identify $\widetilde{M}(\rho_p)$ with its image $i(\widetilde{M}(\rho_p))$ where $i : \widetilde{M}(\rho_p) \to M$ is the inclusion map. The action of group $G(M)$ on (M, ρ_p) defines an equivalence class of $\widetilde{M}(\rho_p)$ under this action: $[\widetilde{M}(\rho_p)] = \{\widetilde{M}((\varphi, \varphi')(\rho_p))/\varphi \in G(M)\}$. It is clear that for each $\varphi \in G(M)$ the leaves of $\widetilde{M}((\varphi, \varphi')(\rho_p))$ form a class of diffeomorphic and parallelizable submanifolds of M. From the definition of $\widetilde{M}((\varphi, \varphi')(\rho_p))$, $\varphi \in G(M)$, it follows that this foliation is embedded in M. Indeed, for $\varphi \in G(M)$ and $x \in M$, we have the relations

$$(\varphi, \varphi'^{-1}(\varphi(x)) = (\varphi^{-1}(x), (\varphi'^{-1}(\varphi(x))) = (\varphi^{-1}(x), (\varphi'^{-1}).$$

Now, we consider an arbitrary leaf $F \in \widetilde{M}(\rho_p)$ and $F_\varphi \in \widetilde{M}((\varphi, \varphi')(\rho_p))$. Clearly, each $\varphi \in G(M)$ defines an isomorphism between the Lie algebras $\chi(F)$ and $\chi(F_\varphi)$ of the vector fields of F and F_φ, respectively. We proved the following theorem:

Theorem 4. *Let (M, ρ_p) be a C^∞-differentiable, paracompact, partial parallelizable manifold, $E(\rho_p)$ the integrable subbundle spanned by ρ_p and $\widetilde{M}(\rho_p)$, $\widetilde{M}((\varphi, \varphi')(\rho_p))$ the embedded foliations defined by $E(\rho_p)$ and $E((\varphi, \varphi')(\rho_p))$, respectively. Consider an arbitrary leaf $F \in \widetilde{M}(\rho_p)$, $F_\varphi \in \widetilde{M}((\varphi, \varphi')(\rho_p))$ and the Lie algebras of vectorfields $\chi(F)$, and $\chi(F_\varphi)$. Then:*

1) The partial parallelizable manifold (M, ρ_p) has an associated $G(M)-$invariant, i.e. the class $[\widetilde{M}(\rho_p)]$.

2) The algebras $\chi(F)$ *and* $\chi(F_\varphi)$ *are isomorphic.*

Proposition 5. *Let* (M,ρ_p) *be a* $C^\infty-$*differentiable, paracompact, partial parallelizable manifold,* $E(\rho_p)$ *integrable subbundle spanned by* ρ_p*, and* $i:\widetilde{M}(\rho_p)\to M$ *the inclusion map of foliation* $\widetilde{M}(\rho_p)$ *of* $E(\rho_p)$*. Then, the total Pontryagin class of* $i^*(TM)$ *is given by*

$$p(i^*(TM)) = i^*(p(T\widetilde{M}(\rho_p)))i^*(p(TM/T\widetilde{M}(\rho_p))).$$

Proof. Using the relations $p(i^*(TM/T\widetilde{M}(\rho_p))) = i^*(p(TM/T\widetilde{M}(\rho_p)))$, $p(i^*(T\widetilde{M}(\rho_p))) = i^*(p(T\widetilde{M}(\rho_p)))$, we obtain

$$p(i^*(TM)) = i^*(p(T\widetilde{M}(\rho_p)))i^*(p(TM/T\widetilde{M}(\rho_p)))\ (\text{cup}-\text{multiplication}).$$

□

4.2. **Homotopy invariants of integrable partial trivial structures.** Since "homotopy invariants" are "homotopy classes", we use in this section the last expression. Our purpose is to obtain some homotopy classes of induced vector bundles over manifold $\widetilde{M}(\rho_p)$. The relation between these classes is presented, too.

