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CONFORMAL TRANSFORMATIONS IN ELECTRO-COMPLEX RANDERS SPACES

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Abstract. In this paper we investigate a class of complex Finsler metrics of Randers type defined by

$$F(z,\mu) = \alpha(z,\mu) + |\beta(z,\mu)|$$

where $\alpha(z,\mu) = \sqrt{a_{i\bar{j}}\mu^i\overline{\mu}^j}$is a Hermitian norm induced by a positive-definite Hermitian metric and $\beta(z,\mu) = \frac{e}{m}A_i(z)\mu^i$ is a complex 1-form. Motivated by the analogies with electromagnetic interactions we refer to these as electro-complex Randers metric and we explore conformal deformations of such spaces.

1. Introduction

Let M be a complex manifold and

(1.1) $$\{x^1, x^2, ..., x^n, x^{n+1}, ..., x^{2n}\}$$

the local real coordinates on M.

1.2 $$\{z^1, z^2, ..., z^n\}$$

are the local complex coordinates where

(1.3) $$z^a = x^{a\cdot} + ix^{n+a}, \overline{z}^a = x^a - ix^{n+a}, 1 \leqslant a \leqslant n$$

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and we will use the Wirtinger operators defined by

$$\frac{\partial}{\partial z^a}=\frac{1}{2}\left(\frac{\partial}{\partial x^a}-i\frac{\partial}{\partial x^{n+a}}\right),\frac{\partial}{\partial \overline{z}^a}=\frac{1}{2}\left(\frac{\partial}{\partial x^a}+i\frac{\partial}{\partial x^{n+a}}\right) \tag{1.4}$$

The sets $\{\frac{\partial}{\partial z^a}\}$ and $\{\frac{\partial}{\partial \overline{z}^a}\}$ are the local frames on the complex tangent bundle $T_z'M$.

Let $\mu\in T_z'M$ be a complex tangent vector, $\mu=\mu^i\frac{\partial}{\partial z^i}$.

We consider $a=a_{i\overline{j}}(z)dz^i\times d\overline{z}^j$ to be a Hermitian positive metric and $b=b_i(z)dz^i$ a differential (1,0)-form and we define the function

$$F(z,\mu)=\alpha(z,\mu)+|\beta(z,\mu)| \tag{1.5}$$

where $\alpha(z,\mu)=\sqrt{a_{i\overline{j}}(z)\mu^i\overline{\mu}^j},\beta(z,\mu)=b_i(z)\mu^i$ and
$|\beta(z,\mu)|=\sqrt{\beta(z,\mu)\overline{\beta(z,\mu)}}$.

We call the function F a complex Randers metric and (M,F) a complex Randers space.

Note that α and β are homogeneous with respect to μ :
$\alpha(z,\lambda\mu)=|\lambda|\,\alpha(z,\mu)\ \beta(z,\lambda\mu)=\lambda\beta(z,\mu)$, for all $\lambda\in C$

We denote $L=F^2$, so $L(z,\lambda\mu)=|\lambda|^2\,L(z,\mu)$ and we calculate

$$\frac{\partial\alpha}{\partial\mu^i}=\frac{1}{2\alpha}a_{i\overline{j}}\overline{\mu}^j,a_{i\overline{j}}\overline{\mu}^j=l_i,\frac{\partial\alpha}{\partial\mu^i}\mu^i=\frac{1}{2}\alpha,\frac{\partial|\beta|}{\partial\mu^i}=\frac{\overline{\beta}}{2|\beta|}b_i,\frac{\partial|\beta|}{\partial\mu^i}\mu^i=\frac{1}{2}\beta$$

$$\frac{\partial L}{\partial\alpha}=2F,\frac{\partial L}{\partial|\beta|}=2F,\frac{\partial^2 L}{\partial\alpha^2}=2,\frac{\partial^2 L}{\partial|\beta|^2}=2,\frac{\partial^2 L}{\partial\alpha\partial|\beta|}=2$$

We consider :

$$b^i=a^{\overline{j}i}b_{\overline{j}},\|b\|^2=a^{\overline{j}i}b_ib_{\overline{j}},\mu_i=\frac{\partial L}{\partial\mu^i}=\frac{\partial L}{\partial\alpha}\frac{\partial\alpha}{\partial\mu^i}+\frac{\partial L}{\partial|\beta|}\frac{\partial|\beta|}{\partial\mu^i}=F\left(\frac{1}{\alpha}l_i+\frac{\overline{\beta}}{|\beta|}b_i\right)$$

