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APPLICATIONS OF GENERALIZED CLOSED SETS DEFINED USING THE FUZZY CLOSURE OF THE α-INTERIOR

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Abstract. In this paper, a new type of fuzzy generalized closed set is defined via fuzzy α-open sets, called $fg\alpha^\theta$-closed set. Complements of $fg\alpha^\theta$-closed sets, called $fg\alpha^\theta$-open sets form a family that is not a topology. Using this new type of generalized closed sets, the corresponding open functions and closed function are introduced and studied. A new separation property generalizing the T_2-property of fuzzy topological space is shown to be invariant to bijective $fg\alpha^\theta$-open functions. Many examples are given, to illustrate the new concepts and to distinguish these from known related concepts.

1. Introduction

In [2, 3], generalized version of fuzzy closed set is introduced and studied. After that different types of fuzzy generalized version of closed sets have been investigated.

Keywords and phrases: Fuzzy α-open set, $fg\alpha^\theta$-closed set, $fg\alpha^\theta$-open function, $fg\alpha^\theta$-closed function, $fT_{g\alpha^\theta}$- space, $fg\alpha^\theta$-T_2-space.
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In this paper taking fuzzy α-open set [6] as a basic tool, we introduce and study $fg\alpha^\theta$-closed set. We have shown that fg-closed sets [3] and $fswg$-closed sets [5] are $fg\alpha^\theta$-closed sets, but the reverse implications are not necessarily true, in general. The notions of fuzzy open function [11] and fuzzy closed function [11] have been introduced by C.K. Wong. Here we introduce the notions of $fg\alpha^\theta$-open function and $fg\alpha^\theta$-closed function and characterize each of these properties in several ways. We also establish the mutual relationships of three classes of generalized open functions, namely $fg\alpha^\theta$-open, fg-open [3] and $fswg$-open functions [5]. Also, we study the mutual relationships of three classes of generalized closed functions, namely $fg\alpha^\theta$-closed, fg-closed [3] and $fswg$-closed functions. [5]. The notion of fuzzy T_2-space was introduced in [8]. In this paper, we introduce the class of $fg\alpha^\theta$-T_2-spaces, that is strictly larger than the class of fuzzy T_2-spaces.
After introducing a new type of generalized version of fuzzy closed set, we introduce and discuss new types of generalized version of fuzzy open and fuzzy closed functions and also a new type of separation axiom is introduced. Some applications of these new classes of functions are established here. Next our aim is to introduce generalized version of fuzzy continuous-like functions using $fg\alpha^\theta$-closed sets as a basic tool. With this idea, our aim is to introduce new types of generalized version of fuzzy separation axioms and compactness.

2. Preliminaries

Throughout this paper (X,τ) or simply by X we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [7]. A fuzzy set A is a function from a non-empty set X into the closed interval $I=[0,1]$, i.e., $A \in I^X$ [12]. The support of a fuzzy set A, denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$ [12]. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement of a fuzzy set A in X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$ [12].

For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [12] while AqB means A is quasi-coincident (q-coincident, for short) with B, if there exists $x \in X$ such that $A(x) + B(x) > 1$ [10]. The negation of these two statements will be denoted by $A \not\leq B$ and $A\not{q}B$ respectively. For a fuzzy point x_t and a fuzzy set A, $x_t \in A$ means $A(x) \geq t$, i.e., $x_t \leq A$.

For a fuzzy set A, clA and $intA$ will stand for fuzzy closure [7] and fuzzy interior [7] of A respectively. A fuzzy set A is called a fuzzy neighborhood (fuzzy nbd, for short) of a fuzzy point x_t if there exists a fuzzy open set U in X such that $x_t \in U \leq A$ [10]. If, in addition, A is fuzzy open, then A is called fuzzy open nbd of x_t [10]. A fuzzy set A is called a fuzzy quasi neighborhood (fuzzy q-nbd, for short) [10] of a fuzzy point x_t in an fts X if there is a fuzzy open set U in X such that $x_t qU \leq A$. If, in addition, A is fuzzy open, then A is called fuzzy open q-nbd [10] of x_t.
A fuzzy set A in X is called
(1) fuzzy regular open [1] if $A = int(clA)$
(2) fuzzy semiopen [1] if $A \leq cl(intA)$
(3) fuzzy α-open [6] if $A \leq int(cl(intA))$.
The complement of a fuzzy α-open set is called fuzzy α-closed [6]. The intersection (resp., union) of all fuzzy α-closed (resp., fuzzy α-open) sets containing (resp., contained in) a fuzzy set A is called fuzzy α-closure [6] (resp., fuzzy α-interior [6]) of A, to be denoted by $\alpha - clA$ (resp., $\alpha - intA$). The collection of all fuzzy regular open (resp. fuzzy semiopen, fuzzy α-open) sets in an fts X is denoted by $FRO(X)$ (resp., $FSO(X)$, $F\alpha O(X)$) and that of fuzzy α-closed sets is denoted by $F\alpha C(X)$.

3. Some properties of $fg\alpha^\theta$-closed Sets

In this section we first introduce the notion of $fg\alpha^\theta$-closed set and establish some of its properties. Then establish the mutual relationships of this newly defined class of sets with the classes of sets defined in [2, 3, 4].

Definition 3.1. Let (X, τ) be an fts and $A \in I^X$. Then A is called $fg\alpha^\theta$-closed set in X if $cl(\alpha - intA) \leq U$ whenever $A \leq U \in \tau$.
The complements of $fg\alpha^\theta$-closed sets are called $fg\alpha^\theta$-open sets.
The collection of all $fg\alpha^\theta$-closed (resp., $fg\alpha^\theta$-open) sets in an fts X is denoted by $FG\alpha^\theta C(X)$ (resp., $FG\alpha^\theta O(X)$).

