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STRONG IDEALS IN QI-ALGEBRAS

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Abstract. The notion of QI-algebras was introduced in 2017 as a generalization of the concept of BI-algebras. In this article, the concepts of strong ideals in QI-algebras are created and its properties are observed. Also, we study some properties of weak ideals and α-ideals in a QI-algebra. Moreover, we prove that the direct product of a family of QI-algebras is a QI-algebra.

1. **Introduction**

Logical algebras are algebraic structures designed to model logical systems, in order to encode inference rules as algebraic identities. Some logical algebras that model non-classical logics are implication algebras, described by preffixes as BCI, BCK, BCH, BI, BH, QI. In 1966, K. Iséki introduced BCI-algebras as models for the so- called BCI-logic, and also Y. Imai and K. Iséki introduced BCK-algebras. In 1967, J. C. Abbott studied implication algebras ([1]). Since then, many types of logical algebras have been introduced and studied.

In 2017, in [6], Borumand Saeid et al. introduced BI-algebras as a generalization of BCI-algebras and, in the same year, R. Kumar Bandaru ([9]) further generalized BI-algebras by introducing the concept of QI-algebra. The study of QI-algebras has be continued in [10, 12, 13, 14, 16].

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Some topics of interest are to examine the internal architecture of each logical algebra, but also to investigate some sub-structures of this algebra, as sub-algebras and ideals.

While in [10], the subject of study was various types of ideals (implicative, fantastic and normal ideals) in right distributive QI-algebras, in the article [16] pseudo-valuations of those algebras are discussed by M. Wojciechowska-Rysiawa. The article [13], written by this author, discusses the concept of positive implicative ideal in QI-algebras and establishes some of its important properties. The article [12] examines the internal architecture of (right distributive) QI-algebras.

S. A. Bhatti [4] (see also [3] and [5]) introduced the notion of strong ideals in BCI-algebras and obtained some results about it. The concept of strong ideals in BH-algebras was introduced in a slightly different way and studied in [2] by S. S. Ahn and J. H. Lee.

The main aim of this this article is to introduce the concept of strong ideal in QI-algebras and to investigate some of its properties. We also introduce and discuss the concepts of weak ideal and α-ideal in QI-algebras.

2. **Preliminaries**

The notion of BI-algebras comes from the (dual) implication algebra. An algebra $\mathfrak{A} =: (A, *, 0)$ of type $(2, 0)$ is called a BI-algebra ([6], Definition 3.1) if the following holds:

(Re) $(\forall x \in A)(x * x = 0)$,

(Im) $(\forall x, y \in A)(x * (y * x) = x)$.

A BI-algebra $\mathfrak{A}$ is said to be right distributive if the following

(DR) $(\forall x, y, z \in A)((x * y) * z = (x * z) * (y * z))$

is valid.

The concept of QI-algebra, as a generalization of the notion of BI-algebra, was introduced, in 2017, by R. Kumar Bandaru. An algebra $\mathfrak{A} =: (A, *, 0)$ of type $(2, 0)$ is called a QI-algebra ([9], Definition 3.1) if the following holds:

(Re) $(\forall x \in A)(x * x = 0)$,

(MR) $(\forall x \in A)(x * 0 = x)$

(QI) $(\forall x, y \in A)(x * (y * (x * y)) = x * y)$.

A QI-algebra $\mathfrak{A}$ is said to be right distributive if, additionally, the formula (DR) is satisfied.

Note that every BI-algebra is a QI-algebra but the converse need not be true ([9], Example 3.2).

The concept of sub-algebras in a QI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ is introduced by a standard way ([14], Definition 3.1). A nonempty subset S of A is a sub-algebra in $\mathfrak{A}$ if it satisfies the condition

(S1) $(\forall x, y \in A)((x \in S \wedge y \in S) \Longrightarrow x * y \in S)$.

It can immediately be concluded that the sub-algebra S in a QI-algebra $\mathfrak{A}$ satisfies the condition

(S0) $0 \in S$.

Indeed, since S is not empty there exists an element $x \in S$. Then, according to (S1) and (Re), we have $x \in S \Longrightarrow 0 = x * x \in S$. We denote the family of all sub-algebras of one QI-algebra $\mathfrak{A} =: (A, *, 0)$ by $\mathfrak{S}(A)$.