Theorem 6. *Let* (M,ρ_p) *be a* C^∞*-differentiable, paracompact, partial parallelizable manifold,* TM *its tangent vector bundle and* $E(\rho_p)$ *the subbundle spanned by* ρ_p*. If* $E(\rho_p)$ *is integrable then:*
1) There exist homotopy classes $\widetilde{c}_m(\widetilde{M}(\rho_p)) = \widetilde{f}_m(\{i^*u(TM)\})$, $\widetilde{c}_{m-p}(\widetilde{M}(\rho_p)) = \widetilde{f}_{m-p}(\{i^*u(TM/T\widetilde{M}(\rho_p))\})$, $\widetilde{c}_p(\widetilde{M}(\rho_p)) = \widetilde{f}_p(\{i^*u(T\widetilde{M}(\rho_p))\})$ *uniquely determined for fixed bijections* $\widetilde{f}_m$, $\widetilde{f}_{m-p}$, $\widetilde{f}_p$*. These classes satisfy relations*

$$\widetilde{c}_m(\widetilde{M}(\rho_p)) = (\widetilde{c}(\widetilde{M}(\rho_p)), \widetilde{c}_{m-p}(\widetilde{M}(\rho_p))),$$

where $u: TM\to T\widetilde{M}(\rho_p)\oplus TM/T(\widetilde{M}(\rho_p))$ *is an isomorphism and* i^*u *denotes the induced isomorphism under embedding* $i:\widetilde{M}(\rho_p)\to M$*.*
2) The homotopy classes of $T\widetilde{M}(\rho_p), TM/T\widetilde{M}(\rho_p)$ *satisfy the relation* $c_m(M) = (c_p(M),\ c_{m-p}(M))$ *where* $c_p(M) = f_p(\{T\widetilde{M}(\rho_p)\})$, $c_{m-p}(M) = f_{m-p}(\{TM/\ T\widetilde{M}(\rho_p)\})$, $c_m(M) = f_m(\{TM\})$ *and* f_p, f_{m-p}, f_m *are bijections given by Lemma 2.2.*

Proof. We study the properties of homotopy classes of induced bundles over manifold $\widetilde{M}(\rho_p)$. Let $i:\widetilde{M}(\rho_p)\to M$ be the embedding and $u: TM\simeq T\widetilde{M}(\rho_p)\oplus TM/T\widetilde{M}(\rho_p)$, the isomorphism given by

Lemma 1. Denote i^*u the induced isomorphism of u under the embedding i and defined by $i^*u : i^*(TM) \to i^*(T\widetilde{M}(\rho_p) \oplus TM/T\widetilde{M}(\rho_p))$, $i^*u(x,y) = (x, u(y)) = (x, u(y_1), u(y_2))$ for $y = (y_1, y_2)$, $y_1 \in T\widetilde{M}(\rho_p)$, $y_2 \in TM/T\widetilde{M}(\rho_p)$. For simplicity we denote:

$$A = \{i^*u(TM)\},\ B = \{i^*u(T\widetilde{M}(\rho_p))\},\ C = \{i^*u(TM/T\widetilde{M}(\rho_p))\}.$$

Then, for fixed bijections $\widetilde{f}_s$ (given by Lemma 2.2), $\widetilde{f}_s : Vect_s(\widetilde{M}(\rho_p)) \to [\widetilde{M}(\rho_p), G_s]$, $s = p, m-p, m$, we obtain the homotopy classes:

$$\widetilde{c}_m(\widetilde{M}(\rho_p)) = \widetilde{f}_m(A), \widetilde{c}_{m-p}(\widetilde{M}(\rho_p)) = \widetilde{f}_{m-p}(C), \widetilde{c}_p(\widetilde{M}(\rho_p)) = \widetilde{f}_p(B).$$