The fundamental tensor of the complex Randers space given by

$$g_{i\overline{j}}=\frac{\partial^2F^2}{\partial\mu^i\partial\overline{\mu}^j} \tag{1.6}$$

can be calculated here as

$$g_{i\overline{j}}=\frac{F}{\alpha}h_{i\overline{j}}+\frac{F}{2|\beta|}b_ib_{\overline{j}}+\frac{1}{2L}\mu_i\mu_{\overline{j}} \tag{1.7}$$

where we denoted

$$h_{i\overline{j}}=a_{i\overline{j}}-\frac{1}{2\alpha^2}l_il_{\overline{j}} \tag{1.8}$$

The d-tensor

$$k_{i\bar{j}} = \frac{\partial^2 F}{\partial \mu^i \partial \overline{\mu}^j} \tag{1.9}$$

is the complex angular metric tensor of the complex Randers space. A direct computation yields

$$k_{i\bar{j}} = \frac{1}{2F}\left(g_{i\bar{j}} - \frac{1}{2L}\mu_i\mu_{\bar{j}}\right). \tag{1.10}$$

2. **Electro-complex Randers space**

We consider a model for complex electrodynamics by considering metrics $F : T'M \to [0, \infty)$ of the form

$$F(z, \mu) = \alpha(z, \mu) + |\beta(z, \mu)| \tag{2.1}$$

where $\alpha(z, \mu) = \sqrt{a_{i\bar{j}}(z)\mu^i\overline{\mu}^j}, \beta(z, \mu) = \frac{e}{m}A_i(z)\mu^i$ with :

$a_{i\bar{j}}(z)$, a Hermitian metric on M;

$A_i(z)$, a complex-valued potential;

$e \in R$, the energy of the particle;

$m \in R$, the mass of the particle;

$\alpha(z, \mu)$ ensures positive-definiteness and smoothness of the metric and $\beta(z, \mu)$ introduces an anisotropic direction dependent perturbation influenced by the field. The real part of $\alpha(z, \mu)$ corresponds to the kinetic energy induced and $|\beta(z, \mu)|$ models the influence of the external electromagnetic potential.

The Finsler space

$$(M, F) = \alpha(z, \mu) + |\beta(z, \mu)|)$$

is called electro-complex Randers space.

The next goal is to find the fundamental tensor $g_{i\bar{j}}$.
In order to do this, we calculate

(2.2)
$$\frac{\partial \alpha}{\partial \mu^i} = \frac{1}{2\alpha}a_{i\bar{k}}\overline{\mu}^k, \frac{\partial \beta}{\partial \mu^i} = \frac{1}{2|\beta|}\frac{e^2}{m^2}A_i\overline{A}_k\overline{\mu}^k, \frac{\partial^2 \alpha^2}{\partial \mu^i \partial \overline{\mu}^k} = a_{i\bar{k}}, \frac{\partial^2 |\beta|^2}{\partial \mu^i \partial \overline{\mu}^k} = \frac{e^2}{m^2}A_i\overline{A}_k$$

By a direct calculation, we obtain

Lemma 1. *The fundamental metric tensor of the electro-complex Randers metric is given by*

$$g_{i\bar{k}} = a_{i\bar{k}}+\frac{e^2}{m^2}A_i\overline{A}_k+2[\frac{1}{2\alpha}a_{i\bar{k}}\left|\beta\right|+\frac{1}{4\alpha\left|\beta\right|}\frac{e^2}{m^2}\left(a_{i\bar{k}}\overline{\mu}^kA_l\mu^l\overline{A_j}+a_{l\bar{j}}\mu^lA_i\overline{A}_k\overline{\mu}^k\right)-$$
$$\tfrac{1}{4\alpha^3}a_{i\bar{l}}\overline{\mu}^la_{k\bar{j}}\mu^k\left|\beta\right|-\tfrac{1}{4|\beta|^3}\tfrac{e^4}{m^4}A_i\overline{A}_l\overline{\mu}^l\overline{A}_jA_k\mu^k\alpha].$$

The nonlinear Chern connection has the coefficients

$$N_i^j = \frac{\partial G^i}{\partial \mu^j}, \tag{2.3}$$

where G^i are the spray coefficients determined from

$$G^i = \tfrac{1}{4}g^{i\bar{k}}\left(\tfrac{\partial^2F^2}{\partial z^j\partial\overline{\mu}^k}+\tfrac{\partial^2F^2}{\partial\mu^j\partial\overline{z}^k}\right).$$

The coefficients for the canonical metrical connection are

$$F_{jk}^i = \frac{1}{2}g^{i\bar{l}}\left(\delta_kg_{j\bar{l}}+\delta_jg_{k\bar{l}}-\delta_{\bar{l}}g_{jk}\right) \tag{2.4}$$

$$C_{jk}^i = \frac{1}{2}g^{i\bar{l}}\left(\partial_kg_{j\bar{l}}+\partial_jg_{k\bar{l}}\right) \tag{2.5}$$

with

$$\delta_j = \frac{\partial}{\partial z^j}-N_j^k\frac{\partial}{\partial\mu^k} \tag{2.6}$$

3. **Conformal transformations in electro-complex Randers spaces**

We consider a conformal change as

$$F^*(z,\mu) = e^{\sigma(z)}F(z,\mu) \tag{3.1}$$

where $\sigma(z)$ is a function of z known as conformal factor.