Remark 3.2. The union and the intersection of two $fg\alpha^\theta$-closed sets may not be so, as it is seen from the following example.

Example 3.3. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Here $F\alpha O(X) = \tau$. Now consider the fuzzy sets B and C defined by $B(a) = 0.6, B(b) = 0.4, C(a) = C(b) = 0.5$. Clearly B and C are

$fg\alpha^\theta$-closed sets in (X,τ). Let $D = B\bigwedge C$. Then $D = A \leq A \in \tau$. But $cl(\alpha - intD) = 1_X \setminus A \nleq A \Rightarrow D$ is not $fg\alpha^\theta$-closed set in X.
Now consider two fuzzy sets E and F defined by $E(a) = 0, E(b) = 0.4, F(a) = 0.5, F(b) = 0$. Then clearly $E, F \in FG\alpha^\theta C(X)$ because $cl(\alpha - intE) = cl(\alpha - intF) = 0_X$. Let $G = E\bigvee F$. Now $G = A \leq A \in \tau$. But $cl(\alpha - intG) = 1_X \setminus A \nleq A \Rightarrow G \notin FG\alpha^\theta C(X)$.

Note 3.4. So we can conclude that the family of all $fg\alpha^\theta$-open sets in an fts (X,τ) does not form a fuzzy topology.

Theorem 3.5. *Let (X,τ) be an fts and $A, B \in I^X$. If $A \leq B \leq cl(\alpha - intA)$ and A is $fg\alpha^\theta$-closed set in X, then B is also $fg\alpha^\theta$-closed set in X.*

Proof. Let $U \in \tau$ be such that $B \leq U$. Then by hypothesis, $A \leq B \leq U$. As A is $fg\alpha^\theta$-closed set in X, $cl(\alpha - intA) \leq U$. Then $cl(\alpha - intA) \leq cl(\alpha - intB) \leq cl(\alpha - int(cl(\alpha - intA))) \leq cl(\alpha - intA) \leq U \Rightarrow B$ is $fg\alpha^\theta$-closed set in X.

Theorem 3.6. *Let (X,τ) be an fts and $A, B \in I^X$. If $int(\alpha - clA) \leq B \leq A$ and A is an $fg\alpha^\theta$-open set in X, then B is also $fg\alpha^\theta$-open set in X.*

Proof. $int(\alpha - clA) \leq B \leq A \Rightarrow 1_X \setminus A \leq 1_X \setminus B \leq 1_X \setminus int(\alpha - clA) = cl(\alpha - int(1_X \setminus A))$ where $1_X \setminus A$ is $fg\alpha^\theta$-closed set in X. By Theorem 3.5, $1_X \setminus B$ is $fg\alpha^\theta$-closed set in $X \Rightarrow B$ is $fg\alpha^\theta$-open set in X.

Theorem 3.7. *Let (X,τ) be an fts and A be a fuzzy set in X. Then A is an $fg\alpha^\theta$-open set in X if and only if $K \leq int(\alpha - clA)$ whenever $K \leq A$ and K is a fuzzy closed set in (X,τ).*

Proof. Let A be an $fg\alpha^\theta$-open set in X and $K \leq A$ where K is a fuzzy closed set in (X,τ). Then $1_X \setminus A \leq 1_X \setminus K$ where $1_X \setminus A$ is $fg\alpha^\theta$-closed set in X and $1_X \setminus K$ is fuzzy open set in (X,τ). By hypothesis, $cl(\alpha - int(1_X \setminus A)) \leq 1_X \setminus K \Rightarrow 1_X \setminus int(\alpha - clA) \leq 1_X \setminus K \Rightarrow K \leq int(\alpha - clA)$.
Conversely, let $K \leq int(\alpha - clA)$ whenever $K \leq A$, $K \in \tau^c$. Then $1_X \setminus A \leq 1_X \setminus K$ where $1_X \setminus K \in \tau$. By hypothesis, $1_X \setminus int(\alpha - clA) \leq 1_X \setminus K \Rightarrow cl(\alpha - int(1_X \setminus A)) \leq 1_X \setminus K \Rightarrow 1_X \setminus A$ is $fg\alpha^\theta$-closed set in X and so A is $fg\alpha^\theta$-open set in X.

Theorem 3.8. *Let (X,τ) be an fts and $A, B \in I^X$. If A is $fg\alpha^\theta$-closed set in X and B is fuzzy closed set in (X,τ) with $A\not{q}B$. Then $cl(\alpha - intA)\not{q}B$.*

Proof. By hypothesis, $A\not{q}B \Rightarrow A \leq 1_X \setminus B \in \tau \Rightarrow cl(\alpha - intA) \leq 1_X \setminus B \Rightarrow cl(\alpha - intA)\not{q}B$.
Now we recall the following definitions from [2, 3, 4] for ready references.

Definition 3.9. Let (X,τ) be an fts and $A \in I^X$. Then A is called
(i) fg-closed set [2, 3] if $clA \leq U$ whenever $A \leq U \in \tau$,
the complement of fg-closed set is called fg-open,
(ii) $fswg$-closed set [4] if $cl(intA) \leq U$ whenever $A \leq U \in FSO(X)$,
the complement of $fswg$-closed set is called $fswg$-open.

Remark 3.10. Fuzzy α-closed sets, $fswg$-closed sets, fg-closed sets are $fg\alpha^\theta$-closed sets. But the reverse implications are not necessarily true, as it is seen from the next example.