The concept of ideal in QI-algebras is determined by the following definition:

Definition 2.1. ([9], Definition 4.1) *A subset J of a QI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ is called an ideal of $\mathfrak{A}$ if the following holds:*

(J0) $0 \in J$,

(J1) $(\forall x, y \in A)((x * y \in J \wedge y \in J) \Longrightarrow x \in J)$.

We denote the family of all ideals of a QI-algebra $\mathfrak{A} =: (A, *, 0)$ by $\mathfrak{J}(A)$.

The following example illustrates the relationship between the concept of sub-algebra and the concept of ideal in QI-algebras: The families $\mathfrak{S}(A)$ and $\mathfrak{J}(A)$ are mutually distinct ([12], Remark 3.1). Additionally, these families are complete lattices by [10], Theorem 4.1 and [12], Theorem 3.1.

Example 2.2. *Let $A = \{0, a, b, c\}$ be a set with the operation given by the table*

$*$	0	a	b	c
0	0	b	a	0
a	a	0	a	0
b	b	b	0	b
c	c	c	b	0

Then $\mathfrak{A} =: (A, *, 0)$ *is a QI-algebra* ([9], *Example 3.2*).

The subsets $S_0 = \{0\}$, $S_3 = \{0, c\}$ *and* $S_4 = \{0, a, b\}$ *are sub-algebras of the QI-algebra* $\mathfrak{A}$. *However, by direct checking it can be determined that the subsets* $S_1 = \{0, a\}$, $S_2 = \{0, b\}$, $S_5 = \{0, a, c\}$ *and* $S_6 = \{0, b, c\}$ *are not sub-algebras of the QI-algebra* $\mathfrak{A}$. *For illustration, for example, for the elements* $0 \in S_6$ *and* $b \in S_6$ *we have* $0*b = a \notin S_6$.

Subsets $J_0 = \{0\}$, $J_1 = \{0, a\}$, $J_3 = \{0, c\}$ *and* $J_5 = \{0, a, c\}$ *of the set* A *are ideals of the QI-algebra* $\mathfrak{A}$. *The subset* $J_2 = \{0, b\}$ *is not an ideal in* $\mathfrak{A}$ *because, for example, for* $b \in J_2$ *we have* $c * b = b \in J_2$ *but* $c \notin J_2$. *The subsets* $J_4 = \{0, a, b\}$ *and* $J_6 = \{0, b, c\}$ *are not ideals in* $\mathfrak{A}$ *because, for example, for* $c \in J_6$ *we have* $a * c = 0 \in J_6$ *but* $a \notin J_6$. *Similarly, we have* $b \in J_4$ *and* $c * b = b \in J_4$ *but* $c \notin J_4$. ∎

Remark 2.3. *As shown in the previous example, the sub-algebra* $S_4 = \{0, a, b\}$ *in* $\mathfrak{A}$ *is not an ideal in* $\mathfrak{A}$, *while the ideal* $J_5 = \{0, a, c\}$ *in* $\mathfrak{A}$ *is not a sub-algebra in* $\mathfrak{A}$. *Therefore, the families* $\mathfrak{S}(A)$ *and* $\mathfrak{J}(A)$ *are not comparable by inclusion. This reconfirms the observation made in* [12], *Remark 3.1.*

The following proposition gives some of the basic properties of QI-algebras.

Proposition 2.4 ([9], Proposition 3.5)**.** *Let* $\mathfrak{A} =: (A, *, 0)$ *be a QI-algebra. Then:*

(1) $(\forall x \in A)(x * (0 * x) = x)$,

(2) $(\forall x, y \in A)(x * y = y \implies x = y)$, ,

(3) $(\forall x \in A)(x * 0 = 0 \implies x = 0)$,

(4) $(\forall x, y \in A)(x * y = x \implies x * (y * x) = x)$

3. **The main results: Strong ideals in QI-algebras**

This section is the central part of this article. First it was shown (Theorem 3.1) that the direct product of a family of QI-algebras is a QI-algebra again. This is followed by the design of the concept of strong ideals in QI-algebras and the study of its properties.

In what follows, we introduce the direct product of a family of QI-algebras. Let $\{(A_i, *_i, 0_i) : i \in I\}$ be a family of QI-algebras. On the cartesian product

$$\prod_{i \in I} A_i =: \{f : I \longrightarrow \cup_{i \in I} A_i \mid (\forall i \in I)(f(i) \in A_i)\},$$

we define the operation $\odot$ as follows

$$(\forall f, g \in \prod_{i\in I} A_i)(\forall \in I)((f \odot g)(i) =: f(i) *_i g(i)),$$

we created the structure $(\prod_{i\in I} A_i, \odot, f_0)$, where f_0 was chosen as follows

$$(\forall i \in I)(f_0(i) =: 0_i).$$

Before we start working with direct products of QI-algebras, we say that the operation determined in this way is well-defined. If a priori we accept conditions that ensure the existence of non-empty direct product, we can prove the following theorem.