It is clear that these classes are uniquely determined for fixed bijections. Now, we describe the relations between these classes. Denote $\widetilde{f}_m = \widetilde{f}_p \oplus \widetilde{f}_{m-p} = (\widetilde{f}_p, \widetilde{f}_{m-p})$ the direct sum of fixed precedent bijections.
Then $\widetilde{c}_m(\widetilde{M}(\rho_p)) = \widetilde{f}_m(A) = (\widetilde{f}_p \oplus \widetilde{f}_{m-p})(B \oplus C) = (\widetilde{f}_p(B), \widetilde{f}_{m-p}(C)) = (\widetilde{c}_p(\widetilde{M}(\rho_p)), \widetilde{c}_{m-p}(\widetilde{M}(\rho_p)))$. Then affirmation (1) follows. Finally, using a similar argument and results from section "Preliminaries", we obtain the relation

$$c_m(M) = (c_p(M), c_{m-p}(M)).$$

□

Briefly speaking: Since $f_s, \widetilde{f}_s$, $s = p, m-p, m$ are bijections, the homotopy classes $c_s(M)$ and $\widetilde{c}_s(\widetilde{M}(\rho_p))$ give a topological interpretation for a certain isomorphism class of bundles.

Analysis of excepted cases $p = 1, p = m$**.** It is clear that $E(\rho_1)$ and $E(\rho_m)$ are integrable subbundles.

Case $p = 1$. Let $\rho_1 = (X)$ be the global partial frame of TM and $\xi_1 = (E(\rho_1), \pi^1, M)$ denote the subbundle of TM spanned by ρ_1, where $E(\rho_1) = \cup_{x \in M} E_x(\rho_1)$, $E_x(\rho_1)$ is the vector space generated by $\rho_1(x) = X(x)$, $x \in M$. The subbundle ξ_1 is trivial, $E(\rho_1) \simeq M \times \mathbb{R}$. Denote $\widetilde{M}(\rho_1)$ the foliation defined by $E(\rho_1)$ and let $u : TM \to T\widetilde{M}(\rho_2) \oplus TM/T\widetilde{M}(\rho_1)$ be the isomorphism given by Lemma 1. The embedding $i : \widetilde{M}(\rho_1) \to M$ induces an isomorphism $i^*u : i^*(TM) \to i^*(T\widetilde{M}(\rho_1) \oplus TM/T\widetilde{M}(\rho_1))$, $i^*u(x,y) = (x, u(y))$, $x \in \widetilde{M}(\rho_1)$, $y \in T\widetilde{M}(\rho_1) \oplus TM/T\widetilde{M}(\rho_1)$. If the bijections $\widetilde{f}_s : Vect_s(\widetilde{M}(\rho_1)) \to [\widetilde{M}(\rho_1), G_s]$, $s = 1, m-1, m$, are fixed, we uniquely obtain the homotopy classes $\widetilde{c}_1(\widetilde{M}(\rho_1)) = \widetilde{f}_1(\{i^*u(TM)(\rho_1)\})$, $\widetilde{c}_{m-1}(\widetilde{M}(\rho_1)) = \widetilde{f}_{m-1}(\{i^*u(T\widetilde{M}(\rho_1))\})$, $\widetilde{c}_m(\widetilde{M}(\rho_1)) = \widetilde{f}_m(\{i^*u(TM)\})$. It is clear that $\widetilde{c}_m(\widetilde{M}(\rho_1)) = (\widetilde{c}_1(\widetilde{M}(\rho_1)))$. A similar result for the homotopy classes of $T\widetilde{M}(\rho_1)$, $TM/T\widetilde{M}(\rho_1)$ is given by relation $c_m(M) = (c_1(M), c_{m-1}(M))$, where $c_1(M) = [h_1]$, $h_1 : M \to G_1$, $h_1 = \text{constant}$.

Case $p = m$. Now, $E(\rho_m) = TM \simeq M \times \mathbb{R}^m$ and therefore this is a singular case.

4.3. **Integrable partial trivial structures defined by foliations.** The notion of parallelizable foliation is inspired by the notion of parallelizable manifold.