Example 3.11. The property of a set being $fg\alpha^\theta$-closed does not imply any of the following properties : fuzzy α-closed, fg-closed, $fswg$-closed
Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X,τ) is an fts. Then $F\alpha O(X) = \tau$. Consider the fuzzy set B defined by $B(a) = 0.5, B(b) = 0.3$. Then $B \leq A \in \tau$. Now $cl(\alpha - intB) = 0_X < A \Rightarrow B$ is $fg\alpha^\theta$-closed set in X, but $clB = 1_X \setminus A \nleq A \Rightarrow B$ is not fg-closed set in X. Consider the fuzzy set C defined by $C(a) = C(b) = 0.5$. Then clearly C is $fg\alpha^\theta$-closed set in X. But $C \leq C \in FSO(X)$ and $cl(intC) = 1_X \setminus A \nleq C \Rightarrow C$ is not $fswg$-closed set in X. We see that $cl(int(clC)) = 1_X \setminus A \nleq C \Rightarrow C$ is not fuzzy α-closed set in X.

Definition 3.12. A fts (X,τ) is called $fT_{g\alpha^\theta}$-space if every $fg\alpha^\theta$-closed set in X is fuzzy closed set in X.
Note 3.13. In $fT_{g\alpha^\theta}$-space, $fg\alpha^\theta$-closed set is fuzzy α-closed, fg-closed, $fswg$-closed.

Now we introduce a new type of generalized version of a neighborhood system in an fts.
Definition 3.14. Let (X,τ) be an fts and x_t, a fuzzy point in X. A fuzzy set A is called $fg\alpha^\theta$-neighborhood ($fg\alpha^\theta$-nbd, for short) of x_t,

if there exists an $fg\alpha^\theta$-open set U in X such that $x_t \in U \leq A$. If, in addition, A is $fg\alpha^\theta$-open set in X, then A is called an $fg\alpha^\theta$-open nbd of x_t.

Definition 3.15. Let (X, τ) be an fts and x_t, a fuzzy point in X. A fuzzy set A is called $fg\alpha^\theta$-quasi neighborhood ($fg\alpha^\theta$-q-nbd, for short) of x_t if there is an $fg\alpha^\theta$-open set U in X such that $x_t qU \leq A$. If, in addition, A is $fg\alpha^\theta$-open set in X, then A is called an $fg\alpha^\theta$-open q-nbd of x_t.

Note 3.16. (i) It is clear from definitions that every $fg\alpha^\theta$-open set is an $fg\alpha^\theta$-open nbd of each of its points. But every $fg\alpha^\theta$-nbd of x_t may not be an $fg\alpha^\theta$-open set containing x_t follows from the next example.
(ii) Also every fuzzy open nbd (resp., fuzzy open q-nbd) of a fuzzy point x_t is an $fg\alpha^\theta$-open nbd (resp., $fg\alpha^\theta$-open q-nbd) of x_t. But the converses are not necessarily true, in general, as it seen from the next example.

Example 3.17. Consider Example 3.3 and the fuzzy point $a_{0.4}$ and the fuzzy set S, defined by $S(a) = 0.5, S(b) = 0.6$. Here S is not an $fg\alpha^\theta$-open set in X. Also $a_{0.4} \in S$. Now consider the fuzzy set B defined by $B(a) = 0.5 = B(b)$. Then B is $fg\alpha^\theta$-open set in X containing $a_{0.4}$. So $a_{0.4} \in B \leq S$ implies that S is an $fg\alpha^\theta$-nbd of $a_{0.4}$.
Next consider the fuzzy point $a_{0.6}$. Then $a_{0.6} qB \leq S \Rightarrow S$ is an $fg\alpha^\theta$-q-nbd of $a_{0.6}$. Again consider the fuzzy set C defined by $C(a) = 0.5, C(b) = 0.3$. Then as $1_X \setminus C$ is $fg\alpha^\theta$-closed set in X, C is an $fg\alpha^\theta$-open set in X. So C is an $fg\alpha^\theta$-open q-nbd of $a_{0.6}$, but not a fuzzy open q-nbd of $a_{0.6}$.

4. $fg\alpha^\theta$-open functions and $fg\alpha^\theta$-closed functions

In this section, we introduce a new type of generalized version of closure-like operator which is seen to be an idempotent operator. Then, using this operator as a basic tool, two types of functions are introduced and characterized.

Definition 4.1. Let (X, τ) be an fts and $A \in I^X$. Then $fg\alpha^\theta$-closure and $fg\alpha^\theta$-interior of A, denoted by $fg\alpha^\theta cl(A)$ and $fg\alpha^\theta int(A)$, are defined as follows:

$fg\alpha^\theta cl(A) = \bigwedge\{F : A \leq F, F \text{ is } fg\alpha^\theta\text{-closed set in } X\}$,
$fg\alpha^\theta int(A) = \bigvee\{G : G \leq A, G \text{ is } fg\alpha^\theta\text{-open set in } X\}$.

Remark 4.2. It is clear from definition that for any $A \in I^X$, $A \leq fg\alpha^\theta cl(A) \leq clA$. If A is $fg\alpha^\theta$-closed set in an fts X, then $A = fg\alpha^\theta cl(A)$. Similarly, $intA \leq fg\alpha^\theta int(A) \leq A$. If A is $fg\alpha^\theta$-open set in an fts X, then $A = fg\alpha^\theta int(A)$. It follows from Remark 3.2 that $fg\alpha^\theta cl(A)$ (resp., $fg\alpha^\theta int(A)$) may not be $fg\alpha^\theta$-closed (resp., $fg\alpha^\theta$-open) set in an fts X.