Theorem 3.1. *The direct product of any family of QI-algebras, determined as above, is a QI-algebra.*

Proof. By direct verification, it can be proved that this structure satisfies the axioms of QI-algebra:

Let $f, g \in \prod_{i\in I} A_i$ be arbitrary elements and $i \in I$. Then, we have:

(Re) $(f \odot f)(i) = f(i) *_i f(i) = 0_i$.

(M) $(f \odot f_0)(i) = f(i) *_i f_0(i) = f(i) *_i 0_i = f(i)$.

(QI) Considering that

$$\begin{aligned}(f \odot (g \odot (x \odot (f \odot g)))(i) &= f(i) *_i (g(i) *_i (f(i) *_i g(i))) = f(i) *_i g(i) \\ &= (f \odot g)(i),\end{aligned}$$

we have that (QI) is a valid formula for the observed structure.

Therefore, the structure $(\prod_{i\in I} A_i, \odot, f_0)$ is a QI-algebra. ∎

This result is strongly related to Theorem 3.16.

In what follows, we will deal with the creation of the concept of a strong ideals in QI-algebras and an examination of its properties. The design of the concept of strong ideals in QI-algebras introduces the following concept of strong ideals in BH-algebras ([2]).

Definition 3.2. *A non-empty subset J of a QI-algebra $\mathfrak{A} =: (A, *, 0)$ is called a strong ideal in $\mathfrak{A}$ if it satisfies (J0) and the following condition:*

(StJ) $(\forall x, y, z \in A)(((x * y) * z \in J \wedge y \in J) \Longrightarrow x * z \in J)$.

The family of all strong ideals in a QI-algebra $\mathfrak{A} =: (A, *, 0)$ is denoted by $\mathfrak{J}_s(A)$.

Remark 3.3. *Sometimes, this class of ideals in logical algebras is called a T-ideal as, for example, in* [11]*, Definition 2.5.*

The set A is a trivial strong ideal in a QI-algebra $\mathfrak{A} =: (A, *, 0)$. So, $A \in \mathfrak{J}_s(A)$.

Proposition 3.4. *Any strong ideal in a QI-algebra* $\mathfrak{A}$ *is an ideal in* $\mathfrak{A}$*. This means* $\mathfrak{J}_s(A) \subseteq \mathfrak{J}(A)$.

Proof. It is clear that J satisfies the condition (J0). Putting $z = 0$ in (StJ), we obtain (J1). ∎

Proposition 3.5. *In every QI-algebra* $\mathfrak{A} =: (A, *, 0)$*, the subset* $\{0\}$ *is a strong ideal in* $\mathfrak{A}$*. So,* $\{0\} \in \mathfrak{J}_s(A)$.

Proof. Let $x, y, z \in A$ be arbitrary elements such that $(x * y) * z = 0$ and $y = 0$. Then $x * z = (x * 0) * z = 0$ in accordance with (M). So, the subset $\{0\}$ is a strong ideal in $\mathfrak{A}$. ∎

Since the family $\mathfrak{J}_s(A)$ is not empty, the following theorem can be proved.

Theorem 3.6. *The family* $\mathfrak{J}_s(A)$ *for an arbitrary QI-algebra* $\mathfrak{A} =: (A, *, 0)$ *is a complete lattice.*

Proof. Let $\{J_k\}_{k \in I}$ be a family of string ideals in $\mathfrak{A}$.

It is clear that $0 \in \bigcap_{k \in I} J_k$ holds. Let $x, y, z \in A$ be arbitrary elements such that $(x * y) * z \in \bigcap_{k \in I} J_k$ and $y \in \bigcap_{k \in I} J_k$. Then, for each $k \in I$, $(x * y) * z \in J_k$ and $y \in J_k$ hold. Thus $x * z \in J_k$ since J_k is a strong ideal in $\mathfrak{A}$. This means $x * z \in \bigcap_{k \in I} J_k$. Therefore, $\bigcap_{k \in I} J_k$ is a strong ideal in $\mathfrak{A}$.