Definition 2. *An electro-complex Randers space equipped with fundamental metric $F^*$is called electro-complex Randers conformally changed space.*

We have $F^*(z,\mu) = \alpha^*(z,\mu)+|\beta^*(z,\mu)|$. So we get

(3.2)

$$\alpha^* = e^{\sigma(z)}\alpha, \beta^*(z,\mu) = e^{\sigma(z)}\beta(z,\mu), \left|\beta^*(z,\mu)\right|^2 = e^{2\sigma(z)}\left|\beta(z,\mu)\right|^2$$

and by a simple calculation we obtain

Theorem 3. *The fundamental metric tensor of the electro-complex Randers conformally changed space is*

$$g^*_{i\bar{j}}(z,\mu) = e^{2\sigma(z)} g_{i\bar{j}}(z,\mu). \tag{3.3}$$

Theorem 4. *The Cartan covariant tensor C^* of the electro-complex Randers conformally changed space is given by*

$$C^*_{ij\bar{k}} = e^{2\sigma(z)} C_{ij\bar{k}}. \tag{3.4}$$

In order to determine the coefficients for the Chern connection we calculate

$$g^{*i\bar{l}} = e^{-2\sigma(z)} g^{i\bar{l}} \tag{3.5}$$

and

$$\frac{\partial g^*_{i\bar{j}}}{\partial z^k} = e^{2\sigma(z)} \left(2\frac{\partial \sigma}{\partial z^k} g_{j\bar{l}} + \frac{\partial g_{j\bar{l}}}{\partial z^k} \right) \tag{3.6}$$

Now we can determine

$$\begin{aligned} (1) \qquad \Gamma^{*i}_{jk} &= g^{*i\bar{l}} \frac{\partial g^*_{i\bar{j}}}{\partial z^k} = e^{-2\sigma(z)} g^{i\bar{l}} e^{2\sigma(z)} \left(2\frac{\partial \sigma}{\partial z^k} g_{j\bar{l}} + \frac{\partial g_{j\bar{l}}}{\partial z^k} \right) 3.7 \\ &= g^{i\bar{l}} 2\frac{\partial \sigma}{\partial z^k} g_{j\bar{l}} + g^{i\bar{l}} \frac{\partial g_{j\bar{l}}}{\partial z^k} = 2\frac{\partial \sigma}{\partial z^k} \delta^i_j + \Gamma^i_{jk} \end{aligned}$$

Theorem 5. *The coefficients for the Chern connection of the electro-complex Randers conformally changed space are*

$$\Gamma^{*i}_{jk} = 2\frac{\partial \sigma}{\partial z^k} \delta^i_j + \Gamma^i_{jk} \tag{3.8}$$

The curvature coefficients of the electro-complex Randers conformally changed space are

$$R^{*i}_{jk\bar{l}} = R^i_{jk\bar{l}} - 2\delta^i_j \frac{\partial^2 \sigma}{\partial z^k \partial z^{\bar{l}}}$$

so, the Ricci curvature is

$$R^{*m}_{jm\bar{l}} = R^m_{jm\bar{l}} - 2\delta^m_j \frac{\partial^2 \sigma}{\partial z^m \partial z^{\bar{l}}}$$

Theorem 6. *The Ricci curvature of the electro-complex Randers conformally changed space is*

$$R^*_{j\bar{l}} = R^m_{jm\bar{l}} - 2n \frac{\partial^2 \sigma}{\partial z^j \partial z^{\bar{l}}}$$

Now we can calculate the scalar Chern curvature
$S^* = g^{*j\bar{l}} R^*_{j\bar{l}} = e^{-2\sigma} g^{j\bar{l}} \left(R^m_{jm\bar{l}} - 2n \frac{\partial^2 \sigma}{\partial z^j \partial z^{\bar{l}}} \right) = e^{-2\sigma} \left(g^{j\bar{l}} R_{j\bar{l}} - 2n\Delta\sigma \right)$
where $\Delta\sigma = g^{j\bar{l}} \frac{\partial^2 \sigma}{\partial z^j \partial z^{\bar{l}}}$.

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