Theorem 4.3. *Let (X, τ) be an fts and $A \in I^X$. Then for a fuzzy point x_t in X, $x_t \in fg\alpha^\theta cl(A)$ if and only if every $fg\alpha^\theta$-open q-nbd U of x_t, UqA.*

Proof. Let $x_t \in fg\alpha^\theta cl(A)$ for any fuzzy set A in an fts X and F be any $fg\alpha^\theta$-open q-nbd of x_t. Then $x_t qF \Rightarrow x_t \not\in 1_X \setminus F$ which is $fg\alpha^\theta$-closed set in X. Then by Definition 4.1, $A \not\leq 1_X \setminus F$ and so there exists $y \in X$ such that $A(y) > 1 - F(y)$. Hence AqF.
Conversely, let for every $fg\alpha^\theta$-open q-nbd F of x_t, FqA. Assume by contrary that $x_t \not\in fg\alpha^\theta cl(A)$. Then by Definition 4.1, there exists an $fg\alpha^\theta$-closed set U in X with $A \leq U$, $x_t \not\in U$. Then $x_t q(1_X \setminus U)$ which being $fg\alpha^\theta$-open set in X is $fg\alpha^\theta$-open q-nbd of x_t. By assumption, $(1_X \setminus U)qA$. Hence $(1_X \setminus A)qA$, which is absurd.

Theorem 4.4. *Let (X, τ) be an fts and $A, B \in I^X$. Then the following statements are true:*
(i) $fg\alpha^\theta cl(0_X) = 0_X$,
(ii) $fg\alpha^\theta cl(1_X) = 1_X$,
(iii) $A \leq B \Rightarrow fg\alpha^\theta cl(A) \leq fg\alpha^\theta cl(B)$,
(iv) $fg\alpha^\theta cl(A \bigvee B) = fg\alpha^\theta cl(A) \bigvee fg\alpha^\theta cl(B)$,
(v) $fg\alpha^\theta cl(A \wedge B) \leq fg\alpha^\theta cl(A) \wedge fg\alpha^\theta cl(B)$, equality does not hold, in general, as follows from Example 3.3,
(vi) $fg\alpha^\theta cl(fg\alpha^\theta cl(A)) = fg\alpha^\theta cl(A)$.

Proof. (i), (ii) and (iii) are obvious.
(iv) From (iii), $fg\alpha^\theta cl(A) \bigvee fg\alpha^\theta cl(B) \leq fg\alpha^\theta cl(A \bigvee B)$.
To prove the converse, let $x_t \in fg\alpha^\theta cl(A \bigvee B)$. Then by Theorem 4.3, for any $fg\alpha^\theta$-open set U in X with $x_t qU$, $Uq(A \bigvee B) \Rightarrow$ there

exists $y \in X$ such that $U(y) + max\{A(y), B(y)\} > 1 \Rightarrow$ either $U(y) + A(y) > 1$ or $U(y) + B(y) > 1$ implies that either UqA or UqB. So either $x_t \in fg\alpha^\theta cl(A)$ or $x_t \in fg\alpha^\theta cl(B)$. Hence $x_t \in fg\alpha^\theta gcl(A) \bigvee fg\alpha^\theta cl(B)$.
(v) Follows from (iii).
(vi) As $A \leq fg\alpha^\theta cl(A)$, for any $A \in I^X$, $fg\alpha^\theta cl(A) \leq fg\alpha^\theta cl(fg\alpha^\theta cl(A))$ (by (iii)).
Conversely, let $x_t \in fg\alpha^\theta cl(fg\alpha^\theta cl(A)) = fg\alpha^\theta cl(B)$ where $B = fg\alpha^\theta cl(A)$. Let U be any $fg\alpha^\theta$-open set in X with x_tqU. Then UqB implies that there exists $y \in X$ such that $U(y) + B(y) > 1$. Let $B(y) = s$. Then y_sqU and $y_s \in B = fg\alpha^\theta cl(A)$. So UqA implies that $x_t \in fg\alpha^\theta cl(A)$. Hence $fg\alpha^\theta cl(fg\alpha^\theta cl(A)) \leq fg\alpha^\theta cl(A)$. Consequently, $fg\alpha^\theta cl(fg\alpha^\theta cl(A)) = fg\alpha^\theta cl(A)$.

Theorem 4.5. *Let (X, τ) be an fts and $A \in I^X$. Then the following statements hold:*
(i) $fg\alpha^\theta cl(1_X \setminus A) = 1_X \setminus fg\alpha^\theta int(A)$
(ii) $fg\alpha^\theta int(1_X \setminus A) = 1_X \setminus fg\alpha^\theta cl(A)$.
Proof (i). Let $x_t \in fg\alpha^\theta cl(1_X \setminus A)$ for a fuzzy set A in an fts (X, τ). It is about assuming by contrary, that $x_t \notin 1_X \setminus fg\alpha^\theta int(A)$. Then $1 - (fg\alpha^\theta int(A))(x) < t$. So $[fg\alpha^\theta int(A)](x) + t > 1$ implies that $fg\alpha^\theta int(A)qx_t$. So there exists at least one $fg\alpha^\theta$-open set $F \leq A$ with x_tqF and so x_tqA. As $x_t \in fg\alpha^\theta cl(1_X \setminus A), Fq(1_X \setminus A)$ Then $Aq(1_X \setminus A)$, which is absurd. Hence

$$fg\alpha^\theta cl(1_X \setminus A) \leq 1_X \setminus fg\alpha^\theta int(A)...(1)$$

Conversely, let $x_t \in 1_X \setminus fg\alpha^\theta int(A)$. Then $1 - [(fg\alpha^\theta int(A)](x) \geq t$. So $x_t \not{q}(fg\alpha^\theta int(A))$. So $x_t \not{q}F$ for every $fg\alpha^\theta$-open set F contained in A ... (2).
Let U be any $fg\alpha^\theta$-closed set in X such that $1_X \setminus A \leq U$. Then $1_X \setminus U \leq A$. Now $1_X \setminus U$ is $fg\alpha^\theta$-open set in X contained in A. By (2), $x_t \not{q}(1_X \setminus U)$ implies that $x_t \in U \Rightarrow x_t \in fg\alpha^\theta cl(1_X \setminus A)$ and so

$$1_X \setminus fg\alpha^\theta int(A) \leq fg\alpha^\theta cl(1_X \setminus A)...(3).$$

Combining (1) and (3), (i) follows.
(ii) Putting $1_X \setminus A$ for A in (i), we get $fg\alpha^\theta cl(A) = 1_X \setminus fg\alpha^\theta int(1_X \setminus A)$. Hence $fg\alpha^\theta int(1_X \setminus A) = 1_X \setminus fg\alpha^\theta cl(A)$.