If we denote by $\mathcal{Z}$ the family of all strong ideals in $\mathfrak{A}$ that contain $\bigcup_{k \in I} J_k$, then $\cap \mathcal{Z}$ is a strong ideal in $\mathfrak{A}$ according to the first part of this proof.

If we put $\sqcap_{k \in I} J_k = \bigcap_{k \in I} J_k$ and $\sqcup_{k \in I} J_k = \cap \mathcal{Z}$, then $(\mathfrak{J}_s(A), \sqcap, \sqcup)$ is a complete lattice. ∎

Corollary 3.7. *Let* $\mathfrak{A} =: (A, *, 0)$ *be a QI-algebra. For each* $x \in A$*, there exists a minimal strong ideal* J_x *in* $\mathfrak{A}$ *that contains* x*.*

Proof. If $\mathcal{Z}$ is the family of all strong ideals in $\mathfrak{A}$ that contain x, then $J_x =: \cap \mathcal{Z}$ is a strong ideal in $\mathfrak{A}$ that contains x. If J is an strong ideal in $\mathfrak{A}$ that contains x, then $J \in \mathcal{Z}$. Thus $J_x \subseteq J$. Therefore, J_x is a minimal strong ideal in $\mathfrak{A}$ that contains x. ∎

The following theorem gives another determination of the concept of strong ideal in QI-algebras.

Theorem 3.8. *Let J be an ideal in a QI-algebra $\mathfrak{A} =: (A, *, 0)$. Then J is a strong ideal in $\mathfrak{A}$ if and only if the following holds*

(StJ1) $(\forall x, y, z \in A)((x*z \in A \setminus J \wedge y \in J) \implies (x*y)*z \in A \setminus J)$.

Proof. Let J be a strong ideal in $\mathfrak{A}$ and let $x, y, z \in A$ be arbitrary elements such that $x*z \notin J$ and $y \in J$. If we assume that $(x*y)*z \in J$, then it would be $x * z \in J$ since J is a strong ideal in $\mathfrak{A}$. We got a contradiction. So, it must be $(x * y) * z \notin J$.

Conversely, let (StJ1) be a valid formula for the ideal J and let $x, y, z \in A$ be such that $(x * y) * z \in J$ and $y \in J$. Assume that $x * z \notin J$. Then, according to (StJ1), there would be $(x * y) * z \notin J$. We got a contradiction. Therefore, $x * z \in J$. This proves that J is a strong ideal in $\mathfrak{A}$. ∎

Analogously to the previous one, the following theorem can be proven:

Theorem 3.9. *Let J be an ideal in a QI-algebra $\mathfrak{A} =: (A, *, 0)$. Then J is a strong ideal in $\mathfrak{A}$ if and only if the following holds*

(StJ2) $(\forall x, y, z \in A)(((x*y)*z \in J \wedge x*z \in A \setminus J) \implies y \in A \setminus J)$.

Proof. Let J be a strong ideal in $\mathfrak{A}$ and let $x, y, z \in A$ be arbitrary elements such that $(x * y) * z \in J$ and $x * z \notin J$. If we assume that $y \in J$, then it would be $x * z \in J$ since J is a strong ideal in $\mathfrak{A}$. We got a contradiction. So, it must be $y \notin J$.

Conversely, let (StJ2) be a valid formula for the ideal J and let $x, y, z \in A$ be such that $(x * y) * z \in J$ and $y \in J$. Assume that $x * z \notin J$. Then, according to (StJ2), there would be $y \notin J$. We got a contradiction. Therefore, $x * z \in J$. This proves that J is a strong ideal in $\mathfrak{A}$. ∎

In what follows we need the following lemma:

Lemma 3.10 ([12], Proposition 3.4)**.** *Let J be an ideal in a right distributive QI-algebra $\mathfrak{A} =: (A, *, 0)$. Then:*

$(\forall x, y \in A)(x \in J \implies x * y \in J)$.

Further on, we have:

Theorem 3.11. *Every ideal in a right distributive QI-algebra is a strong ideal in it.*

Proof. Let J be an ideal in a right distributive QI-algebra $\mathfrak{A} =: (A, *, 0)$ and let $x, y, z \in A$ be such that $(x * y) * z \in J$ and $y \in J$. First, by Lemma 3.10, we have $y \in J \implies y * z \in J$ for arbitrary

$z \in A$. On the other hand, from $(x * z) * (y * z) = (x * y) * z \in J$ and $y * z \in J$ it follows $x * z \in J$ according to (J1) since $\mathfrak{A}$ is a right distributive QI-algebra. Therefore, J is a strong ideal in $\mathfrak{A}$. ∎

Judging by the previous theorem, it makes sense to talk about strong ideals in QI-algebras only in non right distributive QI-algebras.