Definition. A foliation $\mathcal{F}$ of a differential manifold M is parallelizable if each leaf of $\mathcal{F}$ is a parallelizable manifold.

Lemma 7. *Let M be a C^∞-differentiable, paracompact manifold of dimension m, TM its tangent bundle, $\mathcal{F}$ a foliation of M of dimension k, $1 < k < m$. Let $T\mathcal{F}$ be the tangent bundle of $\mathcal{F}$. Then $T\mathcal{F}$ defines a integrable partial trivial structure in TM if and only if $\mathcal{F}$ is a parallelizable foliation.*

Proof. Since every leaf of F is a parallelizable manifold, TF is trivial: $TF \simeq M \times \mathbb{R}^k$. But M is a paracompact manifold, and thus there is an isomorphism of vector bundles: $TM \simeq TF \oplus TM/TF \simeq (M \times \mathbb{R}^k) \oplus TM/TF$. Conversely, if TF defines a partial trivial structure in TM, we have the isomorphism $TF \simeq M \times \mathbb{R}^k$. It follows that TF admits a global frame σ_k. Hence (F, σ_k) is a parallelizable foliation. □

A relation between the partial parallelism of M and parallelizable foliation is given by the following

Lemma 8. *Let M be a C^∞-differentiable, paracompact manifold of dimension m, TM its tangent bundle and $\mathcal{F}$ a foliation of M, $\dim \mathcal{F} = k$, $1 < k < m$. Then $\mathcal{F}$ defines a parallelism on M if and only if $\mathcal{F}$ is a parallelizable foliation.*

Proof. If F is a parallelizable foliation then F defines a partial trivial structure in TM, i.e. $TF \simeq M \times \mathbb{R}^k$. It follows that TF admits a global frame σ_k. Therefore, the couple (M, σ_k) is a parallelizable manifold.

Conversely: Assume that F defines a partial parallelism on M, i.e. a partial frame σ_k of TM. Then σ_k is a frame of TF, and hence F is parallelizable foliation. □

As a consequence of the preceding results, we have the following

Theorem 9. *Let M be a C^∞-differentiable, paracompact, manifold of dimension m, TM its the tangent bundle and p an integer number, $1 < p < m$. Let $\mathcal{F}$ be a foliation of M of dimension p. Then the following properties are equivalent:*

1) The foliation $\mathcal{F}$ is parallelizable;

2) The tangent bundle of $\mathcal{F}$, $T\mathcal{F}$, defines a integrable partial trivial structure in TM;

3) The manifold M is partial parallelizable.

4.4. **Infinitesimal automorphism of integrable partial trivial structures.** In this section, we show that some flow of vector fields conserves the foliation $\widetilde{M}(\rho_p)$ defined by the subbundle $E(\rho_p)$ of a partial parallelizable manifold.

Theorem 10. *Let (M, ρ_p) be a C^∞-differentiable, paracompact, partial parallelizable manifold of dimension m, TM its tangent bundle and p an integer number, $1 < p < m$. Suppose that the subbundle $E(\rho_p)$ is integrable and let $\widetilde{M}(\rho_p)$ be the foliation defined by $E(\rho_p)$. Consider a tangent vector field on M, $Y = Y_1 + Y_2$, where Y_1 is tangent to $\widetilde{M}(\rho_p)$ and Y_2 is transverse to $\widetilde{M}(\rho_p)$. In these conditions, vector fields $[Y, X_i]$, $i = 1, 2, ..., p$, $X_i \in \rho_p$, are infinitesimal automorphisms of $\widetilde{M}(\rho_p)$ if and only if the vector field Y_2 is constant along the leaves of $\widetilde{M}(\rho_p)$.*