Now we introduce a new type of fuzzy open-like function.

Definition 4.6. A function $h : X \to Y$ is called $fg\alpha^{\theta}$-open function if $h(U)$ is $fg\alpha^{\theta}$-open set in Y for every fuzzy open set U in X.

Theorem 4.7. *For a bijective function $h : X \to Y$, the following statements are equivalent:*
(i) h is $fg\alpha^{\theta}$-open,
(ii) $h(intA) \leq fg\alpha^{\theta}int(h(A))$, for all $A \in I^X$,
(iii) for each fuzzy point x_t in X and each fuzzy open set U in X containing x_t, there exists an $fg\alpha^{\theta}$-open set V in Y containing $h(x_t)$ such that $V \leq h(U)$.

Proof (i) $\Rightarrow$ (ii). Let $A \in I^X$. Then $intA$ is a fuzzy open set in X. By (i), $h(intA)$ is $fg\alpha^{\theta}$-open set in Y. Since $h(intA) \leq h(A)$ and $fg\alpha^{\theta}int(h(A))$ is the union of all $fg\alpha^{\theta}$-open sets contained in $h(A)$, we have $h(intA) \leq fg\alpha^{\theta}int(h(A))$.
(ii) $\Rightarrow$ (i). Let U be any fuzzy open set in X. Then $h(U) = h(intU) \leq fg\alpha^{\theta}int(h(U))$ (by (ii)) $\Rightarrow$ $h(U)$ is $fg\alpha^{\theta}$-open set in $Y \Rightarrow h$ is $fg\alpha^{\theta}$-open function.
(ii) $\Rightarrow$ (iii). Let x_t be a fuzzy point in X, and U, a fuzzy open set in X such that $x_t \in U$. Then $h(x_t) \in h(U) = h(intU) \leq fg\alpha^{\theta}int(h(U))$ (by (ii)). Then $h(U)$ is $fg\alpha^{\theta}$-open set in Y. Let $V = h(U)$. Then $h(x_t) \in V$ and $V \leq h(U)$.
(iii) $\Rightarrow$ (i). Let U be any fuzzy open set in X and y_t, any fuzzy point in $h(U)$, i.e., $y_t \in h(U)$. Then there exists a unique $x \in X$ such that $h(x) = y$ (as h is bijective). Then $[h(U)](y) \geq t \Rightarrow U(h^{-1}(y)) \geq t \Rightarrow U(x) \geq t \Rightarrow x_t \in U$. By (iii), there exists an $fg\alpha^{\theta}$-open set V in Y such that $h(x_t) \in V$ and $V \leq h(U)$. Then $h(x_t) \in V = fg\alpha^{\theta}int(V) \leq fg\alpha^{\theta}int(h(U))$. Since y_t is taken arbitrarily and $h(U)$ is the union of all fuzzy points in $h(U)$, $h(U) \leq fg\alpha^{\theta}int(h(U))$ implies that $h(U)$ is $fg\alpha^{\theta}$-open set in Y and hence h is an $fg\alpha^{\theta}$-open function.

Theorem 4.8. *If $h : X \to Y$ is $fg\alpha^{\theta}$-open, bijective function, then the following statements are true:*
(i) for each fuzzy point x_t in X and each fuzzy open q-nbd U of x_t in X, there exists an $fg\alpha^{\theta}$-open q-nbd V of $h(x_t)$ in Y such that $V \leq h(U)$,
(ii) $h^{-1}(fg\alpha^{\theta}cl(B)) \leq cl(h^{-1}(B))$, for all $B \in I^Y$.

Proof (i). Let x_t be a fuzzy point in X and U be any fuzzy open q-nbd of x_t in X. Then $x_t qU = intU$ implies that $h(x_t)qh(intU) \leq fg\alpha^\theta int(h(U))$ (by Theorem 4.7 (i)⇒(ii)) and so there exists at least one $fg\alpha^\theta$-open q-nbd V of $h(x_t)$ in Y with $V \leq h(U)$.
(ii) Let x_t be any fuzzy point in X such that $x_t \notin cl(h^{-1}(B))$ for any $B \in I^Y$. Then there exists a fuzzy open q-nbd U of x_t in X such that $U \not{q} h^{-1}(B)$. Now

$$h(x_t)qh(U)...(1)$$

where $h(U)$ is $fg\alpha^\theta$-open set in Y. Now $h^{-1}(B) \leq 1_X \setminus U$ which is a fuzzy closed set in X. So $B \leq h(1_X \setminus U)$ (as h is injective) $\leq 1_Y \setminus h(U)$. So $B \not{q} h(U)$. Let $V = 1_Y \setminus h(U)$. Then $B \leq V$ which is $fg\alpha^\theta$-closed set in Y. We claim that $h(x_t) \notin V$. It is about assuming by contrary, that $h(x_t) \in V = 1_Y \setminus h(U)$. Then $1 - [h(U)](h(x)) \geq t$ implies that $h(U) \not{q} h(x_t)$, contradicting (1). So $h(x_t) \notin V \Rightarrow h(x_t) \notin fg\alpha^\theta cl(B) \Rightarrow x_t \notin h^{-1}(fg\alpha^\theta cl(B)) \Rightarrow h^{-1}(fg\alpha^\theta cl(B)) \leq cl(h^{-1}(B))$.