Let $\mathfrak{A} =: (A, *, 0_A)$ï€ª and $\mathfrak{B} =: (B, \star, 0_B)$ï€¨ be QI-algebras. A QI-homomorphism is a mapping $f : A \longrightarrow B$ satisfying the condition

$$(\forall x, y \in A)(f(x * y) = f(x) \star f(y)).$$

It is easy to prove that $f(0_A) = 0_B$ holds.

Theorem 3.12. *Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism of QI-algebras. If C is a strong ideal of $\mathfrak{B}$, then $f^{-1}(C)$ is a strong ideal in $\mathfrak{A}$.*

Proof. Since $f(0) = 0$, we have $0 \in f^{-1}(C)$.

Let $x, y, z \in A$ be such that $(x * y) * z \in f^{-1}(C)$ and $y \in f^{-1}(C)$. Then $(f(x)*f(y))*f(z) = f((x*y)*z) \in C$ and $f(y) \in C$. Since C is a strong ideal in $\mathfrak{B}$, it follows from (StJ) that $f(x*z) = f(x)*f(z) \in C$. So that $x * z \in f^{-1}(C)$. Hence $f^{-1}(C)$ is a strong ideal in $\mathfrak{A}$. ∎

Corollary 3.13. *Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism of QI-algebras. Then $Kerf =: \{x \in A : f(x) = 0\}$ is a strong ideal of $\mathfrak{A}$.*

Proof. Since the subset $\{0\}$ is a strong ideal in the QI-algebra $\mathfrak{B}$, by Proposition 3.5, we have that the kernel $Kerf = f^{-1}(\{0\})$ of the homomorphism f is a strong ideal in $\mathfrak{A}$ in accordance with the previous theorem. ∎

Example 3.14. *Let $\mathfrak{A} =: (A, *, 0)$ be a QI-algebra as in Example 2.2.*

Subsets J_0, J_1 and J_5 are strong ideals in $\mathfrak{A}$.

The ideal $J_3 = \{0, c\}$ is not a strong ideal in $\mathfrak{A}$ because, for example, for $x = a$, $y = c$ and $z = 0$ we have $(a * c) * 0 = 0 * 0 = 0 \in J_3$ and $c \in J_3$ but $a * 0 = a \notin J_3$. ∎

Remark 3.15. *The previous example shows that an ideal in a QI-algebra, in the general case, does not have to be a strong ideal in that algebra. So, $\mathfrak{J}_s(A) \subsetneqq \mathfrak{J}(A)$.*

Further on, we have:

Theorem 3.16. *Let $\{(A_i, *_i, 0_i) : i \in I\}$ be a family of QI-algebras, K be a subset of I and let J_i be a strong ideal in $(A_i, *_i, 0_i)$ for each $i \in K$. Then $\prod_{i \in I} T_i$, where $T_i = J_i$ for $i \in K$ and $T_i = A_i$ for $i \in I \backslash K$, is a strong ideal in the QI-algebra $\prod_{i \in I} A_i$.*

Proof. First, it is clear that $f_0 \in \prod_{i\in I} T_i$.

If $K = \emptyset$, then $\prod_{i\in I} T_i = \prod_{i\in I} A_i$, so $\prod_{i\in I} T_i$ is certainly an ideal in $\prod_{i\in I} A_i$. Assume, therefore, that $K \neq \emptyset$.

Let $x, y, z \in \prod_{i\in I} A_i$ be such that $(x \odot y) \odot z \in \prod_{i\in I} T_i$ and $y \in \prod_{i\in I} T_i$. This means $(x(i) *_i y(i)) *_i z(i) \in J_i$ and $y(i) \in J_i$ for each $i \in K$. Then $(x \odot z)(i) = x(i) *_i z(i) \in J_i$ since J_i is a strong ideal in $(A_i, *_i, 0_i)$ for each $i \in K$. Hence $x \odot z \in \prod_{i\in I} T_i$.