Proof. It is clear that the leaves of $\widetilde{M}(\rho_p)$ are parallelizable submanifolds. Consider vector fields $[Y, X_k] = [Y_1, X_k] + [Y_2, X_k]$ where $X_k \in \rho_p$. Let $h = (U, x^k, x^{\hat{k}})$ be a local chart of $\widetilde{M}(\rho_p)$, $k = 1, 2, ..., p$, $\hat{k} = p + 1, ..., m$. Use the representations in the chart h of Y_2 and X_k: $Y_2 = Y_2^{\hat{k}} \dfrac{\partial}{\partial x^{\hat{k}}}$, $X_k = X_k^j \dfrac{\partial}{\partial x^j}$. Then $[Y_2, X_k] = Y_2^{\hat{k}} \dfrac{\partial X^j}{\partial x^{\hat{k}}} \dfrac{\partial}{\partial x^j} - X_k^j \dfrac{\partial Y^{\hat{k}}}{\partial x^j} \dfrac{\partial}{\partial x^{\hat{k}}}$. Because $\frac{\partial}{\partial x^{\hat{k}}}$ are transverse fields to $\widetilde{M}(\rho_p)$, it follows that $[Y_2, X_k]$ is tangent to $\widetilde{M}(\rho_p)$ if and only if $\dfrac{\partial Y_2^{\hat{k}}}{\partial x^j} = 0$, i.e. Y_2 is a constant vector field along the leaves of $\widetilde{M}(\rho_p)$. □

5. Examples

1. As a specific example for the definition 4.1, we will show that there is a wide class of integrable partial trivial structures. It is enough to describe the foliations of some product manifolds.

Let $\mathcal{R}^n = (\mathbb{R}^n, \mathbb{R}^n, A)$ be the real manifold, where the atlas A is defined by $A = (\mathbb{R}^n, 1_{\mathbb{R}^n})$. We also consider the differentiable manifold $\mathcal{R}^n_k = (\mathbb{R}^n, \mathbb{R}^n, B)$, where the atlas B is given by the couple $B = (\mathbb{R}^n, \chi_k)$, $\chi_k : \mathbb{R}^n \to \mathbb{R}^n$, $x = (x_1, x_2, ..., x_n)$, $\chi_k(x) = (x_1^{2k_1+1}, x_2^{2k_2+1}, .., , ..., x_n^{2k_n+1})$, $k = (k_1,\, k_2,...,k_n)$, with $k_1,\, k_2,...,k_n$ positive integers. For each k, the chart $(\mathbb{R}^n, \chi_k)$ is not compatible with the atlas A. The manifolds $\mathcal{R}^n_k$ and $\mathcal{R}^n$ are distinct, but these are diffeomorphic manifolds. Now, let M be an arbitrary C^∞-manifold. Consider differentiable manifolds $M \times \mathcal{R}^n_k$ and $M \times \mathcal{R}^n$. The map $(1_M \times \psi) : M \times \mathcal{R}^n_k \to M \times \mathcal{R}^n$ is C^∞-diffeomorphism. Using preceding notations, we obtain the following

Theorem. 5.1. For each C^∞-differentiable manifold M, the manifolds $M \times \mathcal{R}^n_k$ and $M \times \mathcal{R}^n$ are endowed with integrable partial trivial structures. The leaves of these parallelizable foliations are $\{x\} \times \mathcal{R}^n_k$ and $\{x\} \times \mathcal{R}^n$, respectively, $x \in M$.

2. a) Spheres $S^{4k\pm1}$ are partial parallelizable manifolds.

b) If M is a manifold then $M \times S^\ell$, $\ell = 1, 3, 7$ admits an integrable partial trivial structure.

3. Let G_r be a Lie group and M a differentiable manifold. Then $T(M \times G_r)$ is endowed with and integrable partial trivial structure.

4. If X is a regular vector field on a manifold M, then $\rho_1 = \{X\}$ defines an integrable partial trivial structure in TM.

5. Let G be a Lie transformation group on a manifold M. Group G is a parallelizable manifold and TM is a partial trivial bundle.

6. Let N be a parallelizable manifold, $\dim N = p$ and M a manifold. If $\rho_p = (X_1, X_2, ..., X_p)$ is a global frame of TN, then ρ_p defines an integrable partial trivial structure in $T(M \times N)$.

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