Theorem 4.9. *An injective function $h : X \to Y$ is $fg\alpha^\theta$-open if and only if for each $B \in I^Y$ and F, a fuzzy closed set in X with $h^{-1}(B) \leq F$, there exists an $fg\alpha^\theta$-closed set V in Y such that $B \leq V$ and $h^{-1}(V) \leq F$.*

Proof. Let $B \in I^Y$ and F, a fuzzy closed set in X with $h^{-1}(B) \leq F$. Then $1_X \setminus h^{-1}(B) \geq 1_X \setminus F$ where $1_X \setminus F$ is a fuzzy open set in X implies that $h(1_X \setminus F) \leq h(1_X \setminus h^{-1}(B)) \leq 1_Y \setminus B$ (as h is injective) where $h(1_X \setminus F)$ is an $fg\alpha^\theta$-open set in Y. Let $V = 1_Y \setminus h(1_X \setminus F)$. Then V is $fg\alpha^\theta$-closed set in Y such that $B \leq V$. Now $h^{-1}(V) = h^{-1}(1_Y \setminus h(1_X \setminus F)) = 1_X \setminus h^{-1}(h(1_X \setminus F)) \leq F$.
Conversely, let F be a fuzzy open set in X. Then $1_X \setminus F$ is a fuzzy closed set in X. We have to show that $h(F)$ is an $fg\alpha^\theta$-open set in Y. Now $h^{-1}(1_Y \setminus h(F)) \leq 1_X \setminus F$ (as h is injective). By assumption, there exists an $fg\alpha^\theta$-closed set V in Y such that

$$1_Y \setminus h(F) \leq V...(1)$$

and $h^{-1}(V) \leq 1_X \setminus F$. Therefore, $F \leq 1_X \setminus h^{-1}(V)$ implies that

$$h(F) \leq h(1_X \setminus h^{-1}(V)) \leq 1_Y \setminus V...(2)$$

(as h is injective). Combining (1) and (2), $h(F) = 1_Y \setminus V$ which is an $fg\alpha^\theta$-open set in Y. Hence h is $fg\alpha^\theta$-open function.

Definition 4.10. A function $h : X \to Y$ is called $fg\alpha^\theta$-closed function if $h(A)$ is $fg\alpha^\theta$-closed set in Y for each fuzzy closed set A in X.

Theorem 4.11. *A bijective function $h : X \to Y$ is $fg\alpha^\theta$-closed function if and only if $fg\alpha^\theta cl(h(A)) \leq h(clA)$, for all $A \in I^X$.*

Proof. Let us suppose that $h : X \to Y$ be an $fg\alpha^\theta$-closed function and $A \in I^X$. Then $h(cl(A))$ is $fg\alpha^\theta$-closed set in Y. Since $h(A) \leq h(clA)$ and $fg\alpha^\theta cl(h(A))$ is the intersection of all $fg\alpha^\theta$-closed sets in Y containing $h(A)$, we have $fg\alpha^\theta cl(h(A)) \leq h(clA)$.
Conversely, let for any $A \in I^X$, $fg\alpha^\theta cl(h(A)) \leq h(clA)$. Let U be any fuzzy closed set in X. Then $h(U) = h(clU) \geq fg\alpha^\theta cl(h(U))$ implies that $h(U)$ is an $fg\alpha^\theta$-closed set in Y. Hence h is an $fg\alpha^\theta$-closed function.

Theorem 4.12. *If $h : X \to Y$ is an $fg\alpha^\theta$-closed bijective function, then the following statements hold:*
(i) for each fuzzy point x_t in X and each fuzzy closed set U in X with $x_t \not{q} U$, there exists an $fg\alpha^\theta$-closed set V in Y with $h(x_t) \not{q} V$ such that $V \geq h(U)$,
(ii) $h^{-1}(fg\alpha^\theta int(B)) \geq int(h^{-1}(B))$, for all $B \in I^Y$.

Proof (i). Let x_t be a fuzzy point in X and U be any fuzzy closed set in X with $x_t \not{q} U = clU$ implies that $h(x_t) \not{q} h(clU) \geq fg\alpha^\theta cl(h(U))$ (by Theorem 4.11) $\Rightarrow h(x_t) \not{q} V$ for some $fg\alpha^\theta$-closed set V in Y with $V \geq h(U)$.
(ii). Let $B \in I^Y$ and x_t be any fuzzy point in X such that $x_t \in int(h^{-1}(B))$. Then there exists a fuzzy open set U in X with $U \leq h^{-1}(B)$ such that $x_t \in U$. Then $1_X \setminus U \geq 1_X \setminus h^{-1}(B)$ and so $h(1_X \setminus U) \geq h(1_X \setminus h^{-1}(B))$ where $h(1_X \setminus U)$ is an $fg\alpha^\theta$-closed set in Y. Let $V = 1_Y \setminus h(1_X \setminus U)$. Then V is an $fg\alpha^\theta$-open set in Y and $V = 1_Y \setminus h(1_X \setminus U) \leq 1_Y \setminus h(1_X \setminus h^{-1}(B)) \leq 1_Y \setminus (1_Y \setminus B) = B$ (as h is injective). Now $U(x) \geq t$ implies that $x_t \not{q}(1_X \setminus U)$. Then $h(x_t) \not{q} h(1_X \setminus U)$. So $h(x_t) \leq 1_Y \setminus h(1_X \setminus U) = V$ implies that $h(x_t) \in V = fg\alpha^\theta int(V) \leq fg\alpha^\theta int(B) \Rightarrow x_t \in h^{-1}(fg\alpha^\theta int(B))$. Since x_t is taken arbitrarily, $int(h^{-1}(B)) \leq h^{-1}(fg\alpha^\theta int(B))$, for all $B \in I^Y$.