As shown, $\prod_{i\in I} T_i$ is a strong ideal in $\prod_{i\in I} A_i$. ∎

Example 3.17. *Let $\mathfrak{A} =: (A, *, 0)$ be a QI-algebra as in Example 2.2. Then according to the Theorem 3.1, $\mathfrak{A} \times \mathfrak{A} =: (A \times A, \otimes, (0, 0))$ is a QI-algebra also, where the operation $\otimes$ is defined as follows*

$$(\forall x, y, u, v \in A)((x, y) \otimes (u, v) =: (x * u, y * v)).$$

The subset $J_1 = \{0, a\}$ is a strong ideal in $\mathfrak{A}$ as shown in Example 3.14. The subsets $J_1 \times A$, $A \times J_1$ and $J_1 \times J_1$ are strong ideals in $\mathfrak{A} \times \mathfrak{A}$ according to the Theorem 3.16. ∎

At the end of this section, we prove another specific property of strong ideals in QI-algebras.

Proposition 3.18. *Let J be a strong ideal in a QI-algebra $\mathfrak{A} =: (A, *, 0)$. Then holds*

$$(\forall x, y \in A)(y \in J \implies x * (x * y) \in J).$$

Proof. If we put $z = x * y$ in (WJ), we get

$$0 = (x * y) * (x * y) \in J \wedge y \in J \implies x * (x * y) \in J.$$

This gives $y \in J \implies x * (x * y) \in J$ since $0 \in J$. ∎

4. Conclusion: Weak ideals and α-â2c6'ideals in QI-algebras

In this paper, the concept of strong ideals in QI-algebras is introduced and its important properties are reviewed. In this way, the spectrum of different ideals in this class of logical algebras, introduced and analyzed so far, is supplemented by the concept of strong ideals. As a continuation of previous research on the spectrum of ideals in QI-algebras, that spectrum can be supplemented, for example, by introducing the so-called weak ideals, p-ideals and α-ideals by recognizing their properties as well as their mutual relations. For the sake of illustration, we present the first and third possibilities, since the formula for determining of p-ideals is generally known (See, for example, [15] or [7], Definition 1).

Definition 4.1. *A non-empty subset J of a QI-algebra $\mathfrak{A} =: (A, *, 0)$ is called a weak ideal in $\mathfrak{A}$ if it satisfies the following condition:*

(WJ) $(\forall x, y, z \in A)((x * (y * z) \in J \wedge y \in J) \Longrightarrow x * z \in J)$.

It can be shown that the weak ideal in a QI-algebra also satisfies the condition (J0). Indeed, since J is a nonempty subset of A, there exists some $x \in A$ such that $x \in J$. Then we have $J \ni x = x * 0 = x*(x*x)$ and $x \in J$ from which, by (WJ), it follows that $0 = x*x \in J$.

In addition, we obtain:

Proposition 4.2. *Any weak ideal in a QI-algebra is a sub-algebra in it.*

Proof. Let J be a weak ideal in a QI-algebra $\mathfrak{A}$ and let $x, y \in A$ be arbitrary elements such that $x \in J$ and $y \in J$. Then $x * (y * y) = x * 0 = x \in J$ and $y \in J$. Thus $x * y \in J$ by (WJ). ∎

Proposition 4.3. *Any weak ideal in a QI-algebra is an ideal in it.*

Proof. Putting $z = 0$ in (WJ), we get (J1). ∎

The concept of α-ideal in QI-algebra is given by the following definition:

Definition 4.4. *A non-empty subset J of a QI-algebra $\mathfrak{A}$ is called an α-ideal if, in addition to (J0), it also satisfies the following condition*

(αJ) $(\forall x, y, z \in A)((x * y \in J \wedge y * z \in J) \Longrightarrow x * z \in J)$.

Proposition 4.5. *Any α-ideal in a QI-algebra is an ideal in it.*

Proof. Putting $z = 0$ in (αJ), we get (J1). ∎

Example 4.6. *Let $\mathfrak{A} = (A, *, 0)$ be a QI-algebra as in Example 2.2. The ideals J_0 and J_1 are α-ideals in $\mathfrak{A}$, but the ideal J_3 is not an α-ideal in $\mathfrak{A}$ because we have $0 * c = 0 \in J_3$ and $c * a = c \in J_3$ but $0 * a = b \notin J_3$. The ideal J_5 is also not an α-ideal in $\mathfrak{A}$ because, for example, for $x = 0$, $y = c$ and $z = a$ we have $0 * c = 0 \in J_5$ and $c * a = c \in J_5$ but $0 * a = b \notin J_5$.* ∎

Thus, the family of α-ideals in a QI-algebra differs from the families $\mathfrak{J}(A)$ and $\mathfrak{J}_s(A)$.

In one of the next texts by this researcher, material on weak ideals in QI-algebras will be presented in more detail.

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