Note 4.13. The composition of two $fg\alpha^\theta$-closed (resp., $fg\alpha^\theta$-open) functions need not be so, as it seen from the following example.

Example 4.14. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, C\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4, C(a) = 0.5, C(b) = 0.6$. Then (X, τ_1), (X, τ_2) and (X, τ_3) are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$ and $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly i_1 and i_2 are $fg\alpha^\theta$-closed functions. Let $i_3 = i_2 \circ i_1 : (X, \tau_1) \to (X, \tau_3)$. We claim that i_3 is not $fg\alpha^\theta$-closed function. Now $(1_X \setminus C) \in \tau_1^c, i_3(1_X \setminus C) = 1_X \setminus C \leq A \in \tau_3$. But $cl_{\tau_3} int_{\tau_3}(1_X \setminus C) = 1_X \setminus A \not\leq A$ implies that $1_X \setminus C$ is not $fg\alpha^\theta$-closed set in (X, τ_3). Hence i_3 is not $fg\alpha^\theta$-closed function.
Similarly we can show that the composition of two $fg\alpha^\theta$-open functions may not be so.

Now we recall some definitions from [3, 5, 11] for ready references.

Definition 4.15. Let $h : (X, \tau_1) \to (Y, \tau_2)$ be a function. Then h is called
(i) fuzzy closed (resp., fuzzy open) function [11] if $h(U)$ is fuzzy closed (resp., fuzzy open) set in Y for every fuzzy closed (resp., fuzzy open) set U in X,
(ii) fg-closed (resp., fg-open) function [3] if $h(A)$ is fg-closed set in Y for every fuzzy closed (resp., fuzzy open) set U in X,
(iii) $fswg$-closed (resp., $fswg$-open) function [5] if $h(U)$ is $fswg$-closed (resp., $fswg$-open) set in Y for every fuzzy closed (resp., fuzzy open) set U in X.

Remark 4.16. It is clear from definitions that
(i) fuzzy closed (resp., fuzzy open), fg-closed (resp., fg-open) functions are $fg\alpha^\theta$-closed (resp., $fg\alpha^\theta$-open) function, but not conversely follow from the next examples,
(ii) $f\alpha^\theta g$-closed (resp., $f\alpha^\theta g$-open) function is $fswg$-closed (resp., $fswg$-open) function, but not conversely, as follows from the next example.

Example 4.17. A $fg\alpha^\theta$-closed function is not necessarily a fuzzy closed function, fg-closed function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5 = B(b)$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Here $B \in \tau_1^c, i(B) = B \leq A \in \tau_2$. Then $cl_{\tau_2} int_{\tau_2} B = 0_X < A$. So B is $fg\alpha^\theta$-closed set in (X, τ_2) implies that i is $fg\alpha^\theta$-closed function. But as $B \notin \tau_2^c, i$ is not fuzzy closed function. Again, $B \leq A \in \tau_2$, but

$cl_{\tau_2}B = 1_X \nleq A$ which shows that B is not fg-closed set in (X, τ_2). Hence i is not fg-closed function.

Example 4.18. A $fg\alpha^\theta$-closed function is not necessarily an $fswg$-closed function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = B(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Here $B \in \tau_1^c, i(B) = B < 1_X \in \tau_2$ only and so B is $fg\alpha^\theta$-closed set in (X, τ_2) which proves that i is $fg\alpha^\theta$-closed function. Now $B \leq B \in FSO(X, \tau_2)$. Then $cl_{\tau_2}int_{\tau_2}B = 1_X \setminus A \nleq B$ and so B is not $fswg$-closed set in (X, τ_2). Hence i is not $fswg$-closed function.
In a similar manner we can construct counterexamples for $fg\alpha^\theta$-open functions.

Theorem 4.19. If $h_1 : X \to Y$ is fuzzy closed (resp., fuzzy open) function and $h_2 : Y \to Z$ is $fg\alpha^\theta$-closed (resp., $fg\alpha^\theta$-open) function, then $h_2 \circ h_1 : X \to Z$ is $fg\alpha^\theta$-closed (resp., $fg\alpha^\theta$-open) function.
Proof. Obvious.

5. $fg\alpha^\theta$-OPEN FUNCTIONS AND SEPARATION PROPERTIES

In this section we first introduce the notion of $fg\alpha^\theta$-T_2-space, the class of which is strictly larger than that of fuzzy T_2-space and then establish some applications of $fg\alpha^\theta$-open functions.

We first recall the definition and theorem from [8, 9] for ready references.

Definition 5.1 [8]. An fts (X, τ) is called fuzzy T_2-space if for any two distinct fuzzy points x_α and y_β; when $x \neq y$, there exist fuzzy open sets U_1, U_2, V_1, V_2 such that $x_\alpha \in U_1, y_\beta q V_1, U_1 \not{q} V_1$ and $x_\alpha q U_2, y_\beta \in V_2, U_2 \not{q} V_2$; when $x = y$ and $\alpha < \beta$ (say), there exist fuzzy open sets U and V in X such that $x_\alpha \in U, y_\beta q V$ and $U \not{q} V$.

Theorem 5.2 [9]. *An fts (X, τ) is fuzzy T_2-space if and only if for any two distinct fuzzy points x_α and y_β in X; when $x \neq y$, there exist fuzzy open sets U, V in X such that $x_\alpha q U$, $y_\beta q V$ and $U \not{q} V$; when $x = y$ and $\alpha < \beta$ (say), x_α has a fuzzy open nbd U and y_β has a fuzzy open q-nbd V such that $U \not{q} V$.*
Now we introduce the following concept.

Definition 5.3. An fts (X, τ) is called $fg\alpha^\theta$-T_2-space if for any two distinct fuzzy points x_t and y_s in X the following hold:

(1) If $x \neq y$, then there exist $fg\alpha^\theta$-open sets U, V in X such that $x_t qU$, $y_s qV$ and $U \not q V$;

(2) If $x = y$ and $t < s$ (say), then x_t has an $fg\alpha^\theta$-open nbd U and y_s has an $fg\alpha^\theta$-open q-nbd V such that $U \not q V$.

Remark 5.4. Clearly, every fuzzy T_2-space is an $fg\alpha^\theta$-T_2-space, but the converse is not necessarily true, as follows from the next example.

Example 5.5. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X\}$. Then (X, τ) is an fts. Clearly (X, τ) is not a fuzzy T_2-space. Here every fuzzy set in (X, τ) is $fg\alpha^\theta$-open set in (X, τ). Clearly it is $fg\alpha^\theta$-T_2-space.

Theorem 5.6. *If a bijective function $h : X \to Y$ is $fg\alpha^\theta$-open function from a fuzzy T_2-space X onto an fts Y, then Y is $fg\alpha^\theta$-T_2-space.*

Proof. Let z_t and w_s be two fuzzy points in Y. Since h is bijective, there exist unique x, y in X such that $h(x) = z, h(y) = w$, i.e., $h(x_t) = z_t, h(y_s) = w_s$.

Case I. Suppose $z \neq w$. Then $x \neq y$. Since X is fuzzy T_2-space, there exist fuzzy open sets U, V in X such that $x_t qU, y_s qV$ and $U \not q V$. Then $h(x_t)(= z_t)qh(U), h(y_s)(= w_s)qV$ and $h(U) \not q h(V)$ where $h(U)$ and $h(V)$ are $fg\alpha^\theta$-open sets in Y as h is an $fg\alpha^\theta$-open function [Indeed, $h(U)qh(V)$ so that there exists $p \in Y$ such that $[h(U)](p) + [h(V)](p) > 1$ and so $U(h^{-1}(p)) + V(h^{-1}(p)) > 1$ where $h^{-1}(p) \in X$ which shows that UqV, a contradiction].

Case II. Suppose $z = w$ and $t < s$ (say). Then $x = y$ and $t < s$. Since X is fuzzy T_2-space, there exist a fuzzy open nbd U of x_t and a fuzzy open q-nbd V of y_s such that $U \not q V$. Then $h(x_t) \in h(U), h(y_s)qh(V)$ and $h(U) \not q h(V)$ where $h(U), h(V)$ are $fg\alpha^\theta$-open sets in Y, i.e., $h(U)$ is an $fg\alpha^\theta$-open nbd of z_t, $h(V)$ is an $fg\alpha^\theta$-open q-nbd of w_s and $h(U) \not q h(V)$. Consequently, Y is $fg\alpha^\theta$-T_2-space.

Similarly we can prove the following theorem.

Theorem 5.7. *If a bijective function $h : X \to Y$ is $fg\alpha^\theta$-open function from a fuzzy T_2-space X onto an $fT_{g\alpha^\theta}$-space Y, then Y is fuzzy T_2-space.*

References

[1] Azad, K.K.; **On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity**, *J.Math. Anal. Appl.*, 82 (1981), 14-32.

[2] Balasubramanian, G. and Sundaram, P.; **On some generalizations of fuzzy continuous functions**, *Fuzzy Sets and Systems*, 86 (1997), 93-100.

[3] Bhattacharyya, Anjana; $fg^*\alpha$**-continuous functions in fuzzy topological spaces**, *International Journal of Scientific and Engineering Research*, Vol. 4, Issue 8 (2013), 973-979.

[4] Bhattacharyya, Anjana; **Fuzzy regular generalized α-closed sets and fuzzy regular generalized α-continuous functions**, *Advances in Fuzzy Mathematics*, Vol. 12, No. 4 (2017), 1047-1066.

[5] Bhattacharyya, Anjana; $fswg$-**Closed Set and Its Applications**, *Bull. Cal. Math. Soc.*, Vol. 114, No. 1 (2022), 715-734.

[6] Bin Shahna, A.S.; **On fuzzy strong semicontinuity and fuzzy precontinuity**, *Fuzzy Sets and Systems*, Vol. 44 (1991), 303-308.

[7] Chang, C.L.; **Fuzzy topological spaces**, *J. Math. Anal. Appl.*, Vol. 24 (1968), 182-190.

[8] Hutton, B. and Reilly, I.; **Separation axioms in fuzzy topological spaces**, *Fuzzy Sets and Systems*, Vol. 31 (1980), 93-104.

[9] Mukherjee, M.N. and Ghosh, B.; **On nearly compact and θ-rigid fuzzy sets in fuzzy topological spaces**, *Fuzzy Sets and Systems*, Vol. 43 (1991), 57-68.

[10] Pu, Pao Ming and Liu, Ying Ming; **Fuzzy topology I. neighborhood structure of a fuzzy point and Moore-Smith Convergence**, *J. Math Anal. Appl.*, Vol. 76 (1980), 571-599.

[11] Wong, C.K.; **Fuzzy points and local properties of fuzzy topology**, *J. Math. Anal. Appl.*, Vol. 46 (1974), 316-328.

[12] Zadeh, L.A.; **Fuzzy Sets**, *Inform. Control*, Vol. 8 (1965), 338-353.

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